Contraintes qualitatives pour le raisonnement sur le temps et l'espace

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par

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Introduction

Thématique de recherche centrale en Informatique depuis de nombreuses décennies, le raisonnement concernant le temps et l'espace a notamment été étudié dans des domaines tels que la compréhension du langage naturel, la spécification et la vérification de programmes et de systèmes, les systèmes de gestion de base de données temporelles et spatiales, les systèmes d'informations géographiques (SIG), la planification temporelle et spatiale, *etc.*

Dans les années 80, ALLEN [All81, All83] propose un formalisme pour raisonner sur le temps qui deviendra un modèle pour de nombreux autres formalismes dévolus au raisonnement sur le temps et l'espace. ALLEN représente les entités temporelles (actions, activités, événements) par des intervalles de la droite et considère 13 relations de base entre ces intervalles représentant des positions particulières entre entités temporelles. La relation de base $meets$, par exemple, est constituée de l'ensemble des paires d'intervalles telles que la borne supérieure du premier intervalle coïncide avec la borne inférieure du second. Cette relation peut être utilisée pour représenter le fait qu'une activité se termine à l'instant où une autre débute. L'originalité de l'approche d'ALLEN ne réside pas réellement dans le fait de considérer les 13 relations de base correspondant à toutes les configurations possibles des quatre bornes de deux intervalles mais plutôt dans la manière dont est représenté et géré l'ensemble des informations temporelles concernant un système.

En effet, ALLEN représente les informations temporelles par un ensemble de contraintes. Chacune de ces contraintes est définie par un ensemble de relations de base et représente les positions relatives possibles entre deux entités temporelles du système à modéliser. Dans la suite de ce rapport, nous appellerons réseau de contraintes qualitatives un tel ensemble de contraintes (RCQ en abrégé). Un RCQ est une description qualitative d'un ensemble de configurations possibles d'entités puisqu'elle ne fait explicitement aucune référence à des données quantitatives ou métriques. Malgré tout, chaque relation de base correspond à une abstraction d'un ensemble de configurations pouvant être décrites quantitativement. Une telle abstraction peut être souhaitable lorsque les informations à représenter ne sont pas suffisamment précises ou sont qualitatives par nature. Un RCQ permet également de représenter des informations incomplètes ou incertaines puisque chaque contrainte est définie par un ensemble de relations de base.

Étant donné un RCQ, ALLEN propose un mécanisme d'inférence basé sur une table de composition (appelée également table de transitivité). Dans le cadre du calcul des intervalles, la table de composition décrit, pour chaque couple de relations de base r et r', l'ensemble des relations de base pouvant être satisfaites par deux intervalles x et y lorsqu'il existe un troisième intervalle z tel que x et z satisfont r et z et y satisfont r'. La méthode d'inférence proposée consiste à itérer sur tous les triplets de variables v_i, v_j, v_k du RCQ l'opération consistant à supprimer de l'ensemble de relations de base définissant la contrainte entre v_i et v_k, les relations de base non possibles du fait des relations de base permises entre v_i et v_k et celles entre v_k et v_j. Ces relations de base non possibles sont déduites de la table de composition.

De plus, le traitement est réalisé jusqu'à ce qu'un point fixe soit obtenu. En fait, ALLEN applique une méthode de filtrage des contraintes à l'aide de l'opération de composition (appelée aujourd'hui faible composition ou composition algébrique et notée dans le suite par $\diamond$). Le sous-RCQ obtenu est équivalent au RCQ initial et admet une propriété de cohérence locale que nous appellerons $\diamond$-cohérence.

La méthode proposée par ALLEN n'est pas complète pour décider de la cohérence ou non d'un RCQ. En effet, elle permet de supprimer des relations de base non possibles mais n'enlève pas dans le cas général toutes les relations de base non possibles. En fait, cette méthode de calcul de la fermeture d'un RCQ par faible composition est une méthode polynomiale tandis que le problème de décider de la cohérence d'un RCQ du calcul des intervalles est un problème NP-complet. Compte tenu de ces résultats de complexité, de nombreuses études ont eu pour objectif de caractériser des ensembles de relations de base du calcul des intervalles pour lesquels le problème de la cohérence est un problème polynomial et pouvant notamment se résoudre à partir de la méthode du calcul de la fermeture par faible composition. Aujourd'hui, une cartographie complète des classes traitables et de celles qui ne le sont pas a été tracée pour le calcul d'ALLEN [KJJ03] . Définir un RCQ sur un fragment traitable du calcul apporte l'assurance de pouvoir décider de sa cohérence en temps polynomial.

De manière générale, pour décider du problème de la cohérence d'un RCQ, un algorithme de recherche avec retour arrière peut être mis en œuvre. À chaque étape de la recherche, une contrainte est sélectionnée puis définie par une des relations de base la composant. De plus, le calcul de la fermeture par faible composition peut être réalisé afin de supprimer des relations non possibles. L'algorithme s'arrête lorsqu'un scénario (un RCQ défini par des relations singletons) fermé par faible composition est caractérisé ou lorsque l'ensemble des choix pris lors de la recherche ont conduit à un échec. Dans le premier cas, la cohérence du RCQ initial a été caractérisée. Dans le second cas, nous pouvons affirmer qu'il est non cohérent. Cet algorithme de recherche a été rendu beaucoup plus efficace par l'utilisation de classes traitables [LR92, Neb96, Neb97]. En effet, à chaque étape de la recherche, plutôt que d'instancier la contrainte sélectionnée par chacune des relations singletons correspondant à ses relations de base, nous pouvons réaliser une instanciation par des sous-relations de la contrainte issues d'une classe traitable. Le facteur de branchement dans l'arbre de recherche se trouve ainsi diminué.

Le calcul des intervalles d'ALLEN a initié une multitude de définitions de formalismes dévolus au raisonnement sur le temps et sur l'espace. Un des points communs de ces formalismes est qu'ils sont tous définis à partir d'un ensemble de relations de base complètes et mutuellement exclusives représentant des configurations qualitatives particulières d'entités temporelles ou spatiales. De plus, pour tous ces formalismes, des RCQ sont utilisés pour représenter l'ensemble des informations temporelles ou spatiales du système considéré. Le raisonnement se fait également par résolution de contraintes à l'aide du calcul de la fermeture par faible composition.

Dans le chapitre suivant, nous définissons formellement ce qu'est un formalisme qualitatif pour le temps et l'espace utilisant des ensembles de contraintes comme langage de représentation des connaissances. Nous donnons quelques exemples de formalismes qualitatifs représentatifs compte tenu du choix des entités et des relations de base considérées. Nous posons quelques définitions et propriétés concernant les RCQ. Nous décrivons également nos travaux [BCL03b]**[CL04a]**[p103] concernant l'axiomatisation en logique du premier ordre des relations de base du calcul des points cycliques [IC00] et du calcul des intervalles cycliques [BO00], travaux réalisés en collaboration avec PHILIPPE BALBIANI et GÉRARD LIGOZAT.

Le deuxième chapitre est consacré au problème de la cohérence des RCQ. Dans un premier temps, nous détaillons l'algorithme de recherche le plus efficace utilisé afin de résoudre ce problème. Dans un deuxième temps, nous présentons certains de nos travaux concernant ce problème :

- nos études consacrées à la recherche de classes traitables pour certains formalismes qualitatifs (le calcul des intervalles généralisés [BCFO98a, BCFO98b, BCL00]**[Con04]**[p115], le calcul des rectangles et celui des n-pavés [BCF98, BCF99a, BCF99d], le calcul des n-points [BCF99c, BC02b], le calcul des points cycliques [BCL03b], le calcul INDU [BCL03a]**[BCL06]**[p141]),

- notre étude concernant la notion de contraintes éligibles et la notion de contraintes gelées [CLS07],
- nos travaux dévolus aux cohérences locales appelées $\overset{\diamond}{f}$-cohérences **[CL10]**[p203],
- nos recherches concernant la résolution du problème de la cohérence des RCQ par traduction en CSP discrets [DCLS07] et en problème SAT [CD07, CD08],
- nos travaux concernant les décompositions arborescentes de RCQ [Con11, CC11]**[CD11]**[p233].

Ces différents travaux ont été réalisés en collaboration avec Philippe Balbiani, Assef Chmeiss, Christophe Lecoutre, Gérard Ligozat, Lakhdar Saïs, ainsi qu'avec Mahmoud Saade et Dominique D'Almeida que j'ai co-encadrés lors de leur thèse.

En première partie du troisième chapitre, nous décrirons nos travaux **[BC02a]**[p85] concernant une logique spatio-temporelle correspondant intuitivement à la logique propositionnelle temporelle linéaire pour laquelle les propositions sont définies par des contraintes qualitatives. Nous présenterons également un travail concernant un sous-langage de cette logique spatio-temporelle [CLT05]. Ces différents travaux ont été réalisés en collaboration avec Philippe Balbiani, Gérard Ligozat, Mahmoud Saade et Stavros Tripakis.

La seconde partie du troisième chapitre est consacrée à des travaux [CKS08, CKMS09b, CKMS09c] **[CKMS09a, CKMS10b]**[p165,p185] concernant la problématique de la fusion des RCQ. Dans le cadre de la thèse de Nicolas Schwind co-encadré par Souhila Kaci, Pierre Marquis et moi-même, nous avons défini différentes familles d'opérateurs de fusion de RCQ que nous décrirons.

Pour conclure, nous décrirons un ensemble de perspectives de recherche concernant nos travaux.

Chapitre 1

Les formalismes qualitatifs pour le temps et l'espace à base de contraintes

Sommaire

UN formalisme qualitatif pour le temps et l'espace à base de contraintes considère un ensemble de relations de base représentant des configurations temporelles ou spatiales particulières entre des entités. À partir de ces relations peuvent être définis des problèmes de satisfaction de contraintes particuliers permettant de s'abstraire de descriptions utilisant des valeurs numériques qui sont parfois difficiles, voire impossibles à réaliser pour certaines applications. Une représentation qualitative d'informations temporelles ou spatiales permet également de mettre en œuvre des méthodes de raisonnement efficaces comme nous le verrons par la suite.

Le premier formalisme qualitatif pour le temps est issu du système proposé par ALLEN en 1981 [All81]. L'objectif d'ALLEN était de définir un système permettant de représenter des connaissances temporelles dans le cadre de la compréhension du langage naturel et de pouvoir raisonner sur ces connaissances, en particulier lors de l'ajout de nouvelles informations. Le modèle de temps retenu par ALLEN utilise des intervalles de la droite pour représenter des entités temporelles telles que des actions, des événements ou bien encore des activités. Un ensemble de relations (appelées relations de base) correspondant à des configurations qualitatives particulières entre deux intervalles peut alors être considéré. Dans cet ensemble, se trouve par exemple la relation $precedes$ qui est satisfaite par deux intervalles dans le cas où la borne supérieure du premier intervalle est strictement plus petite que la borne inférieure du second. Cette relation permet de représenter le fait qu'une activité se déroule strictement avant une autre. ALLEN utilise un réseau de contraintes afin de représenter l'ensemble des connaissances temporelles d'un système à modéliser. Ce réseau spécifie des relations de base possibles pour chaque couple d'entités temporelles. Un mécanisme de raisonnement proposé consiste en une propagation des contraintes réalisée avec une table de transitivité (appelée aujourd'hui table de (faible) composition) à partir de laquelle peuvent être déduites toutes les configurations possibles de trois intervalles. Compte tenu de son compromis entre pouvoir d'expressivité et efficacité de raisonnement, le calcul des intervalles d'ALLEN a été utilisé dans de nombreux domaines applicatifs ou théoriques de l'Informatique tels que la planification [AK83, PA87, All91, Dor95], le traitement du langage naturel [SC88], les bases de données temporelles [Sno87, CZ98] , les bases de données multimédia [LG93], la gestion de documents mulitmédia [LE08], la biologie molléculaire [GS93] et le workflow [LSPG06]. Depuis trois décennies, de nombreux formalismes inspirés du calcul d'ALLEN ont été proposés dans le cadre du raisonnement temporel mais également dans le cadre du raisonnement spatial. Dans les années 90, a été notamment défini le calcul RCC [RC89, RCC92a, RCC92b, Ege91] permettant de raisonner sur les positions relatives d'entités spatiales à l'aide de huit relations de base d'ordre topologique. Le calcul RCC a connu un succès aussi important que le calcul d'ALLEN. Ce formalisme est utilisé dans de nombreuses applications, en particulier des applications concernant les SIG (Systèmes d'Informations Géographiques).

Dans ce chapitre, nous introduisons les différents éléments définissant un formalisme qualitatif à base de contraintes : les relations de base et les relations disjonctives ainsi que les différentes opérations qui leurs sont associées. Nous décrivons quelques formalismes qualitatifs représentatifs, aussi bien du point de vue des entités prises en compte (points, intervalles, régions, ...) que du point de vue du type des relations de base considérées (relations de type directionnel, relations de type topologique,...). Après avoir introduit les réseaux de contraintes qualitatives et certaines propriétés les concernant, nous décrivons nos travaux [BCF98, BCL03b]**[CL04a]**[p103] concernant l'axiomatisation en logique du premier ordre des relations du calcul des rectangles [BCFO98a], de celles du calcul des points cycliques [IC00] et de celles du calcul des intervalles cycliques [BO00].

1.1 Relations de base et relations disjonctives des formalismes qualitatifs

Un formalisme qualitatif considère un ensemble fini B de relations non vides appelées relations de base ou encore relations atomiques. Ces relations sont définies sur un domaine D et sont de même arité a (avec $a > 1$). Les éléments de D sont utilisés pour représenter les entités temporelles ou spatiales du système. D est généralement un ensemble non fini. Les relations de B permettent de distinguer et de caractériser des configurations qualitatives entre deux ou plusieurs entités. Tout tuple de a éléments de D satisfait une et une seule relation de base de l'ensemble de B (les relations de B sont dites complètes et mutuellement exclusives). Formellement, l'ensemble B satisfait les deux propriétés suivantes :

- (1) $\bigcup\{r \in \mathsf{B}\} = \overbrace{\mathsf{D} \times \ldots \times \mathsf{D}}^{\text{a fois}} = \mathsf{D}^a$ (complétude),
- (2) $\forall r, r' \in \mathsf{B}$ tels que $r \neq r', r \cap r' = \emptyset$ (exclusion mutuelle).

Une relation (complexe) d'un formalisme qualitatif correspond à une union de relations de base. Nous définissons l'ensemble $\mathcal{A}$ par l'ensemble des relations correspondant à toutes les unions des relations de base. Formellement, $\mathcal{A}$ est défini par $\mathcal{A} = \{\bigcup E : E \subseteq \mathsf{B}\}$. Il est habituel de représenter une relation $r_1 \cup \ldots \cup r_j$ (avec $r_i \in \mathsf{B}$ pour tout $i \in \{1, \ldots, j\}$) de $\mathcal{A}$ par l'ensemble des relations de base $\{r_1, \ldots, r_j\}$ qui la composent. De ce fait, lorsqu'il n'y a pas d'ambiguïté, nous ne ferons pas de distinction entre $\mathcal{A}$ et 2^{B} par la suite. Ainsi, l'ensemble 2^{B} représentera l'ensemble des relations du formalisme qualitatif considéré. Parmi les relations de 2^{B}, nous notons Ψ la relation appelée relation totale ou relation universelle et contenant toutes les relations de base de B. La relation vide, notée $\emptyset$, correspond à la relation ne contenant aucune des relations de base. De plus, nous supposons qu'il existe une relation de 2^{B}, notée par Id, correspondant à la relation identité sur D^a. Notons que, dans la plupart des formalismes qualitatifs, cette relation correspond à une seule relation de base.

Lorsque les relations de base sont d'arité 2 nous utiliserons une notation infixée, dans le cas contraire une notation préfixée. Ainsi, pour des relations de base d'arité 2, deux éléments $x, y \in \mathsf{D}$ satisfont la relation $r \in \mathsf{B}$, notée $x\ r\ y$, si et seulement si $(x, y) \in r$. Pour $R \in 2^{\mathsf{B}}$ et $x, y \in \mathsf{D}$, x et y satisfont R, noté $x\ R\ y$, lorsqu'il existe une relation de base $r \in R$ tel que $x\ r\ y$. La relation totale Ψ est toujours satisfaite par deux éléments, tandis que la relation vide ne l'est jamais. Lorsque l'arité a des relations est supérieure à 2, les notations $r(x_1, \ldots, x_a)$, $R(x_1, \ldots, x_a)$, avec $r \in \mathsf{B}$, $R \in 2^{\mathsf{B}}$ et $x_1, \ldots, x_a \in D$ correspondent respectivement à la satisfaction de r et R par $x_1, \ldots, x_a$.

Dans la suite, nous donnons quelques exemples d'ensembles de relations de base de formalismes qualitatifs. Nous considérons tout d'abord des exemples de formalismes prenant en compte des points comme entités de base, puis des formalismes considérant des intervalles, enfin des formalismes basés sur des régions.

Le calcul des instants. Le calcul des instants [VK86, VKB90] également appelé algèbre des points permet de représenter le positionnement d'entités temporelles ponctuelles telles que des événements en considérant trois relations de base : $x\ precedes\ y$ (x se réalise avant y), $x\ follows\ y$ (x se réalise après y) et $x\ same\ y$ (x et y se réalisent au même instant), voir Figure 1.1. Les trois relations de base considérées par l'algèbre des points sont interprétées sur un ensemble muni d'une relation d'ordre linéaire. En considérant les points de la droite des nombres rationnels muni de la relation d'ordre habituelle $<$, les trois relations de base $precedes$, $follows$ et $same$ se définissent de la manière suivante : $precedes = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x < y\}$, $follows = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : y < x\}$ et $same = \{(x, y) \in \mathbb{Q} \times \mathbb{Q} : x = y\}$. À partir de ces trois relations de base, nous pouvons définir huit relations complexes correspondant à l'ensemble $2^{\mathsf{B}} = \{\{precedes, follows, same\}, \{precedes, follows\}$, $\{precedes, same\}, \{follows, same\}, \{precedes\}$, $\{follows\}, \{same\}, \emptyset\}$ Considérons, par exemple la relation $\{follows, precedes\}$. Cette relation permet d'exprimer qu'un instant se réalise avant ou après un autre instant mais pas en même temps. De plus, $x \in \mathbb{Q}$ et $y \in \mathbb{Q}$ satisfont cette relation si et seulement si, $x \neq y$. Désormais, pour des raisons de concision, nous utiliserons les symboles $<$, $>$ et $=$ pour référencer respectivement les relations de base $precedes$, $follows$ et $same$.

FIGURE 1.1 – Les relations de base du calcul des instants.

Le calcul des directions cardinales. Dévolu au raisonnement spatial, le calcul des directions cardinales proposé par LIGOZAT [Lig98a, Lig98b] est une extension au plan du calcul des instants. Les entités du domaine D sont donc les points du plan muni d'un repère orthogonal. La position relative entre deux points est déterminée par les relations de base de l'algèbre des points issues des projections des points sur les deux axes. Nous obtenons ainsi 9 positions relatives qualitatives possibles entre deux points correspondant à l'ensemble de relations de base $\mathsf{B} = \{EQ, E, N, S, W, NE, NW, SW, NW\}$, voir Figure1.2. À partir de ces 9 relations sont définies les 2^9 relations du calcul des directions cardinales.

Le calcul des n-points et le calcul α-étoile. Le calcul des n-points [BC02b] est une généralisation du calcul des instants à l'espace euclidien de dimension n avec $n \geq 1$. Les entités considérées sont les points de l'espace euclidien de dimension n muni d'un repère orthogonal. Chaque relation de base est définie par un n-uplet formé de n relations de base du calcul des instants. Le $i^{\text{ème}}$ élément du n-uplet, avec $1 \leq i \leq n$,

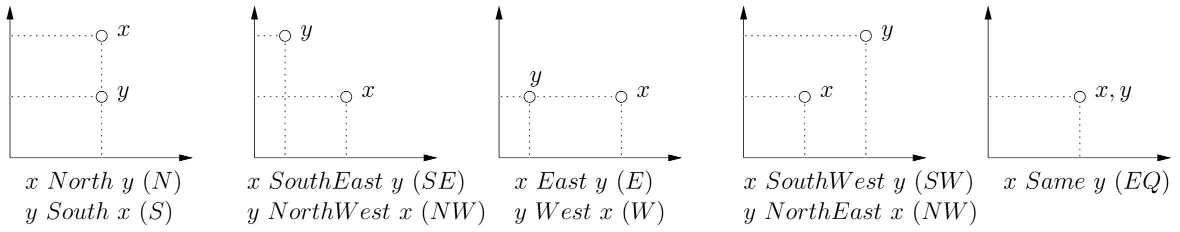


FIGURE 1.2 – Les relations de base du calcul des directions cardinales.

correspond à la relation de base du calcul des instants satisfaite par les projections orthogonales des deux points sur le $i^{\text{ème}}$ axe. Dans la figure 1.3 se trouve illustrés deux 3-points satisfaisant la relation de base $(<, <, =)$. Le calcul des n-points est basé sur 3^n relations de base. Le calcul des instants et le calcul des directions cardinales correspondent respectivement au calcul des n-points pour $n = 1$ et $n = 2$.

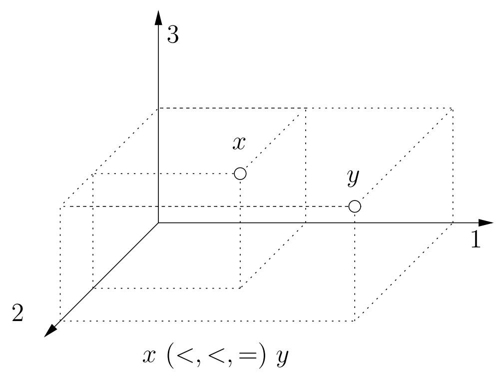
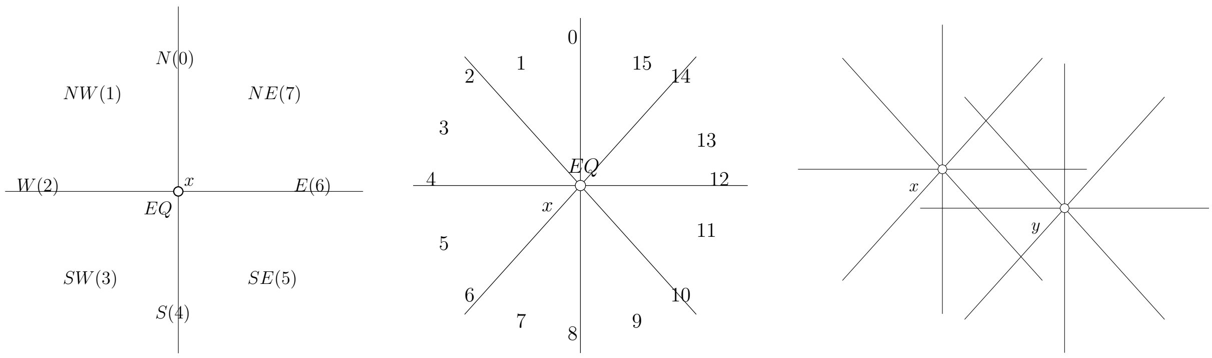


FIGURE 1.3 – Une relation de base du calcul des 3-points.

Le calcul α-étoile [Mit02, Mit04], avec α un entier strictement positif, est également une généralisation du calcul des directions cardinales. Il permet une description plus fine des positionnements des entités spatiales. Étant donné un point du plan x et une relation de base r du calcul des directions cardinales, la satisfaction de r par x et un deuxième point y impose que le point y soit dans une zone particulière du plan, voir Figure 1.4(a). Ces différentes zones forment une partition du plan et sont délimitées par 4 demi-droites d'origine x, deux demi-droites consécutives formant un angle de 90 degrés (nous supposons le plan orienté dans le sens trigonométrique). Le calcul α-étoile généralise ce partitionnement en considérant non plus 4 demi-droites mais α demi-droites d'origine x. Chaque paire de demi-droites consécutives forme un angle de $(360/\alpha)$ degrés. Chaque région correspond à une relation de base du calcul α-étoile qui est ainsi basé sur $(2.\alpha) + 1$ relations. En dehors de la relation identité désignée par EQ, chaque autre relation de base est identifiée par un nombre compris entre 0 et $(2.\alpha) - 1$. Les figures 1.4(a) et 1.4(b) représentent respectivement ce partitionnement pour le calcul 4-étoile et celui pour le calcul 8-étoile. Dans la figure 1.4(c) sont illustrés deux points x et y du plan tels que $x\ 1\ y$ et $y\ 5\ x$ dans le cadre du calcul 4-étoile, et $x\ 3\ y$ et $y\ 11\ x$ dans le cadre du calcul 8-étoile. Les relations de base référencées par un nombre pair correspondent à des demi-droites, celles référencées par un nombre impair à des zones coniques de dimension 2. Notons que le calcul des directions cardinales correspond au calcul 4-étoile.



(a) (b) (c)

FIGURE 1.4 – Partitionnement du plan par les relations de base du calcul des directions cardinales (du calcul 4-étoile) (a) et par celles du calcul 8-étoile (b). Deux points x et y du plan (c).

Calcul des instants basé sur un ordre partiel. En suivant le modèle du temps utilisé dans de le cadre de certaines logiques temporelles [Eme90a], BROXVALL et JONSSON [BJ00] réinterprètent le calcul des instants en considérant une structure temporelle non linéaire. Formellement, ils considèrent un ensemble d'éléments D muni d'un ordre partiel $\leq$. À partir de cet ordre partiel et du prédicat d'égalité sont définies quatre relations de base $(<, >, =, \|)$ de la manière suivante : la relation $<$ est définie par $\{(x, y) \in \mathsf{D} \times \mathsf{D} : x \leq y \text{ et } x \neq y\}$, $>$ est définie par $\{(x, y) \in \mathsf{D} \times \mathsf{D} : y \leq x \text{ et non } x = y\}$, la relation $\|$ correspond à l'ensemble de valeurs $\{(x, y) \in \mathsf{D} \times \mathsf{D} : \text{ non } x \leq y \text{ et } \text{ non } y \leq x\}$ et enfin $=$ correspond à la relation identité sur D. Le calcul des instants sur un ordre partiel permet notamment de représenter des relations temporelles concernant des événements intervenant dans des systèmes distribués non pourvus d'une horloge globale. BROXVALL et JONSSON considèrent également le cas particulier de la sous-classe des ordres partiels satisfaisant la propriété $\forall x, y, z \in \mathsf{D}$, si $x \| y$ et $y \leq z$ alors $x \| z$ stipulant que deux points temporels incomparables ne peuvent avoir un successeur commun. Intuitivement, cette condition interdit la jonction de deux branches de temps dans le futur afin d'obtenir un modèle de temps arborescent dans le futur.

Le calcul des points cycliques. Le calcul des points cycliques que nous avons proposé et étudié dans [BCL03b] est basé sur des relations de base ternaires permettant de raisonner sur les positions relatives d'entités ponctuelles sur un cercle orienté du plan. Étant donnés trois points x, y et z du cercle, six positions relatives peuvent être distinguées (Figure 1.5) : y est rencontré strictement après x et strictement avant z en suivant l'orientation du cercle ($\mathsf{B}_{abc}(x, y, z)$), y est rencontré strictement après z et strictement avant x en suivant l'orientation du cercle ($\mathsf{B}_{acb}(x, y, z)$), x et y sont égaux et distincts de z ($\mathsf{B}_{aac}(x, y, z)$), x et z sont égaux et distincts de y ($\mathsf{B}_{aba}(x, y, z)$), z et y sont égaux et distincts de x ($\mathsf{B}_{baa}(x, y, z)$) et, x, y, z sont trois points égaux ($\mathsf{B}_{aaa}(x, y, z)$). Dans un contexte de représentation spatiale, les relations des points cycliques permettent par exemple de capturer et de représenter les positions relatives d'objets observés au travers d'un tour d'horizon panoramique. De manière générale, les relations de base du calcul des points cycliques peuvent être définies à partir d'un ensemble D muni d'une relation d'ordre cyclique dense. Formellement une relation d'ordre cyclique sur D est une relation ternaire $\prec$ satisfaisant les propriétés suivantes, pour tout $x, y, z, t \in \mathsf{D}$:

- non $\prec (x, y, y)$ (P1) ; si $\prec (x, y, z)$ et $\prec (x, z, t)$ alors $\prec (x, y, t)$ (P2 - Transitivité) ;
- si $x \neq y$ et $x \neq z$ alors $y = z$ ou $\prec (x, y, z)$ ou $\prec (x, z, y)$ (P3 - Totalité) ;
- $\prec (x, y, z)$ ssi $\prec (y, z, x)$ ssi $\prec (z, x, y)$ (P4 - Cyclicité) ;

- si $x \neq y$ alors il existe z tel que $\prec (x, z, y)$ et il existe z tel que $\prec (x, y, z)$ (P5 - Densité) ;
- il existe $x, y \in \mathsf{D}$ tels que $x \neq y$ (P6).

Dans le cadre d'un cercle orienté $\mathcal{C}$, nous pouvons formellement définir D par l'intervalle $[0, 360[$ de $\mathbb{Q}$. Chaque point du cercle $\mathcal{C}$ est défini par le nombre rationnel de cet intervalle correspondant à l'angle formé par une droite horizontale et la droite passant par le centre de $\mathcal{C}$ et ce point. La relation d'ordre cyclique $\prec$ peut se définir par $\prec = \{(x, y, z) \in \mathsf{D} : x < y < z$ ou $y < z < x$ ou $z < x < y\}$ avec $<$ la relation d'ordre linéaire habituelle sur les nombres rationnels. Les six relations de base du calcul des points cycliques peuvent se définir à partir de la relation d'ordre cyclique $\prec$ et le prédicat d'égalité. Nous avons par exemple $\mathsf{B}_{abc}(x, y, z)$ si et seulement si $\prec (x, y, z)$ et $\mathsf{B}_{aab}(x, y, z)$ ssi $x = y$ et non $x = z$.

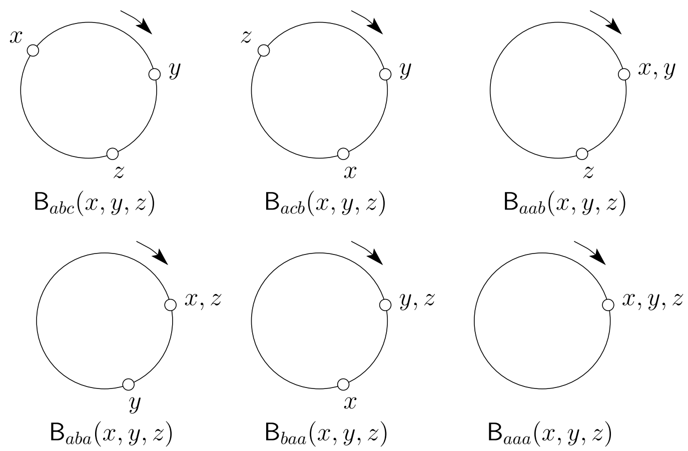


FIGURE 1.5 – Relations de base du calcul des points cycliques.

Nous allons maintenant décrire sommairement quelques relations de base de formalismes qualitatifs dont les domaines sont des intervalles.

Le calcul des intervalles. Le calcul des intervalles proposé par ALLEN [All81, All83] est certainement le plus connu des formalismes qualitatifs et celui ayant reçu le plus d'attention. Outre qu'il est le premier formalisme qualitatif proposé et étudié, sa notoriété provient notamment de son intérêt pour la représentation d'informations temporelles dans de nombreuses applications de l'Informatique. ALLEN représente des entités temporelles de type activité ou action par des intervalles de la droite et considère 13 relations de base binaires : $\mathsf{B} = \{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$ (voir Figure 1.6). Chacune de ces relations correspond à une configuration particulière des quatre bornes de deux intervalles et permet de représenter une position relative qualitative particulière entre deux entités temporelles. La relation de base $meets$ correspond par exemple à la configuration dans laquelle la deuxième borne du premier intervalle est égale à la première borne du second intervalle et permet de représenter une situation où une activité se termine exactement au même instant où une deuxième activité débute. Formellement, pour le calcul d'ALLEN, le domaine D peut être défini par l'ensemble des intervalles de la droite des rationnels : $\mathsf{D} = \{x = (x^-, x^+) \in \mathbb{Q} \times \mathbb{Q} : x^- < x^+\}$. Chaque relation de base peut être définie par des contraintes sur les bornes des intervalles. Ainsi, la relation de base $starts$ se définit par $starts = \{(x, y) \in \mathsf{D} \times \mathsf{D} : x^- = y^-$ et $y^+ > x^+\}$ et la relation $meets$ par $meets = \{(x, y) \in \mathsf{D} \times \mathsf{D} : x^+ = y^-\}$. Notons que dans [All81] ALLEN ne considère que 9 relations de base, les relations de base d, s, f (respectivement di, si, fi) étant regroupées dans une unique relation.

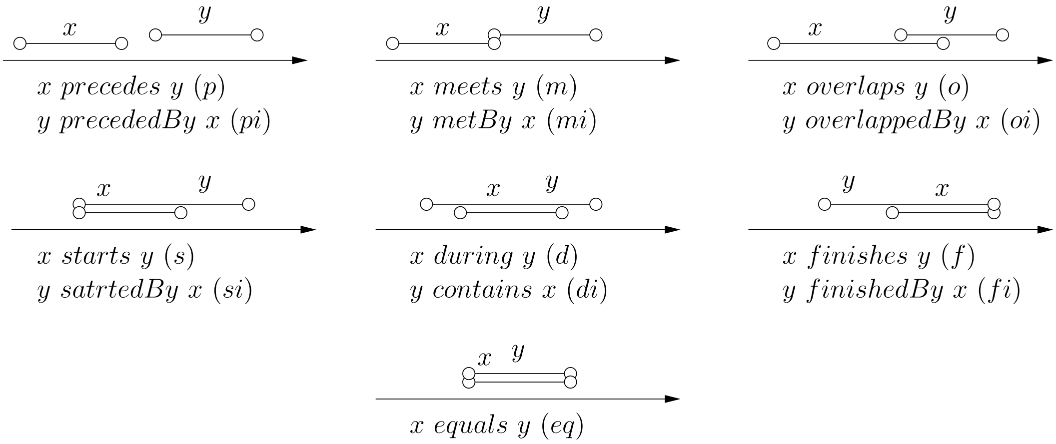


FIGURE 1.6 – Relations de base du calcul des intervalles.

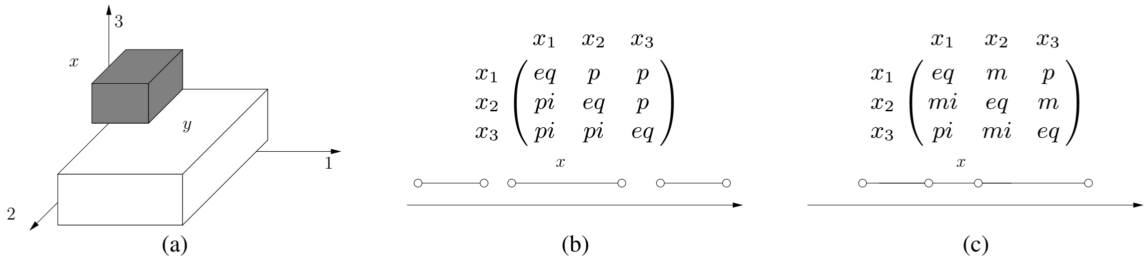


FIGURE 1.7 – (a) Deux 3-pavés x et y satisfaisant la relation de base $(overlaps, starts, metBy)$, (b) un intervalle généralisé de LADKIN et (c) un intervalle généralisé de LIGOZAT.

Le calcul des n-pavés et les calculs des intervalles généralisés. Le calcul des n-pavés (avec $n \geq 1$) que nous avons étudié dans [BCF99d] est une généralisation naturelle du calcul des intervalles à un espace euclidien de dimension n muni d'une base orthogonale. Les entités spatiales considérées, appelées n-pavés, sont les pavés dont les côtés sont parallèles aux axes du repère. Chaque relation de base de ce calcul est binaire et caractérisée par un n-uplet de relations de base du calcul des intervalles. Le $i^{\text{ème}}$ élément de ce n-uplet est la relation de base satisfaite par les intervalles issus des projections orthogonales des deux pavés sur le $i^{\text{ème}}$ axe. La figure 1.7 (a) représente deux 3-pavés satisfaisant la relation de base $(overlaps, starts, metBy)$. Le calcul des n-pavés admet 13^n relations de base. Le calcul des 1-pavés correspond au cas particulier du calcul des intervalles et le calcul des 2-pavés à celui du calcul des rectangles [BCF98]. D'autres extensions naturelles du calcul des intervalles consistent à considérer comme entités des ensembles d'intervalles satisfaisant des positions relatives particulières. Ces entités sont appelées intervalles généralisés. Les intervalles généralisés de LADKIN [Lad86] correspondent à des tuples d'intervalles pour lesquels deux intervalles consécutifs satisfont la relation $before$ du calcul des intervalles. Ceux considérés par LIGOZAT dans [Lig91] peuvent être définis par des tuples d'intervalles pour lesquels chaque couple d'intervalles consécutifs satisfait la relation $meets$. La relation qualitative entre deux intervalles généralisés est déterminée par les relations de base du calcul d'ALLEN satisfaite entre chaque paire d'intervalles constituant les intervalles généralisés. Dans **[Con04]**[p115], nous définissons et étudions un calcul englobant le calcul des n-pavés et les calculs des intervalles généralisés de Ladkin et de Ligozat. Dans ce calcul, les intervalles généralisés considérés sont des n-uplets d'intervalles *a priori* sans contrainte sur leur structure. Chaque relation de base est définie par une matrice de relations de base du calcul des intervalles. Cette matrice permet de décrire la position relative entre deux intervalles généralisés en spécifiant la relation satisfaite par chaque paire d'intervalles composée d'un intervalle du premier intervalle généralisé et d'un intervalle du second intervalle généralisé. Ces relations de base permettent également de structurer un intervalle généralisé en utilisant la relation appropriée qu'il doit satisfaire. Les figures 1.7 (a) et (b) illustrent les deux relations de base permettant de contraindre un 3-intervalle généralisé à être un intervalle généralisé considéré par LADKIN et par LIGOZAT respectivement.

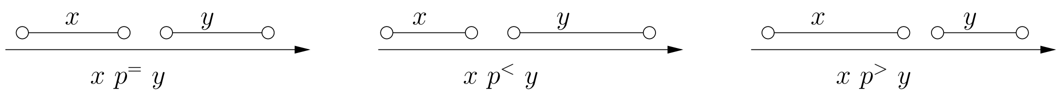


FIGURE 1.8 – Raffinement de la relation $precedes$ du calcul d'Allen en trois relations de base du calcul INDU.

Le calcul INDU. Le calcul INDU est un formalisme temporel proposé par PUJARI *et al.* [PKS99] considérant également des intervalles comme entités temporelles. Il prend en compte des informations concernant les durées relatives des intervalles en plus des informations exprimées par les relations du calcul d'ALLEN. Le calcul INDU est basé sur 25 relations de base. Chacune de ces relations de base est issue d'un raffinement d'une relation de base du calcul des intervalles et peut-être définie par un couple de relations (i, p). La première relation i de ce couple est la relation de base du calcul d'Allen satisfaite par les deux intervalles. La seconde relation p est une relation de base du calcul des points représentant la relation satisfaite par les durées des deux intervalles ($>$ correspond au cas où le premier intervalle a une durée strictement supérieure à celle du second intervalle, $<$ correspond au cas où le premier intervalle a une durée strictement inférieure à celle du second intervalle, $=$ correspond au cas où les deux intervalles ont la même durée). La relation de base du calcul INDU définie par le couple (i, p) sera notée i^p dans le suite. Ainsi, l'ensemble des relations de base du calcul INDU correspond à l'ensemble $\mathsf{B} = \{eq^=, p^<, p^>, p^=, pi^<, pi^>, pi^=, m^<, m^>, m^=, mi^<, mi^>, mi^=, o^<, o^>, o^=, oi^<, oi^>, oi^=, s^<, si^>, d^<, di^>, f^<, fi^>\}$. Les relations de base d'ALLEN p, pi, o, oi, m, mi se raffinent en trois relations de base du calcul INDU (voir Figure 1.8 pour une illustration concernant la relation $precedes$) tandis que les relations de base $s, si, f, fi, d, di, =$ sont inchangées malgré leur renommage.

Le calcul des intervalles cycliques et le calcul des intervalles sur un temps arborescent. Le calcul des intervalles cycliques proposé par BALBIANI et OSMANI [BO00] considère les positions relatives possibles entre deux intervalles du cercle orienté. Les relations de base de ce calcul sont au nombre de 16 et correspondent à toutes les configurations possibles de quatre bornes de deux intervalles définis sur un ordre cyclique, voir Figure 1.9. Certaines relations de base de ce calcul ne peuvent pas être satisfaites par deux intervalles définis sur un ordre linéaire. Considérons par exemple la relation de base mmi (meets-metBy). Cette relation correspond à la configuration où la seconde borne (respectivement la première borne) du premier intervalle est égale à la première borne (respectivement la deuxième borne) du second intervalle. Cette relation de base peut être satisfaite par deux intervalles du cercle orienté mais pas par deux intervalles de la droite.
Comme pour le calcul des instants, les relations du calcul des intervalles ont également été interprétées sur un modèle de temps arborescent [Euz98, RW04]. Avec ce type de modèle, 6 nouvelles configurations qualitatives peuvent être caractérisées lorsque deux intervalles se trouvent sur deux *branches* de temps distinctes. Ces nouvelles relations sont représentées par la figure 1.10. Nous avons par exemple la relation

$initiallyMeets$ (iM) qui est satisfaite par deux intervalles x et y lorsque la borne inférieure de x se réalise strictement avant la borne inférieure de y et lorsque les bornes supérieures de x et y ne sont pas comparables. En utilisant les relations de base précédemment introduites dans le cadre du calcul des instants sur un modèle de temps arborescent, nous avons $x \; iM \; y$ si et seulement si, $x^- < x^+$, $y^- < y^+$, $x^- < y^-$ et $x^+ \parallel y^+$.

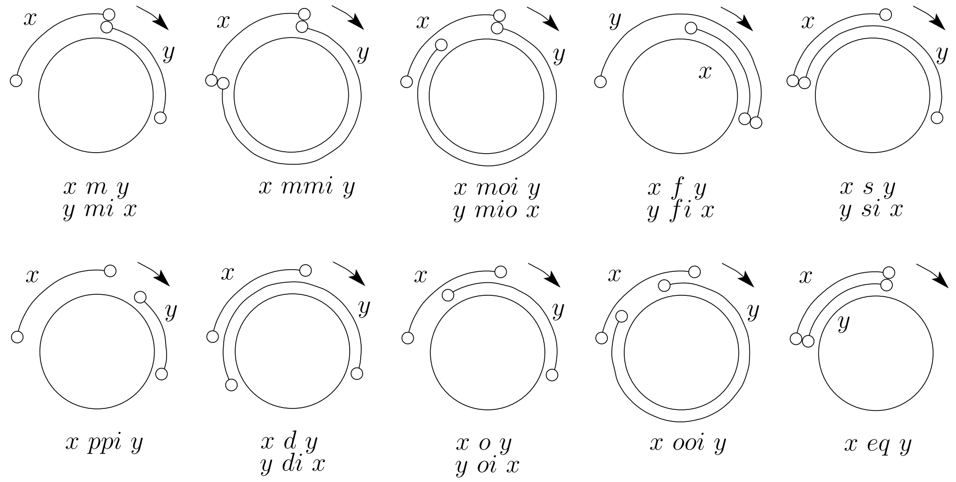


FIGURE 1.9 – Les relations de base du calcul des intervalles cycliques.

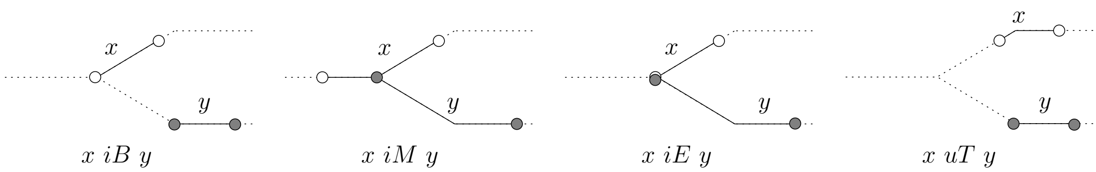


FIGURE 1.10 – Des relations de base du calcul des intervalles pour un temps arborescent dans le futur : $initiallyBefore(iB)$, $initiallyMeets(iM)$, $initiallyEquals(iE)$ et $unrelatedTo(uT)$.

Le calcul des régions RCC8. Dans le domaine du raisonnement spatial, RANDELL *et al.* [RC89, RCC92a, RCC92b] ont proposé une théorie du premier ordre appelée Region Connection Calculus (RCC). RCC est une axiomatisation de l'espace considérant des régions comme entités spatiales primitives basée sur la relation binaire $C(x, y)$: la région x est connectée à la région y. L'interprétation standard de RCC considère les ensembles réguliers d'un espace topologique où $C(x, y)$ est interprété par : les fermetures topologiques de x et y partagent au moins un point. Compte tenu de l'axiomatisation de RCC [RCC92b], chaque région possède un complémentaire. Ainsi, l'ensemble de tous les points est exclu de l'ensemble des régions. De plus, du fait de l'interprétation topologique de C, il n'est pas possible de distinguer les ensembles ouverts, les ensembles fermés et les ensembles semi-ouverts. Nous pouvons donc considérer uniquement l'un de ces ensembles pour définir les régions, l'ensemble des ensembles fermés par exemple. RANDELL et al. définissent à partir du prédicat C différentes autres relations binaires :

- $DC(x, y) \stackrel{\text{Déf.}}{\equiv} \neg C(x, y)$ (x est déconnecté de y) ;

- $P(x,y) \overset{\text{Déf.}}{\equiv} \forall z[C(z,x) \rightarrow C(z,y)]$ (x est une partie de y) ;
- $PP(x,y) \overset{\text{Déf.}}{\equiv} P(x,y) \wedge \neg P(y,x)$ (x est une partie propre de y) ;
- $EQ(x,y) \overset{\text{Déf.}}{\equiv} P(x,y) \wedge P(y,x)$ (x et y sont identiques) ;
- $O(x,y) \overset{\text{Déf.}}{\equiv} \exists z[P(z,x) \wedge P(z,y)]$ (x chevauche y) ;
- $PO(x,y) \overset{\text{Déf.}}{\equiv} O(x,y) \wedge \neg P(x,y) \wedge \neg P(y,x)$ (x chevauche partiellement x) ;
- $EC(x,y) \overset{\text{Déf.}}{\equiv} \neg O(x,y) \wedge C(x,y)$ (x est extérieurement connecté à y) ;
- $TPP(x,y) \overset{\text{Déf.}}{\equiv} PP(x,y) \wedge \exists z[EC(z,x) \wedge EC(z,y)]$ (x est une partie propre tangentielle de y) ;
- $NTPP(x,y) \overset{\text{Déf.}}{\equiv} PP(x,y) \wedge \neg\exists z[EC(z,x) \wedge EC(z,y)]$ (x est une partie propre non tangentielle de y) ;
- $P^{\sim}(x,y) \overset{\text{Déf.}}{\equiv} P(y,x)$; $PP^{\sim}(x,y) \overset{\text{Déf.}}{\equiv} PP(y,x)$;
- $TPP^{\sim}(x,y) \overset{\text{Déf.}}{\equiv} TPP(y,x)$; $NTPP^{\sim}(x,y) \overset{\text{Déf.}}{\equiv} NTPP(y,x)$.

L'ensemble des huit relations $\{DC, EC, PO, EQ, TPP, NTPP, TPP^{\sim}, NTPP^{\sim}\}$ correspond à l'ensemble des relations de base du formalisme qualitatif appelée RCC8. Une illustration graphique de ces relations est donnée par la figure 1.11. Les relations de base de RCC8 permettent de raisonner sur la nature des relations topologiques entre des régions représentant des objets d'un système à modéliser. En définissant la relation PP (respectivement la relation $PP^{\sim}$) par l'union des deux relations TPP et $NTPP$ (respectivement $TPP^{\sim}$ et $NTPP^{\sim}$), l'ensemble $\{DC, EC, EQ, PP, PP^{\sim}\}$ constitue l'ensemble des relations de base d'un formalisme moins expressif appelé RCC5. Dans le cadre de certaines applications ou études, les relations topologiques de RCC8 sont interprétées sur des régions très spécifiques (polygones, rectangles, régions convexes,...). Par exemple, dans [Ege05], EGENHOFER considère les régions d'une sphère et montre que des configurations particulières entre régions peuvent être réalisées sur la sphère et pas dans le plan. Dans certains formalismes, les relations de base de RCC8 ont également été combinées avec des relations considérant des aspects autres que topologiques. Par exemple, dans [GR98] GEREVINI et RENZ considèrent comme entités spatiales des régions de $\mathbb{R}^n$ auxquelles est associée une taille. En plus des relations topologiques de RCC8, ils prennent en compte le rapport qualitatif entre les tailles des régions. Comme pour le formalisme INDU, les relations du calcul des points sont utilisées pour représenter ce rapport. La relation de base $<$ permettra par exemple d'indiquer que la taille de la première région est strictement inférieure à la taille de la seconde région. Une autre combinaison de relations a été proposée par LI *et al.* [LLR09] qui considérent les relations de base de RCC8 ainsi que des relations permettant d'exprimer l'orientation relative entre deux régions du plan. Ces relations sont les relations de base du calcul des directions cardinales que nous présentons dans le paragraphe suivant.

Le calcul des directions cardinales sur les régions. Le calcul des directions cardinales sur les régions a été proposé par GOYAL et EGENHOFER [GE01, SK04]. Ce formalisme permet de raisonner sur l'orientation relative de régions dans le plan (ensembles bornés fermés réguliers et connexes du plan). Étant donnée une région x du plan muni d'un repère orthogonal, à l'aide des quatres droites prolongeant les côtés du plus petit rectangle englobant contenant x (le plus petit rectangle contenant x et dont les côtés sont parallèles aux axes), nous pouvons définir 9 zones particulières du plan, voir Figure 1.12(a). Ces 9 zones sont notées NW,N,SE,W,O,E,SW,S,SE et sont appelées tuiles. La relation de base satisfaite par une région y et la région x est déterminée par l'ensemble des tuiles qui s'intersectent avec y. Elle est notée par les références de ces tuiles séparées par le symbole deux points. La figure 1.12(b) représente une configuration où la région y satisfait avec la région x la relation de base W:SW:S et la région x satisfait avec la région y la relation de base N:NE:O:E. Dans la figure 1.12(c), la région y satisfait avec x la relation de base W:O:SW:S. Une des particularités du calcul des directions cardinales sur les régions est que la transposée d'une relation de base ne correspond pas à une relation de base. Considérons par

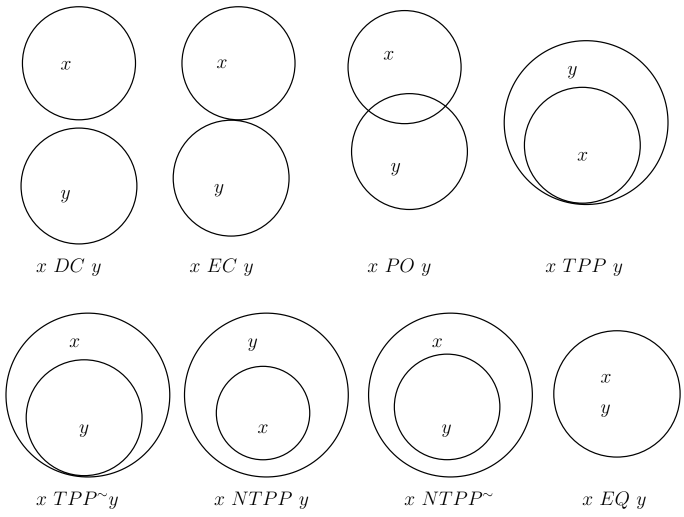
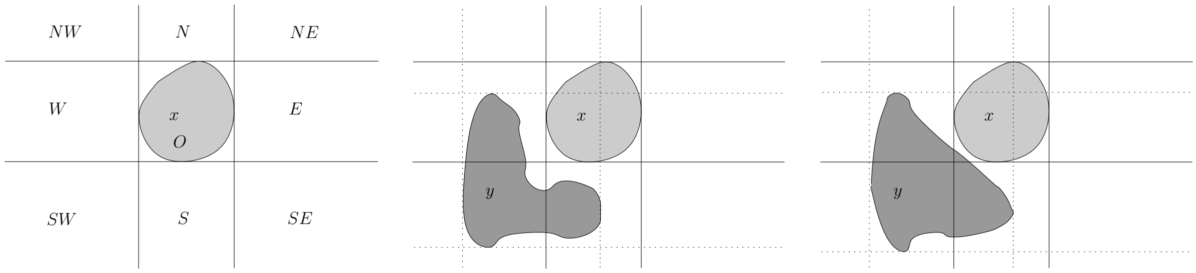


FIGURE 1.11 – Illustration des relations de base de RCC8.



(a) (b) (c)

FIGURE 1.12 – Division du plan par une région x pour le calcul des directions cardinales sur les régions (a), x N:NE:O:E y et y W:SW:S x (b), x N:NE:O:E z et z W:O:SW:S x.

exemple deux régions x et y telles que x satisfait avec y la relation de base N:NE:O:E. En examinant de nouveau la figure 1.12, nous pouvons constater que la relation de base pouvant être satisfaite entre y et x n'est pas unique. En effet, elle peut être la relation W:SW:S (Figure 1.12(b)) mais également la relation W:O:SW:S (Figure 1.12(c)).

1.2 Les opérations relationnelles

Comme ensemble d'ensembles de relations de base, l'ensemble 2^{B} est muni des opérations ensemblistes d'union ($\cup$) et d'intersection ($\cap$). Étant données deux relations $R, S \in 2^{\mathsf{B}}$, $R \cup S$ correspond à la relation de 2^{B} constituée des relations de base de B appartenant à R ou à S, $R \cap S$ est la relation de 2^{B} formée des relations de base appartenant à la fois à R et à S. En tant qu'ensemble de relations, l'ensemble 2^{B} est également muni d'opérations relationnelles que nous allons décrire.

Opérations pour des relations d'arité 2 **(inverse et composition faible).** Lorsque l'ensemble B est un ensemble de relations de base binaires, 2^{B} est muni de l'opération unaire inverse ($^{\smile}$) et de l'opération binaire appelée opération de faible composition ou opération de composition algébrique, notée $\diamond$ dans

la suite. Dans le cas général, l'opération inverse associe à chaque relation de base r la relation $r^{\smile}$ appartenant à 2^{B} définie par $r^{\smile} = \{r' \in \mathsf{B} : \exists x, y \in \mathsf{D}$ avec $x\ r\ y$ et $y\ r'\ x\}$. L'opération inverse est généralisée à l'ensemble 2^{B} par $R^{\smile} = \bigcup_{r \in R} r^{\smile}$ pour tout $R \in 2^{\mathsf{B}}$. Notons que nous avons pour tout $x, y \in \mathsf{D}$ et $R \in 2^{\mathsf{B}}$, si $x\ R\ y$ alors $y\ R^{\smile}\ x$. L'inverse n'est pas forcément vraie dans le cas général. Pour la plupart des formalismes, il existe pour chaque relation de base $r \in \mathsf{B}$ une relation de base de B correspondant à la transposée de r, *i.e.* la relation $\{(y, x) : (x, y) \in r\}$. Dans le cadre de ces formalismes, pour toute relation de base $r \in \mathsf{B}$, nous avons $r^{\smile}$ qui ne contient qu'une seule relation de base, la transposée de r. D'autre part, nous avons pour tout $x, y \in \mathsf{D}$ et $R \in 2^{\mathsf{B}}$, $x\ R\ y$ si et seulement si $y\ R^{\smile}\ x$. À notre connaissance, il n'existe que le calcul des directions cardinales sur les régions précédemment présenté pour lequel les éléments de la transposée d'une relation de base sont répartis dans plusieurs relations de base. Dans ce cadre précis, le calcul de l'opération inverse des relations de base a donné lieu à des études particulières [CF04].
La relation inverse d'une relation de 2^{B} peut se calculer à partir d'une table des inverses dans laquelle pour chaque relation de base r est donnée la relation $r^{\smile}$. Dans la Figure 1.1 sont représentées les tables des inverses de quelques uns des formalismes présentés précédemment. En considérant, par exemple, la relation $\{p, m, oi\}$ du calcul des intervalles et la table des inverses pour ce calcul, nous pouvons facilement établir l'égalité suivante : $\{p, m, oi\}^{\smile} = \{pi, mi, o\}$.

(a)

r	$r^{\smile}$
$<$	$>$
$>$	$<$
$=$	$=$

(b)

r	$r^{\smile}$
$<$	$>$
$>$	$<$
$=$	$=$
$\parallel$	$\parallel$

(c)

r	$r^{\smile}$
N	S
NW	SE
W	E
SW	NE
S	N
SE	NW
E	W
NE	SW
EQ	EQ

(d)

r	$r^{\smile}$
b	bi
bi	b
o	oi
oi	o
m	mi
mi	m
d	di
di	d
si	s
s	si
f	fi
fi	f
eq	eq

(e)

r	$r^{\smile}$
DC	DC
EC	EC
PO	PO
TPP	$TPP^{\sim}$
$TPP^{\sim}$	TPP
$NTPP$	$NTPP^{\sim}$
$NTPP^{\sim}$	$NTPP$
EQ	EQ

TABLE 1.1 – Table des inverses du calcul des instants (a), du calcul des instants sur des ordres partiels (b), du calcul des directions cardinales (c), du calcul des intervalles (d) et du calcul RCC8 (e).

Étant données deux relations de base $r, r' \in \mathsf{B}$ et trois entités $x, y, z \in \mathsf{D}$ telles que x et y satisfont r et, y et z satisfont r', seules certaines relations de base de B peuvent être éventuellement satisfaites par x et z. Ce sous-ensemble de relations de base correspond à la relation de 2^{B} résultant de la composition faible entre r et r', dénotée par $r \diamond r'$. Considérons par exemple les deux relations de base o ($overlaps$) et f ($finishes$) satisfaites respectivement par x et y et par, y et z, avec x, y, z trois intervalles de la droite des rationnels. Seules trois positions relatives différentes sont possibles entre x et z, voir Figure 1.13. Ces trois configurations qualitatives correspondent aux relations de base $overlaps$, $starts$ et $during$, ainsi nous avons $o \diamond f = \{o, s, d\}$. Formellement, pour tout $r, r' \in \mathsf{B}$, $r \diamond r' = \{r'' : \exists x, y, z \in \mathsf{D}$ avec $x\ r\ y, y\ r'\ z$ et $x\ r''\ z\}$. L'opération de faible composition est généralisée aux relations de 2^{B} par $R \diamond R' = \bigcup_{r \in R, r' \in R'} r \diamond r'$ pour tout $R, R' \in 2^{\mathsf{B}}$. Le calcul de la faible composition de deux relations de 2^{B} s'effectue à partir d'une table de composition dans laquelle sont stockées les compositions faibles de chaque couple de relations de base. À titre d'illustration, les tables de composition du calcul des instants

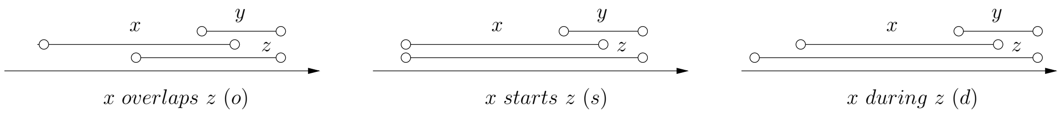


FIGURE 1.13 – Trois configurations possibles pour trois intervalles x, y, z tels que $x\ o\ y$ et $y\ f\ z$.

et du calcul des intervalles sont représentées par la figure 1.2. À partir de la table de composition du calcul des intervalles, nous pouvons par exemple facilement établir les égalités suivantes : $\{o, di\} \diamond \{m, s\} = (o \diamond m) \cup (o \diamond s) \cup (di \diamond m) \cup (di \diamond s) = \{p\} \cup \{o\} \cup \{di, o, fi\} \cup \{di, o, fi\} = \{p, o, d, fi\}$. En considérant

(a)

$\diamond$	$<$	$>$	$=$
$<$	$<$	$<, >, =$	$<$
$>$	$<, >, =$	$>$	$>$
$=$	$<$	$>$	$=$

(b)

$\diamond$	p	pi	d	di	o	oi	m	mi	s	si	f	fi	eq
p	p	Ψ	p d o m s	p	p	p d o m s	p	p do m s	p	p	p d o m s	p	p
pi	Ψ	pi	pi d oi mi fi	pi	pi d oi mi fi	pi	pi d oi mi fi	pi	pi d oi mi fi	pi	pi	pi	pi
d	p	pi	d	Ψ	p d o m s	pi d oi mi f	p	pi	d	pi d oi mi f	d	p d o m s	d
di	p di o m fi	pi di oi mi si	d di s si f fi o oi eq	di	di o fi	di oi si	di o fi	di oi fi	di o fi	di	di oi si	di	di
o	p	pi di oi mi si	d o s	p di o m fi	p o m	eq d di o oi s si f fi	p	di oi si	o	di oi fi	d o s	p o m	o
oi	p di o m fi	pi	d oi f	pi di oi mi si	eq d di o oi s si f fi	pi oi mi	di o fi	pi	d oi f	pi oi mi	oi	di oi si	oi
m	p	pi di oi mi si	d o s	p	p	d o s	p	eq f fi	m	m	d o s	p	m
mi	p di o m fi	pi	d oi f	pi	d oi f	pi	eq s si	pi	d oi f	pi	mi	mi	m
s	p	pi	d	p di o m fi	p o m	d oi f	p	mi	s	eq s si	d	p o m	s
si	p di o m fi	pi	d oi f	di	di o fi	oi	di oi fi	mi	eq s si	si	oi	di	si
f	p	pi	d	pi di oi mi si	p o s	pi oi mi	m	pi	d	pi oi mi	f	eq f fi	f
fi	p	pi di oi mi si	d o s	di	o	di oi s	m	di oi s	o	di	eq f fi	fi	fi
eq	p	pi	d	di	o	oi	m	mi	s	si	f	fi	eq

TABLE 1.2 – Table de composition du calcul des instants (a) et table de composition du calcul des intervalles (b).

le composition relationnelle usuelle définie par $R \circ R' = \{(x, z) \in \mathsf{D} \times \mathsf{D} : \exists y$ avec $x\ R\ y$ et $y\ R'\ z\}$, nous pouvons remarquer que dans le cas général $R \circ R' \subseteq R \diamond R'$, d'où le nom de composition faible pour l'opérateur $\diamond$. Pour des formalismes tels que le calcul des instants ou le calcul des intervalles interprétés sur la droite des rationnels, nous avons l'égalité $R \circ R' = R \diamond R'$ pour tout $R, R' \in 2^{\mathsf{B}}$. Pour d'autres formalismes tels que le calcul INDU, le calcul RCC8 ou encore le calcul des intervalles interprété sur la droite des entiers, nous avons pour certaines relations R et R', une inclusion stricte : $R \circ R' \subset R \diamond R'$. Comme illustration, considérons les deux relations singletons $\{p^=\}$ et $\{m^=\}$ du formalisme INDU. En prenant les trois intervalles x, y, z de la droite des rationnels définis par $x = [0, 4]$, $y = [6, 10]$ et $z = [10, 14]$ nous avons $x\ p^=\ y$, $y\ m^=\ z$ et $x\ p^=\ z$. Il s'ensuit que $p^= \in \{p^=\} \diamond \{m^=\}$. Considérons maintenant les deux intervalles $x' = [5, 7]$ et $z' = [8, 10]$, nous avons $x' p^= z'$. Ainsi, $x' (\{p^=\} \diamond \{m^=\})\ z'$. De plus, il est clair qu'il n'existe pas d'intervalle y' tel que $x'\ p^=\ y'$, $y'\ m^=\ z'$ et $x'\ p^=\ z'$ puisque la longueur de y' doit être de 2 et y doit être placé strictement après x' et avant y' (un emplacement de longueur 1). Nous avons donc, $(x', z') \notin \{p^=\} \circ \{m^=\}$. De tout cela découle l'inclusion stricte $\{p^=\} \circ \{m^=\} \subset \{p^=\} \diamond \{m^=\}$. La composition faible peut être utilisée dans le cadre de tout formalisme qualitatif contrairement à la composition relationnelle usuelle qui ne peut pas toujours être définie par des unions de relations de base. Pour une étude générale concernant la composition faible, le lecteur peut se reporter à [RL05]. Cette opération a également été étudiée dans le contexte spécifique de certains formalismes. Par exemple, dans [LY03], une étude est menée afin de déterminer s'il existe des modèles de RCC8 pour lesquels la composition usuelle et la composition faible sont similaires ; la réponse est négative. Comme autre exemple, nous pouvons citer les travaux réalisés dans le cadre du calcul des directions cardinales sur les régions afin de définir la table de composition [SK01, SK04]. Notons que la structure algébrique constituée de l'ensemble des relations 2^{B}, des opérations d'inverse $^{\smile}$ et de faible composition $\diamond$, de la relation d'identité Id et de la relation totale Ψ, forme pour certains formalismes une algèbre relationnelle au sens de TARSKI [Tar41, TG87]. C'est par exemple le cas pour le calcul des intervalles et le calcul des instants mais pas pour d'autres formalismes tels que le calcul INDU. Le lecteur peut se reporter à [LR04] pour une discussion concernant ce sujet.

Opérations pour des relations d'arité quelconque (rotation, permutation et composition faible). Dans le cadre de la thèse de MAHMOUD SAADE encadrée par PIERRE MARQUIS et moi-même, nous avons porté une partie de nos travaux sur l'étude de formalismes qualitatifs basés sur des relations de base d'arité a quelconque, avec $a > 1$. Nous avons en particulier formellement défini et étudié des opérations relationnelles nécessaires au raisonnement dans le cadre de ces formalismes.

Avant de détailler les opérations de rotation, permutation et composition faible, nous considérons une hypothèse générale permettant de représenter l'égalité entre deux entités de D. Pour tout $i, j \in \{1, \ldots, a\}$ nous supposons qu'il existe une relation de 2^{B} correspondant à la relation diagonale sur i et j, notée Δ_{ij} et définie par $\Delta_{ij} = \{(x_1, \ldots, x_a) \in \mathsf{D}^a : x_i = x_j\}$. Remarquons que la relation identité correspond à l'intersection de l'ensemble des diagonales. En considérant le calcul des points cycliques, nous avons à titre d'illustration $\Delta_{12} = \Delta_{21} = \{B_{aab}, B_{aaa}\}$ et $\Delta_{13} = \Delta_{31} = \{B_{aba}, B_{aaa}\}$.

L'opération inverse étendue au cas de relations de base d'arité $n > 1$ quelconque se décline en deux opérations : l'opération de permutation ($\looparrowright$) et l'opération de rotation ($\curvearrowright$). Pour chaque relation de base $r \in \mathsf{B}$, nous supposons que la permutation de r défnie par $r^{\looparrowright} = \{(x_1, \ldots, x_a, x_{a-1}) : r(x_1, \ldots, x_{a-1}, x_a)\}$ et la rotation de r définie par $r^{\curvearrowright} = \{(x_2, \ldots, x_a, x_1) : r(x_1, x_2, \ldots, x_a)\}$, appartiennent également à l'ensemble des relations de base B. Les opérations de permutation et de rotation sont définies sur 2^{B} de la manière suivante : pour tout $R \in 2^{\mathsf{B}}$, $R^{\looparrowright} = \{r^{\looparrowright} : r \in R\}$ et $R^{\curvearrowright} = \{r^{\curvearrowright} : r \in R\}$. La table 1.2(a) et la table 1.2(b) correspondent respectivement à la table de permutation et la table de rotation des relations de base du calcul des points cycliques. À partir de ces deux tables, la permutation et la rotation de toute relation du calcul des points cycliques peuvent être calculées. Nous avons par exemple $\{B_{aab}, B_{abc}\}^{\looparrowright} = \{B_{aab}^{\looparrowright}, B_{abc}^{\looparrowright}\} = \{B_{aba}, B_{acb}\}$ et $\{B_{aab}, B_{abc}\}^{\curvearrowright} = \{B_{aab}^{\curvearrowright}, B_{abc}^{\curvearrowright}\} = \{B_{aba}, B_{abc}\}$.

(a)

a	B_{aaa}	B_{aab}	B_{aba}	B_{baa}	B_{abc}	B_{acb}
$a^{\looparrowright}$	B_{aaa}	B_{aba}	B_{aab}	B_{baa}	B_{acb}	B_{abc}

(b)

a	B_{aaa}	B_{aab}	B_{aba}	B_{baa}	B_{abc}	B_{acb}
$a^{\curvearrowright}$	B_{aaa}	B_{aba}	B_{baa}	B_{aab}	B_{abc}	B_{acb}

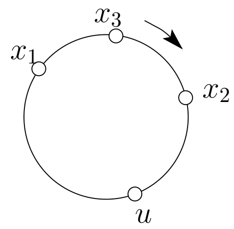
TABLE 1.3 – Table de permutation (a) et table de rotation (b) des relations de base du calcul des points cycliques.

Pour les formalismes qualitatifs binaires, la composition faible permet, étant données trois entités x_1, x_2, u pour lesquelles sont connues la relation satisfaite par x_1 et u et, la relation satisfaite par u et x_1, de savoir quelles sont les relations de base pouvant être satisfaites par x_1 et x_2. De la même manière, pour un formalisme qualitatif d'arité $a > 1$, la composition faible permet étant données $a + 1$ entités $x_1, \ldots, , x_a, u$ pour lesquelles sont connues la relation satisfaite par $x_1, \ldots, x_{a-1}, u$, celle satisfaite par

$x_1, \ldots, x_{a-2}, u, x_n, \ldots$, celle satisfaite par $u, x_2, \ldots, x_a$, d'en déduire les relations de base pouvant être satisfaites par $x_1, \ldots, x_a$. Formellement, pour un formalisme qualitatif d'arité $a > 1$, l'opération de faible composition est une opération d'arité a définie de la manière suivante :

- soient $r_1, \ldots, r_a \in \mathsf{B}, \diamond(r_1, \ldots, r_a) = \{r \in \mathsf{B} : r(x_1, \ldots, x_a)$ et $\exists u \in \mathsf{D}$ avec $r_1(x_1, \ldots, x_{a-1}, u)$, $r_2(x_1, \ldots, x_{a-2}, u, x_n), \quad \ldots, r_a(u, x_2, \ldots, x_a)\}$.
- Soient $R_1, \ldots, R_a \in 2^{\mathsf{B}}$, $\diamond(R_1, \ldots, R_a) = \bigcup_{r_1 \in R_1, \ldots, r_a \in R_n} \diamond(r_1, \ldots, r_a)$.

À titre d'illustration, considérons quatre points d'un cercle orienté x_1, x_2, x_3, u et les trois relations de base B_{abc}, B_{acb}, et B_{acb} satisfaites respectivement par x_1, x_2, u, x_1, u, x_3 et u, x_2, x_3. La seule configuration qualitative possible entre x_1, x_2 et x_3 est représentée par la figure 1.14(a). Elle correspond à la relation de base B_{acb}. Ainsi, $\diamond(B_{abc}, B_{acb}, B_{acb}) = \{B_{acb}\}$.



(a)

R_1	B_{aaa}	B_{aaa}	B_{aab}	B_{aab}	B_{aab}	B_{aab}	B_{aba}	B_{abc}
R_2	B_{aaa}	B_{aab}	B_{aba}	B_{abc}	B_{baa}	B_{acb}	B_{aab}	B_{abc}
R_3	B_{aaa}	B_{aab}	B_{baa}	B_{acb}	B_{aba}	B_{abc}	B_{abc}	B_{acb}
$\diamond(R_1, R_2, R_3)$	$\{B_{aaa}\}$	$\{B_{aab}\}$	$\{B_{aaa}\}$	$\{B_{aab}\}$	$\{B_{aab}\}$	$\{B_{aab}\}$	$\{B_{abc}\}$	$\{B_{abc}\}$

(b)

FIGURE 1.14 – (a) la seule configuration qualitative possible lorsque $B_{abc}(x_1, x_2, u)$, $B_{acb}(x_1, u, x_3)$, et $B_{acb}(u, x_2, x_3)$, (b) la table de faible composition du calcul des points cycliques.

Différentes propriétés concernant les différentes opérations présentées ont été mises en évidence dans la thèse de M. SAADE [Saa08].

1.3 Les réseaux de contraintes qualitatives (RCQ)

Dans le cadre des formalismes qualitatifs, les informations temporelles ou spatiales concernant les positions relatives des différentes entités du système peuvent être représentées par un ensemble de contraintes définies à l'aide d'un réseau de contraintes qualitatives, RCQ en abrégé. Chaque contrainte représente un ensemble de configurations qualitatives acceptables entre des entités. Elle est définie par un ensemble de relations de base. Formellement, pour un formalisme qualitatif basé sur l'ensemble de relations de base B et d'arité a, un RCQ est défini de la manière suivante :

Définition 1 *Un* RCQ $\mathcal{N}$ *est un couple* (V, C) *où :*

- $V = \{v_1, \ldots, v_n\}$ *est un ensemble de* n *variables représentant les entités temporelles ou spatiales du système ;*
- C *est une application qui associe à chaque tuple* $(v_{i_1}, \ldots, v_{i_a})$ *de variables de* V *une relation* $C(v_{i_1}, \ldots, v_{i_a}) \in 2^{\mathsf{B}}$*, dénotée également par* $C_{i_1 \ldots i_a}$ *ou* $\mathcal{N}[i_1, \ldots, i_a]$*.* C *est telle que pour tout tuple* $(v_{i_1}, \ldots, v_{i_a})$ *de variables de* V*, nous avons* $C(v_{i_1}, \ldots, v_{i_{a-1}}, v_{i_a}) = C(v_{i_1}, \ldots, v_{i_a}, v_{i_{a-1}})^{\looparrowright}$

et $C(v_{i_1}, \ldots, v_{i_{a-1}}, v_{i_a}) = C(v_{i_a}, v_{i_1}, \ldots, v_{i_{a-1}})^{\curvearrowright}$. *De plus, pour tout* k, l *avec* $0 < k < l \leq a$ *si* $v_{i_k} = v_{i_l}$ *alors* $C(v_{i_1}, \ldots, v_{i_a}) \subseteq \Delta_{i_k i_l}$ *(voir page 20 pour les définitions de l'opération de rotation (*$\curvearrowright$*), l'opération de permutation (*$\looparrowright$*), et les relations diagonales).*

Lorsque les relations de base sont binaires, les conditions posées sur l'application C se simplifient du fait que les opérations de rotation et de permutation correspondent à l'opération d'inverse. D'autre part, la seule relation diagonale est la relation identité. Ainsi, pour des relations binaires, nous avons $C(v, v') = C(v', v)^{\smile}$ et $C(v, v) \subseteq \mathsf{Id}$ pour tout $v, v' \in V$.

À titre d'illustration, la figure 1.15 représente trois RCQ définis dans différents calculs. Remarquons qu'une contrainte n'est pas représentée lorsqu'elle peut être obtenue par l'opération d'inverse, par l'opération de permutation ou encore par l'opération de rotation. En outre, nous ne représentons pas les contraintes correspondant aux relations diagonales. Le RCQ $\mathcal{N}_1 = (V, C)$ est défini dans le calcul des intervalles et sur quatre variables v_1, v_2, v_3, v_4 représentant des intervalles de la droite. La contrainte $C_{23} = \{d, o, fi\}$ stipule que la position relative entre les intervalles représentés par v_2 et v_3 doit correspondre à la relation de base $during$ (d) ou $overlaps$ (o) ou bien encore $finishedBy$ (fi). Les RCQ $\mathcal{N}_2$ et $\mathcal{N}_3$ sont respectivement définis sur le calcul RCC8 et le calcul des points cycliques. $\mathcal{N}_3$ définit des contraintes ternaires. Il est représenté par un hypergraphe orienté. La contrainte C_{243} stipule que seules les relations de base B_{abc}, B_{acb} et B_{aab} peuvent être satisfaites par les points du cercle orienté représentés par v_2, v_4 et v_3.

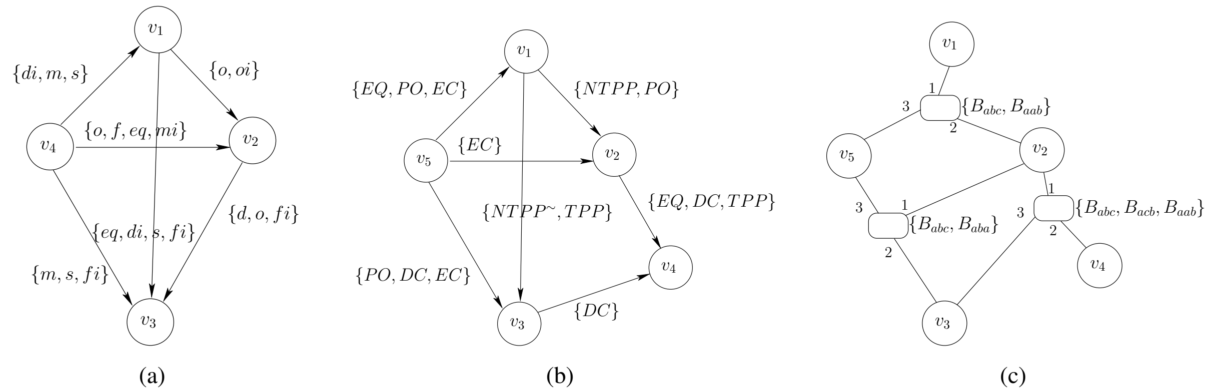


FIGURE 1.15 – (a) un RCQ $\mathcal{N}_1$ du calcul des intervalles, (b) un RCQ $\mathcal{N}_2$ du calcul RCC8 et (c) un RCQ $\mathcal{N}_3$ du calcul des points cycliques.

Dans la suite de ce rapport, nous supposerons un ensemble de relations de base B donné. De plus, pour des raisons de clarté et de concision, nous supposerons que les relations de base considérées sont binaires. Les concepts, propriétés et définitions peuvent dans la plupart du cas être étendus de manière naturelle au cas de relations de base d'arité supérieure. Nous donnons maintenant quelques définitions concernant les RCQ :

Définition 2 *Soit* $\mathcal{N} = (V, C)$ *un* RCQ.

- *Une solution partielle de* $\mathcal{N}$ *sur* $V' \subseteq V$ *est une application* σ *de* V' *vers* D *telle que pour chaque couple de variables* (v_i, v_j) *de* V', $\sigma(v_i)$ *et* $\sigma(v_j)$ *satisfont* C_{ij}, i.e. *il existe une relation de base* $r \in C_{ij}$ *telle que* $(\sigma(v_i), \sigma(v_j)) \in r$.
- *Une solution de* $\mathcal{N}$ *est une solution partielle sur* V.
- $\mathcal{N}$ *est cohérent si, et seulement si,* $\mathcal{N}$ *admet une solution.*
- *Deux* RCQ *définis sur le même ensemble de variables sont équivalents si, et seulement si, ils admettent les mêmes solutions.*

- *Un* RCQ $\mathcal{N}' = (V', C')$ *est un sous-*RCQ *de* $\mathcal{N}$*, dénoté par* $\mathcal{N}' \subseteq \mathcal{N}$*, si, et seulement si,* $V = V'$ *et* $C'_{ij} \subseteq C_{ij}$ *pour tout* $v_i, v_j \in V$ *(dans le cas où il existe* v_i, v_j *avec* $C'_{ij} \subset C_{ij}$ *nous parlerons de sous-*RCQ *strict, et nous le dénoterons par* $\mathcal{N}' \subset \mathcal{N}$*).*
- *Un* RCQ *atomique est un* RCQ *tel que chacune de ses contraintes est définie par une seule relation de base.*
- *Un scénario* $\mathcal{S}$ *de* $\mathcal{N}$ *est un sous-*RCQ *atomique de* $\mathcal{N}$*.*
- *Une relation de base* r *est dite incohérente pour la contrainte* C_{ij} *si, et seulement si, il n'existe pas de scénario cohérent* $\mathcal{S}$ *de* $\mathcal{N}$ *tel que* $\mathcal{S}[i, j] = r$*.*

Pour un scénario cohérent donné, il existe un ensemble, généralement non fini, de solutions correspondant à ce scénario. La figure 1.16 représente des solutions de chacun des RCQ de la figure 1.15 et les scénarios correspondant.

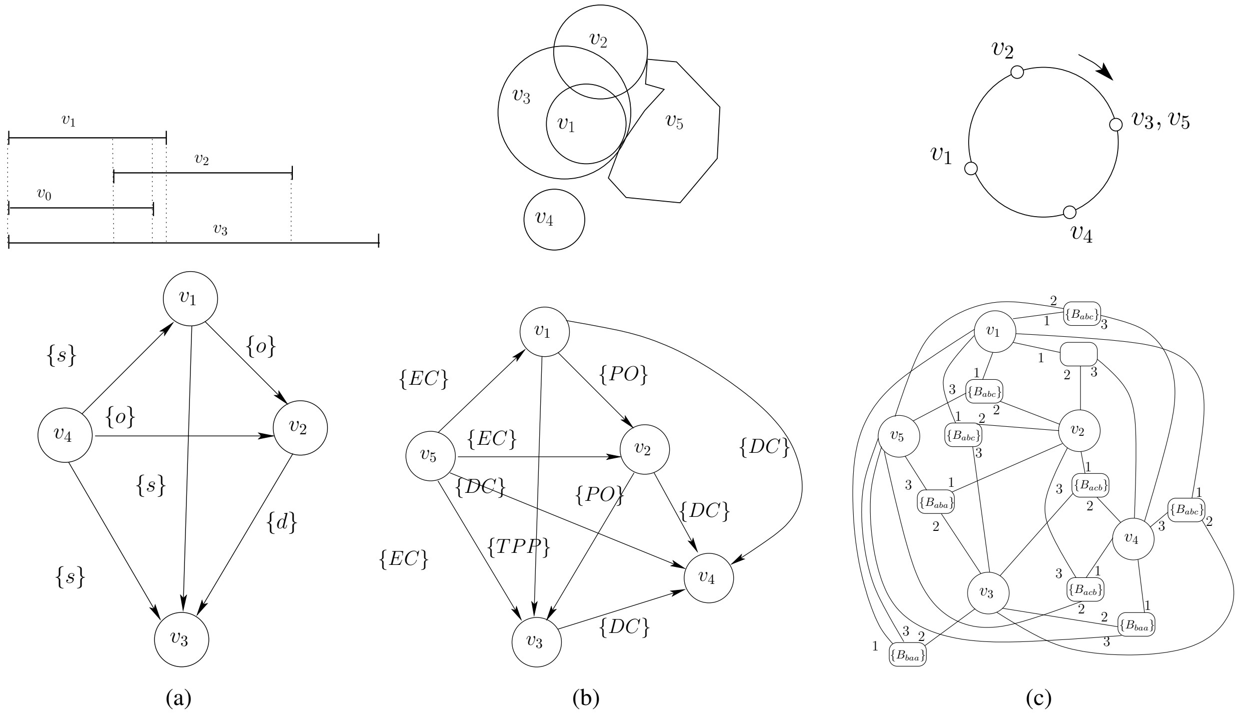


FIGURE 1.16 – Une solution et un scénario de $\mathcal{N}_1$ (a), de $\mathcal{N}_2$ (b) et de $\mathcal{N}_3$ (c).

Étant donné un RCQ, le principal problème est le problème de la cohérence : déterminer si ce RCQ admet ou n'admet pas au moins une solution. Pour la plupart des formalismes qualitatifs, ce problème est un problème NP-complet. En considérant les formalismes présentés précédemment, le problème de la cohérence est uniquement polynomial pour le calcul des instants [VK86]. Pour les autres formalismes, ce problème est NP-complet [RN97, BCF98, Lig98b, BO00, SK05]. Classiquement, les méthodes pour résoudre le problème de la cohérence consiste en la recherche d'un scénario et non pas d'une solution d'un RCQ. En effet, pour la plupart des formalismes, une solution peut être construite facilement en temps polynomial à partir d'un scénario. En règle générale, nous disposons également d'une méthode polynomiale pour décider du problème de la cohérence des scénarios du formalisme. Une méthode simple et brutale pour décider de la cohérence d'un RCQ consiste à énumérer et à tester la cohérence des scénarios du RCQ.

Un nombre important de travaux de recherche ont été et sont menés afin de proposer des méthodes pour résoudre ce problème. Les plus efficaces [LR92, Neb96, Neb97] consistent à réaliser une recherche avec retour arrière combinée avec d'une part une méthode de filtrage des contraintes utilisant l'opération de faible composition, et d'autre part un découpage des contraintes à l'aide de classes de relations dites traitables. Dans le chapitre suivant, cette approche sera détaillée, ainsi que nos contributions concernant la résolution du problème de la cohérence des RCQ.

Avant cela, nous allons faire le lien entre les RCQ et des langages logiques basés sur une axiomatisation des relations de base.

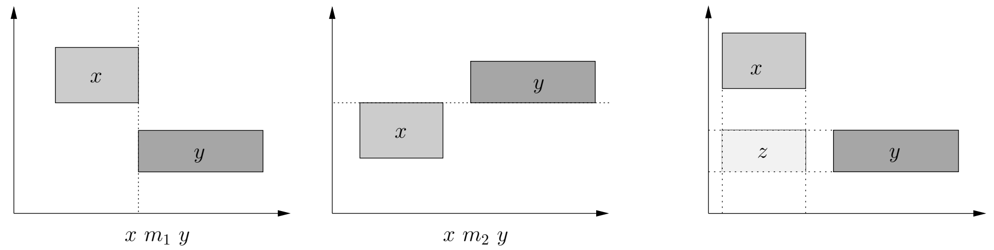
1.4 Axiomatisation des relations de base en logique du premier ordre

De nombreuses axiomatisations en logique du premier ordre de structures temporelles ont été proposées et étudiées dans le passé. Les relations considérées dans ces travaux correspondent ou permettent d'exprimer les relations de base prises en compte par certains formalismes qualitatifs. Concernant le calcul des intervalles, nous pouvons citer par exemple les travaux de ALLEN et HAYES [AH85] et LADKIN [Lad87] où sont définies et étudiées des axiomatisations en logique du premier ordre de la relation $meets$ à partir de laquelle les douze autres relations de base du calcul des intervalles peuvent être définies. D'autres axiomatisations équivalentes considérant d'autres prédicats sur les intervalles ont été proposées. VAN BENTHEM [van83] propose par exemple une axiomatisation des relations d'inclusion et de précédence sur les intervalles. La première de ces relations correspond à l'union des relations de base $starts, finishes, during$ du calcul des intervalles, tandis que la seconde correspond à la relation $before$. Dans [Tsa87], TSANG considère la relation de précédence et la relation de chevauchement sur les intervalles (qui correspond à l'union de toutes les relations de base exceptées les relations $precedes$ et son inverse). Une étude complète de ces différentes axiomatisations est réalisée dans [Haj96]. Dans le cadre spatial, des axiomatisations de certaines relations de base ont également été définies. Nous avons, par exemple, le calcul RCC8 qui comme nous l'avons vu précédemment a ses relations de base interprétées dans un modèle de la théorie appelée Region Connection Calculus proposée par RANDELL *et al.* [RC89, RCC92a, RCC92b].

Une axiomatisation en logique du premier ordre des relations de base considérées permet d'obtenir un langage plus riche que les réseaux de contraintes qualitatives. Muni d'un tel langage, le problème de la cohérence revient dans certains cas à considérer la validité d'une formule particulière. Il est également possible d'établir la table de composition faible du formalisme qualitatif considéré en testant la validité de formules particulières ou bien encore d'établir des propriétés telles que la composition faible et la composition usuelle coïncident. Définir un calcul à partir d'une axiomatisation comme dans le cadre de RCC8 permet également de considérer une classe d'interprétations (de domaines) plutôt que de considérer un domaine particulier.

Nous allons maintenant décrire nos travaux concernant des axiomatisations que nous avons proposées dans le cadre du calcul des rectangles, du calcul des points cycliques et du calcul des intervalles cycliques.

Dans le cadre du calcul des rectangles, nous avons proposé une théorie en logique du premier ordre des rectangles du plan [BCF98] afin de caractériser les modèles du calcul des rectangles. Cette théorie considère comme entités primitives les rectangles et les deux prédicats m_1 et m_2 entendus pour représenter respectivement les relations : *le côté droit du premier rectangle et le côté gauche du second rectangle sont sur la même droite verticale* et *le côté haut du premier rectangle et le côté bas du second rectangle sont sur la même droite horizontale*, voir Figure 1.17(a). L'axiomatisation de la relation m_1 et celle de m_2 correspondent à celle proposée pour la relation $meets$ des intervalles [AH85, Lad87]. Ainsi, un modèle des rectangles est une structure $(\mathcal{R}, m_1, m_2)$ avec $(\mathcal{R}, m_1)$ et $(\mathcal{R}, m_2)$ deux modèles des intervalles. Parmi ces modèles, nous considérons les modèles dits normaux qui sont les modèles des rectangles tels que (1) $\equiv_1 \cap \equiv_2$ est la relation identité sur $\mathcal{R}$ et (2) $\equiv_1 \circ \equiv_2$ est la relation universelle sur $\mathcal{R}$, avec $\equiv_i = (m_i^{-1} \circ m_i) \cap (m_i \circ m_i^{-1})$ pour $i \in \{1, 2\}$. La propriété (1) exprime intuitivement le fait que deux rectangles ayant les mêmes projections verticales et les mêmes projections horizontales sont identiques. La propriété (2) correspond au fait que pour tout couple de rectangles x et y, il existe un rectangle z tel que les projections de z et x selon la verticale sont égales et que les projections de z et y selon l'horizontale sont égales, voir Figure 1.17(b). Notre principal résultat correspond à la propriété suivante : tout modèle dénombrable des rectangles normal est isomorphe au modèle des rectangles du plan des rationnels (rectangles dont les côtés sont horizontaux et verticaux dans le plan des rationnels). Remarquons qu'à partir des relations m_1 et m_2 peuvent s'exprimer chacune des 169 relations de base de l'algèbre des rectangles.



(a) (b)

FIGURE 1.17 – (a) Illustration des deux relations m_1 et m_2 et (b) illustration de la propriété (2).

Dans [BCL03b], nous avons étudié l'axiomatisation en logique du premier ordre de la relation d'ordre cyclique dense $\prec$ (donnée précédemment en page 11) à partir de laquelle peuvent s'exprimer les relations de base du calcul des points cycliques. Nous avons, tout d'abord, établi que deux ordres cycliques denses dénombrables sont isomorphes. Tout ordre cyclique dense est donc isomorphe à l'ordre cyclique dense $(\mathbb{Q}, \prec)$ définit par $\prec (x, y, z)$ si et seulement si $x < y < z$ ou $y < z < x$ ou $z < x < y$. Nous pouvons également en déduire que la théorie du premier ordre basée sur les 6 axiomes définissant un ordre cyclique dense est complète. Dans cette étude, nous avons également défini une méthode d'élimination des quantificateurs pour cette théorie. À partir de cette méthode, nous pouvons transformer toute formule en une formule sans quantificateur équivalente et avec les mêmes variables libres. Ainsi, pour une formule sans variable libre, notre méthode permet de décider si cette formule est conséquence ou non de la théorie de l'ordre cyclique dense. Cette méthode peut être également utilisée afin de résoudre le problème de la cohérence d'un RCQ $\mathcal{N} = (V, C)$ du calcul des points cycliques. En effet, il suffit d'appliquer notre méthode d'élimination des quantificateurs sur la formule $\Phi_{\mathcal{N}} = \exists(v_1 \ldots \exists v_n)(\bigwedge_{v_i,v_j,v_k \in V}\{\phi(C_{ijk})\})$ avec n le nombre de variables de V et $\phi(C_{ijk})$ la formule résultant de la traduction de la relation C_{ijk} à l'aide des prédicats $\prec, =$. Considérons par exemple que $C_{ijk} = \{\mathsf{B}_{abc}, \mathsf{B}_{aab}\}$, nous aurons $\phi(C_{ijk}) = \prec (v_i, v_j, v_k) \vee (v_i = v_k \wedge \neg(v_x = v_z))$.

Dans **[CL04a]**[p103], nous avons proposé une axiomatisation en logique du premier ordre, notée CycInt, de la relation *meets* (m) du calcul des intervalles cycliques. Les autres relations de base de ce calcul peuvent être définies à partir de cette relation et du prédicat d'égalité. Nous avons par exemple :

$$u \; mmi \; v \; \equiv_{\text{Déf.}} \exists w, x, y, z \; w \; m \; x \; m \; y \; m \; z \; m \; w \wedge z \; m \; u \; m \; y \wedge x \; m \; v \; m \; w,$$

où une formule de la forme $v_1 \; m \; v_2 \; m \; v_3 \ldots$ est une abréviation de $v_1 \; m \; v_2 \wedge v_2 \; m \; v_3 \ldots$. CycInt est composé de huit axiomes. Ces axiomes sont motivés par des propriétés intuitives du modèle attendu que sont les intervalles du cercle. Certains sont des adaptations au cas cyclique d'axiomes proposés dans le cadre d'axiomatisations de la relation *meets* du calcul des intervalles. Nous ne donnons et commentons que les deux premiers axiomes, l'axiomatisation complète étant décrite dans **[CL04a]**[p103]. Nous utiliserons l'abréviation $X(u, v, w, x) = u \; \mathbf{m} \; v \wedge w \; \mathbf{m} \; x \wedge (u \; \mathbf{m} \; x \; w \; \mathbf{m} \; v)$. Intuitivement, $X(u, v, w, x)$ est satisfaite lorsque les deux intervalles cycliques de chacun des couples (u, v) et (w, x) se rencontrent et les rencontres se réalisent en un même point. Les axiomes $\mathbf{A1}$ et $\mathbf{A2}$ de CycInt sont les suivants :

- **A1.** $\forall u, v, w, x, y, z\ X(u, v, w, x) \wedge X(y, z, w, x) \rightarrow X(u, v, y, z)$,
- **A2.** $\forall u, v, w, x, y, z\ X(u, v, w, x) \wedge X(y, u, x, z) \rightarrow \neg u\ \mathbf{m}\ x \wedge \neg x\ \mathbf{m}\ u$.

Le premier axiome exprime l'intuition qu'étant donnés trois couples d'intervalles se rencontrant, si un couple d'intervalles se rencontre en un même point qu'un deuxième couple d'intervalles et si de plus ce deuxième couple d'intervalles se rencontre en un même point qu'un troisième couple d'intervalles, alors nous avons le premier et le troisième couples d'intervalles qui se rencontrent en un même point. Le deuxième axiome stipule que deux intervalles cycliques avec les mêmes bornes ne peuvent pas être en relation $meets$. En définissant des passerelles entre les modèles de CycInt et les ordres cycliques, nous montrons que les modèles dénombrables de CycInt sont isomorphes. À partir de ce résultat, nous pouvons en déduire que la résolution du problème de la cohérence d'un RCQ du calcul des intervalles cycliques conduira à un résultat unique quelque soit le modèle de CycInt utilisé comme interprétation des relations de base. De plus, nous pouvons également utiliser un prouveur de théorème pour résoudre le problème de la cohérence d'un RCQ. Considérons par exemple le RCQ $\mathcal{N} = (V, C)$ du calcul des intervalles cycliques avec $V = \{v_1, v_2, v_3\}$, $C_{12} = \{ppi, mi\}$, $C_{13} = \{m, mi\}$ et $C_{23} = \{o\}$. $\mathcal{N}$ est cohérent si, et seulement si, nous pouvons déduire la formule $\phi = (\exists v_1, v_2, v_3)((\phi(v_1\ ppi\ v_2) \vee \phi(v1\ mi\ v2)) \wedge (\phi(v1\ m\ v3) \vee \phi(v1\ mi\ v3)) \wedge \phi(v2\ o\ v3))$ à partir de CycInt (ϕ correspond à la traduction d'une contrainte dans le langage de la logique du premier ordre utilisant le prédicat m et le prédicat d'égalité).

1.5 Conclusion

Dans ce chapitre, nous avons introduit de manière générale les formalismes qualitatifs pour le temps et l'espace à base de contraintes. Nous avons notamment présenté la structure algébrique sur laquelle s'appuie un tel formalisme : l'ensemble des relations de base à partir duquel sont définies les relations complexes, ainsi que les différentes opérations algébriques (l'opération d'inverse, l'opération de faible composition, l'opération de permutation et l'opération de rotation). À travers différents exemples représentatifs, nous avons pu mettre en avant les différents types de relations considérés par les formalismes qualitatifs pour le temps et l'espace. Nous avons notamment présenté des formalismes considérant comme entités primitives des points, des intervalles, des rectangles ou bien encore des régions et comme relations des relations d'arité 2 ou 3 permettant d'exprimer des positions relatives en s'attachant à des aspects d'orientation, de précédence ou encore de topologie. La liste des formalismes qualitatifs présentés n'est pas exhaustive et de nombreux autres calculs ont été proposés (le calcul des intervalles orientés [Ren01], le calcul $\mathcal{OPRA}$ [MRW00], le calcul issu de la combinaison de RCC8 et du calcul des directions cardinales, le calcul combinant RCC8 et l'algèbre des rectangles [LLR09], le calcul $\mathcal{LR}$ [SN04], ...). Nous avons également défini les réseaux de contraintes qualitatives (RCQ) et le problème principal associé, le problème de la cohérence. Le chapitre suivant sera entièrement dévolu à ce problème et à nos travaux de recherche le concernant. Nous verrons en particulier comment grâce aux différentes opérations algébriques définies dans cette section peut être définie une méthode efficace de résolution de ce problème. Le problème de la cohérence d'un RCQ est un problème de satisfaction de contraintes particulier. Nous avons également vu qu'il peut être considéré comme une formule de la logique du premier ordre quantifiée existentiellement à l'aide d'une axiomatisation des relations de base. Concernant cette approche logique des formalismes qualitatifs, nous avons présenté nos travaux sur l'axiomatisation des relations de base du calcul des points cycliques et sur celle du calcul des intervalles cycliques.

Chapitre 2

Résolution de réseaux de contraintes qualitatives

Sommaire

Dans ce chapitre, nous nous focalisons sur la résolution du problème de la cohérence des RCQ. Dans un premier temps, nous présentons la méthode de fermeture par faible composition [All83]. Cette méthode permet de supprimer d'un RCQ certaines relations de base non cohérentes à l'aide de l'opération de faible composition. Le RCQ résultant possède une propriété de cohérence locale appelée $\diamond$-cohérence. La méthode de fermeture par faible composition peut être utilisée comme prétraitement d'une recherche de scénario cohérent ou comme méthode de filtrage des contraintes lors de cette recherche. Nous décrivons également l'algorithme de recherche le plus efficace dans le cadre de la résolution de RCQ. Cet algorithme [Neb96, Neb97], en dehors du fait qu'il utilise, comme méthode de filtrage des contraintes, la méthode de fermeture par faible composition, utilise également une classe de relations dite traitable. Cette classe permet dans certains cas de considérablement diminuer le facteur de branchement de l'arbre de recherche par rapport à une recherche brutale considérant une à une les relations de base définissant une contrainte. Dans un second temps, nous présentons quelques uns de nos travaux concernant cet algorithme et de manière générale la résolution de RCQ.

L'expression de RCQ à l'aide de relations appartenant à une classe traitable permet de s'assurer que la résolution de ces RCQ peut être réalisée en temps raisonnable (temps polynomial). D'autre part, une classe traitable pour laquelle la $\diamond$-cohérence est complète peut être utilisée avantageusement lors de la résolution du problème de la cohérence sur tout le formalisme. La caractérisation de classes traitables est donc une étude fondamentale systématiquement réalisée sur les différents formalismes qualitatifs pour le temps et pour l'espace proposés dans la littérature. Dans la section 2.2, nous décrivons notre approche et nos résultats concernant la recherche de classes traitables réalisée dans le cadre de différents formalismes qualitatifs : le calcul des intervalles généralisés [BCFO98a, BCFO98b, BCL00]**[Con04]**[p115], les calculs des rectangles et des n-pavés [BCF98, BCF99a, BCF99d], le calcul des n-points [BCF99c, BC02b], le calcul des points cycliques [BCL03b] et le calcul INDU [BCL03a]**[BCL06]**[p141].

La section 2.3 est en partie dévolue à la notion de contraintes éligibles que nous avons présentée dans [CLS07]. Cette notion permet de cerner les contraintes devant être considérées et découpées en sous-relations de la classe traitable lors de l'application de l'algorithme de recherche. Elle peut être utilisée dans le cadre de la définition de la condition d'arrêt de la recherche et dans celui de la définition d'une heuristique sélectionnant la prochaine contrainte à traiter. Remarquons que la notion d'éligibilité a été prise en compte dans une implémentation du solveur de contraintes qualitatives GQR [GWW08] proposée par GANTNER *et al.*, rendant ce solveur plus performant [WW09].

Dans la section 2.4, nous présentons la notion de contraintes gelées [CLS07]. Une contrainte gelée est simplement une contrainte ne pouvant pas être modifiée lors de l'application de l'algorithme de recherche (méthode de filtrage comprise). Comme nous le verrons, geler des contraintes peut être réalisé dans des situations particulières et permet d'obtenir un algorithme de recherche plus efficace.

La section 2.5 est consacrée à la description d'une nouvelle famille de cohérences locales proposée dans **[CL10]**[p203]. Chaque cohérence locale de cette famille est définie par une application f associant à chaque relation de 2^{B} un ensemble de relations. La cohérence locale associée à une application f donnée est appelée $\overset{\diamond}{f}$-cohérence. Comme nous le verrons, la plus faible des cohérences de cette famille est la $\diamond$-cohérence et la plus forte, une cohérence locale similaire à la cohérence SAC étudiée dans le cadre des CSP discrets [DB97, BD08]. Pour certaines applications f, un calcul de la fermeture par $\overset{\diamond}{f}$-cohérence peut être avantageusement réalisé lors d'un prétraitement à la résolution d'un RCQ en lieu et place d'un calcul de la fermeture par composition faible.

Les RCQ peuvent être transformés en CSP discrets de manière équivalente pour certains formalismes qualitatifs. Dans la section 2.6, nous présentons cette transformation et des propriétés la concernant. Nous présentons également une approche que nous avons proposée dans [DCLS07] et qui consiste à résoudre le problème de la cohérence d'un RCQ à l'aide d'une recherche de solutions d'un CSP discret issu d'une relaxation de sa transformation en CSP discret.

Dans la littérature, des transformations du problème de la cohérence des RCQ en problème SAT ont également été étudiées [PTS06, PTS08, WW09]. Dans [CD07, CD08], nous en proposons une que nous décrivons dans la section 2.7. L'originalité de cette traduction en problème SAT réside dans l'exploitation des classes traitables correspondant aux relations convexes.

Récemment, nous avons étudié une approche de résolution de RCQ à l'aide de décompositions arborescentes [Con11]**[CD11]**[p233]. Cette approche permet de ne pas traduire en problème SAT l'ensemble des contraintes d'un RCQ. D'autre part, elle nous a permis de revisiter l'algorithme de recherche habituellement utilisé pour résoudre un RCQ et de définir un nouvel algorithme de résolution plus efficace [Con11, CC11]. Ces travaux sont présentés dans la section 2.8.

Avant de conclure, nous décrivons les implémentations réalisées au cours de ces études. Nous nous focaliserons en particulier sur la boîte à outils QAT (Qualitative Algebra Tools) réalisée en collaboration avec MAHMOUD SAADE et GÉRARD LIGOZAT.

2.1 Algorithme de recherche efficace basé sur la ⋄-cohérence et les classes traitables

2.1.1 Méthode de fermeture par faible composition et la ⋄-cohérence

La méthode de fermeture par faible composition permet de supprimer certaines relations de base d'un RCQ avec la garantie qu'elles ne peuvent pas être satisfaites par une solution. Étant donné un RCQ $\mathcal{N} = (V, C)$, l'application de cette méthode consiste à itérer l'opération de triangulation suivante :

$$C_{ij} \leftarrow C_{ij} \cap (C_{ik} \diamond C_{kj}),$$

sur tous les triplets de variables $v_i, v_j, v_k \in V$, jusqu'à ce qu'un point fixe soit obtenu. Du fait de la définition de l'opération de faible composition, cette méthode est saine puisqu'elle ne supprime des contraintes que des relations de base ne pouvant pas être satisfaites par une solution. Dans le cas général, elle n'est pas complète pour le problème de la cohérence. En effet, elle ne supprime pas forcément toutes les relations de base incohérentes des différentes contraintes. Toutefois, lorsque l'application de la méthode de fermeture par faible composition conduit à un RCQ contenant la contrainte vide, nous pouvons conclure que le RCQ initial est non cohérent. Dans le cas contraire, nous ne pouvons pas affirmer la cohérence de ce RCQ. Nous allons maintenant introduire la cohérence locale de RCQ appelée ⋄-cohérence. Elle est définie de la manière suivante :

Définition 3 *Soit un* RCQ $\mathcal{N} = (V, C)$. $\mathcal{N}$ *est dit ⋄-cohérent ou fermé par faible composition si et seulement si* $C_{ij} \subseteq C_{ik} \diamond C_{kj}$, $\forall v_i, v_j, v_k \in V$.

Le RCQ obtenu en appliquant la méthode de fermeture par faible composition sur un RCQ $\mathcal{N}$ est unique. Ce RCQ appelé fermeture de $\mathcal{N}$ par ⋄-cohérence et noté $\diamond(\mathcal{N})$ correspond au plus grand (pour $\subseteq$) sous-RCQ ⋄-cohérent de $\mathcal{N}$. Étant donnés deux RCQ $\mathcal{N}$ et $\mathcal{N}'$ définis sur un même ensemble de variables, nous avons les propriétés suivantes concernant la fermeture par ⋄-cohérence :

- $\diamond(\mathcal{N}) \subseteq \mathcal{N}$;
- $\diamond(\mathcal{N})$ est équivalent à $\mathcal{N}$;
- $\diamond(\diamond(\mathcal{N})) = \diamond(\mathcal{N})$;
- si $\mathcal{N}' \subseteq \mathcal{N}$ alors $\diamond(\mathcal{N}') \subseteq \diamond(\mathcal{N})$.

La ⋄-cohérence est très proche de la chemin-cohérence définie dans le cadre des CSP [Mac77, MF85, Mon74]. En effet, dans le cas où la faible composition et la composition relationnelle classique correspondent aux mêmes opérations, ces deux cohérences locales coïncident. Dans le cas général, la ⋄-cohérence implique la $(0, 3)$-cohérence (il existe une solution partielle pour tout triplet de variables). Les algorithmes permettant d'implémenter de manière efficace la méthode de fermeture par faible composition sont largement inspirés des algorithmes permettant d'obtenir la propriété de chemin-cohérence dans le cadre des CSP, en particulier des algorithmes PC1 et PC2 [Mac77, MF85, Mon74].

Ceux basés sur PC1 réalisent dans une boucle principale les opérations de triangulation pour l'ensemble des triplets de variables du RCQ. Cette boucle sera de nouveau réalisée jusqu'à ce qu'aucune des contraintes du RCQ ne soit modifiée. La complexité en temps de ces algorithmes est en $O(n^5)$ avec n le nombre de variables. Cette complexité temporelle peut être rabaissée en $O(n^3)$ en considérant des algorithmes basés sur PC2. Ces algorithmes utilisent une structure de données permettant de sauvegarder les paires ou les triplets de variables pour lesquels il est nécessaire de réaliser des opérations de triangulation, évitant ainsi de réaliser des opérations de triangulation inutiles (*i.e.* des opérations de triangulations ne supprimant pas de relations de base de la contrainte considérée).

Considérons la fonction FFC2P (Fermeture par Faible Composition) donnée à titre d'exemple. Cette fonction inspirée de PC2 utilise une queue sauvegardant les couples de variables (v_i, v_j) pour lesquels

Function FFC2P($\mathcal{N}$)

in : $\mathcal{N} = (V, C)$ un RCQ avec $n = |V|$.
output : $\diamond(\mathcal{N})$.

begin
  $Q \leftarrow \{(v_i, v_j) : 1 \leq i \leq j \leq n\}$; /* Initialisation de Q */
  while $Q \neq \emptyset$ **do**
    $(v_i, v_j) \leftarrow next(Q)$; /* Sélection d'un couple (v_i, v_j) à traiter */
    $Q \leftarrow Q \setminus \{(v_i, v_j)\}$;
    foreach $v_k \in V$ **do**
      $R \leftarrow C_{kj} \cap (C_{ki} \diamond C_{ij})$; /* Opération de triangulation sur (v_k, v_i, v_j) */
      if $C_{kj} \neq R$ **then**
        $C_{kj} \leftarrow R$; $C_{jk} \leftarrow R^{\smile}$;
        if $k \leq j$ **then**
          $Q \leftarrow Q \cup \{(v_k, v_j)\}$;
        else
          $Q \leftarrow Q \cup \{(v_j, v_k)\}$;
      $R \leftarrow C_{ik} \cap (C_{ij} \diamond C_{jk})$; /* Opération de triangulation sur (v_i, v_j, v_k) */
      if $C_{ik} \neq R$ **then**
        $C_{ik} \leftarrow R$; $C_{ik} \leftarrow R^{\smile}$;
        if $i \leq k$ **then**
          $Q \leftarrow Q \cup \{(v_i, v_k)\}$;
        else
          $Q \leftarrow Q \cup \{(v_k, v_i)\}$;
  return $\mathcal{N}$;

il sera nécessaire de réaliser les opérations de triangulation $C_{kj} \leftarrow C_{kj} \cap (C_{ki} \diamond C_{ij})$ et $C_{ik} \leftarrow C_{ik} \cap (C_{ij} \diamond C_{jk})$ pour toute variable $v_k \in V$, avec $\mathcal{N} = (V, C)$ le RCQ donné en paramètre. La fonction FFC2P a une complexité spatiale en $O(n^2)$. Il est également possible de sauvegarder les triplets de variables correspondant aux opérations de triangulations à réaliser plutôt que des couples de variables. Dans ce cas, une structure de données contiendra les triplets (v_i, v_j, v_k) pour lesquels il sera nécessaire de réaliser l'opération de triangulation $C_{ij} \leftarrow C_{ij} \cap (C_{ik} \diamond C_{kj})$. Les algorithmes obtenus ont une complexité spatiale en $O(n^3)$.

Dans le cadre des algorithmes basés sur PC2, différentes heuristiques concernant l'ordre dans lequel sont réalisées les différentes opérations de triangulation ont été proposées. Ces heuristiques peuvent diminuer considérablement le nombre global d'opérations de triangulation à effectuer. Une heuristique efficace va tenter de sélectionner au cours du traitement comme prochain couple (ou triplet) de variables à traiter, le couple (ou triplet) maximisant le nombre de relations de base supprimées lors des opérations de triangulation correspondantes. L'efficacité d'une heuristique dépend également du temps et de l'espace mémoire nécessaire à son traitement, ceci devant être de moindre coût compte tenu du nombre important d'opérations de triangulation à réaliser dans le cas général. Dans le cas d'une structure de données manipulant des paires de variables, une heuristique simple et efficace consiste à sélectionner un couple de variables (v_i, v_j) dont la relation correspondante est de plus petite cardinalité par rapport aux relations des autres couples sélectionnables.

Une autre heuristique efficace proposée par MANCHAK et VAN BEEK [BM96] dans le cadre du calcul des intervalles utilise une pondération statique de chaque relation de base. Un poids est attribué à chacune des relations de base de B de la manière suivante. Tout d'abord est calculée, pour chaque relation de base, la somme des cardinalités des relations obtenues par composition faible de la relation de base considérée avec chacune des relations du calcul. Dans un second temps, une mise à l'échelle de ces sommes est réalisée afin de définir le poids de chacune des relations de base : la valeur 1 correspond à la plus petite somme obtenue, la valeur 2 à la somme suivante, et ainsi de suite. Les poids obtenus pour les 13 relations de base du calcul des intervalles sont les suivants : 1 pour eq, 2 pour s, si, f, fi, m, mi, 3 pour p, pi, di et 4 pour d, o, oi. La pondération d'une relation complexe est obtenue en sommant les poids des relations de base la composant. Ainsi, le poids de la relation $\{p, m, o, eq\}$ est de $3 + 2 + 4 + 1$, *i.e.* 11. Intuitivement, plus le poids d'une relation est faible, plus la relation est estimée être restrictive lors d'une opération de triangulation. L'heuristique choisira parmi les couples de variables sélectionnables, un couple de variables (v_i, v_j) minimisant le poids de C_{ij}. Dans [CLS06b], nous avons défini une heuristique basée sur une pondération légèrement différente. En effet, nous attribuons les différents poids à partir de statistiques réalisées sur la table de composition. Pour chaque relation de base, nous réalisons la somme des cardinalités des relations présentes sur la ligne ou la colonne ayant pour entrée cette relation de base. Puis, à partir d'une mise à l'échelle des sommes obtenues, nous affectons un poids à chaque relation de base. Pour le calcul des intervalles, nous obtenons une pondération qui diffère de celle de MANCHAK et VAN BEEK par le fait par exemple que le poids de la relation de base d est 3 à la place de 4. L'avantage de notre pondération est qu'elle nécessite un faible coût de calcul même pour un nombre de relations de base important et peut donc s'étendre à tout formalisme qualitatif.

Dans le cadre de la thèse MAHMOUD SAADE, nous avons réalisé une étude expérimentale comparative complète des différents algorithmes implémentant la fermeture par faible composition. Nous avons comparé des algorithmes basés sur PC1 et des algorithmes basés PC2 utilisant différentes structures de données et heuristiques. Lors de cette étude, nous avons introduit de nouvelles heuristiques correspondant essentiellement à la combinaison d'heuristiques existantes. De plus, nous avons introduit un nouvel algorithme basé à la fois sur PC1 et sur PC2. Cet algorithme réalise tout d'abord une boucle principale dans laquelle sont effectuées les opérations de triangulation pour l'ensemble des triplets des variables du RCQ. Il réalise, dans un deuxième temps, un traitement similaire aux algorithmes basés sur PC2 à la différence que la structure de données des couples ou triplets à traiter ne contient initialement que les couples ou triplets correspondant aux contraintes modifiées lors de la première phase du traitement. L'intérêt de cet algorithme est d'être un compromis entre des algorithmes basés sur PC1 moins efficaces en terme de temps et des algorithmes basés sur PC2 plus gourmands en espace mémoire. Pour plus de détails, le lecteur pourra se reporter à [CLS06b, Saa08].

2.1.2 Algorithme de recherche

Nous allons maintenant présenter l'algorithme de recherche avec retour arrière proposé par LADKIN et REINEFELD [LR92] et NEBEL [Neb96, Neb97]. À chaque étape de la recherche, cet algorithme utilise la méthode de fermeture par faible composition comme méthode de filtrage des contraintes afin de supprimer des relations de base non cohérentes. De plus, il utilise une classe traitable pour minimiser le facteur de branchement lors de la recherche. Avant de donner et commenter cet algorithme, définissons la notion de classes traitables.

Une classe est un sous-ensemble $\mathcal{C}$ de 2^{B}, fermé pour les différentes opérations que sont l'intersection, l'inverse et la faible composition. Nous dirons qu'un RCQ $\mathcal{N} = (V, C)$ est défini sur une classe $\mathcal{C}$ lorsque $C_{ij} \in \mathcal{C}$ pour tout $v_i, v_j \in V$. Une classe $\mathcal{C}$ est dite traitable lorsque le problème de la cohérence des RCQ définis sur $\mathcal{C}$ est un problème polynomial. Les classes traitables qui nous intéressent plus particulièrement ici sont les classes pour lesquelles le problème de la cohérence peut être résolu au moyen de la méthode de fermeture par faible composition, *i.e.* dont tout RCQ $\diamond$-cohérent non trivialement incohérent est cohérent. Pour une telle classe traitable $\mathcal{C}$, nous dirons dans la suite que la $\diamond$-cohérence est complète pour $\mathcal{C}$.

La fonction Cohérence (voir page 32) est basée sur l'algorithme de recherche le plus efficace et le plus

Function Cohérence($\mathcal{N}$) : Booléen

in : $\mathcal{N} = (V, C)$, un RCQ.
output : $true$ si $\mathcal{N}$ est cohérent, $false$ sinon.

1 **begin**
2 $\mathcal{N} \leftarrow \diamond(\mathcal{N})$; /* Filtrage par fermeture par faible composition */
3 **if** $\mathcal{N} = \bot$ **then**
4 **return** *false* ; /* Non cohérence détectée */
5 Sélectionne $(v_i, v_j) \in V \times V$ avec $i < j$ tel que (v_i, v_j) non déjà sélectionné;
6 **if** *un tel couple n'existe pas* **then**
7 **return** *true* ; /* Plus de contraintes à traiter, cohérence détectée */
8 Partager C_{ij} en sous-relations $R_1, \ldots, R_k \in \mathcal{C}$;
9 **foreach** $l \in 1, \ldots, k$ /* Recherche pour chacune des sous-relations */
10 **do**
11 $C_{ij} \leftarrow R_l$; $C_{ji} \leftarrow R_l^{\smile}$;
12 **if** Cohérence($\mathcal{N}$) **then**
13 **return** *true* ; /* Cohérence détectée pour la sous-relation r_l */
14 **return** *false* ; /* Non cohérence détectée pour chaque sous-relation */

couramment utilisé afin de résoudre un RCQ. Son traitement nécessite une classe traitable $\mathcal{C}$ pour laquelle la $\diamond$-cohérence est complète. La fonction Cohérence est récursive et considère une à une les différentes contraintes du RCQ $\mathcal{N}$ passé en paramètre (ligne 5). Avant la sélection de la prochaine contrainte à traiter, la méthode de fermeture par faible composition est appliquée afin de supprimer de $\mathcal{N}$ des relations de base non possibles et de rendre celui-ci $\diamond$-cohérent (ligne 2). Suite à ce traitement, si une contrainte de $\mathcal{N}$ est vide, la non cohérence de $\mathcal{N}$ est détectée. Dans le cas contraire, une nouvelle contrainte C_{ij} est sélectionnée (ligne 5), puis traitée de la manière suivante. Elle est tout d'abord découpée en sous-relations $R_1, \ldots, R_k$ appartenant à la classe $\mathcal{C}$. Ainsi, nous avons $C_{ij} = \bigcup_{l \in \{1,\ldots,k\}} R_l$ avec $R_l \in \mathcal{C}$ pour tout $l \in \{1, \ldots, k\}$. Pour que ce découpage soit toujours réalisable, il est nécessaire que $\mathcal{C}$ contienne l'ensemble des relations singletons de 2^{B}. Dans le cas général, ce découpage n'est pas forcément unique. Par la suite, de manière itérative la contrainte C_{ij} est définie par une des sous-relations issues du découpage avant de réaliser un appel récursif à la fonction Cohérence. À la ligne 14, la non cohérence du RCQ courant est détectée du fait que chacune des sous-relations n'a pas abouti à la caractérisation d'un sous-RCQ cohérent. Dans le cas où il n'existe plus de contrainte à traiter (ligne 6), la fonction Cohérence a caractérisé un sous-RCQ $\diamond$-cohérent du RCQ initial, défini sur $\mathcal{C}$ et ne contenant pas la relation vide comme contrainte. Nous pouvons donc déduire que ce RCQ et le RCQ initial sont cohérents (pourvu que la $\diamond$-cohérence soit complète pour $\mathcal{C}$). Ainsi, nous avons la propriété suivante :

Théorème 1 *Soit $\mathcal{C}$ une classe traitable contenant l'ensemble des relations singletons pour laquelle la $\diamond$-cohérence est complète. Nous avons : la fonction* Cohérence *est saine et complète pour le problème de la cohérence des* RCQ *définis sur* 2^{B}.

Notons que si la classe $\mathcal{C}$ ne contient pas l'ensemble des relations singletons, la fonction Cohérence permet de résoudre le problème de la cohérence des RCQ définis sur l'ensemble des relations constitué des unions de relations de $\mathcal{C}$. De plus, lorsque les RCQ définis sur la classe traitable $\mathcal{C}$ ne sont pas résolus par fermeture par faible composition mais par une autre méthode polynomiale, un appel de cette dernière méthode doit être réalisé afin de s'assurer de la cohérence du RCQ courant (ajout d'une instruction à la ligne 6 de la fonction Cohérence).

Les implémentations les plus efficaces concernant le calcul de la fermeture par faible composition (ligne 2) sont réalisées à partir d'algorithmes incrémentaux basés sur PC2. Ces algorithmes initialisent la structure des données par les couples ou les triplets devant être traités suite à la modification réalisée à ligne 11 de la fonction Cohérence. Pour obtenir la version incrémentale de la fonction FFC2P donnée dans la section précédente, il nous faudrait seulement substituer la ligne 2 par une initialisation de Q de la forme $Q \leftarrow \{(v_i, v_j)\}$ avec C_{ij} la contrainte modifiée lors de la recherche (ligne 11 de la fonction Cohérence).

Dans [Neb96, Neb97], NEBEL réalise une étude expérimentale complète d'un algorithme similaire à la fonction Cohérence. Cette étude réalisée sur des RCQ du calcul des intervalles considère la classe des relations ORD-Horn que nous présenterons dans la section suivante. Il montre que le facteur de branchement de l'arbre de recherche est très fortement réduit du fait de l'utilisation de cette classe. En supposant que les relations du calcul des intervalles soient uniformément distribuées, la facteur de branchement moyen (théorique) est de $2,533$ pour la classe des relations ORD-Horn contre $6,5$ lorsqu'on utilise les relations singletons pour le découpage des contraintes. Comme nous le verrons par la suite, ces résultats ont conduit à de très nombreux travaux afin de caractériser des classes traitables dans différents formalismes qualitatifs.

2.2 Recherche de classes traitables

Pour de nombreux formalismes qualitatifs, des études ont été menées afin de caractériser des classes ou des ensembles de relations traitables [VK86, VP87, NB94, Neb95, DJ96, Kou96, DJ97, JD97, RN97, BCF98, BCFO98a, BCF99a, RN99, Ren99, BCF99d, BCL00, BO00, Ren01, GN02, BCL03b, BCL03a, RW04, BCL06]. Nous nous bornerons à décrire quelques classes traitables du calcul des intervalles et des résultats concernant nos propres travaux.

Les études concernant le calcul des intervalles ont permis de réaliser une cartographie complète des classes traitables et de celles qui ne le sont pas [KJJ03]. Les relations convexes [Nök91, VKB90] et les relations ORD-Horn [NB94, Neb95, NB95], également appelées relations préconvexes [Lig94, Lig96], forment deux classes traitables de ce calcul pour lesquelles la méthode de fermeture par faible composition résout le problème de la cohérence. Différentes approches permettent de caractériser ces ensembles de relations. La première caractérisation des relations ORD-Horn données par NEBEL et BÜRCKERT [NB94, NB95] utilisent la notion de clauses ORD. Une clause ORD est une clause contenant uniquement des littéraux de la forme $a = b$, $a \leq b$ et $a \neq b$ (avec a, b des bornes d'intervalles). En considérant les littéraux de la forme $a = b$ et $a \leq b$ comme des littéraux positifs et les littéraux de la forme $a \neq b$ comme des littéraux négatifs, les relations ORD-Horn sont les relations du calcul des intervalles pouvant s'exprimer par des ensembles de clauses ORD de Horn (clauses ORD avec au plus un littéral positif). La relation $\{p, o, d\}$ est, par exemple, une relation ORD-Horn puisque la satisfaction de cette relation de base entre deux intervalles $x = (x^-, x^+)$ et $y = (y^-, y^+)$ peut s'exprimer avec l'ensemble de clauses ORD-Horn suivant :

$$x^- \leq x^+, x^- \neq x^+, y^- \leq y^+, y^- \neq y^+, x^+ \leq y^+, x^+ \neq y^+, x^- \neq y^-, y^- \leq x^- \vee y^- \neq x^+.$$

Les relations ORD-Horn sont au nombre de 868, soit plus de 10% des 8192 relations du calcul des intervalles. La classe des relations ORD-Horn est la seule classe maximale traitable contenant les 13 relations singletons du calcul des intervalles. Ainsi, pour toute classe contenant les 13 relations singletons, soit le problème de la cohérence la concernant est NP-complet, soit cette classe est incluse dans la classe des relations ORD Horn ; le problème de la cohérence correspondant est alors polynomial. Des classes maximales traitables ne contenant pas toutes les relations singletons ont été caratérisées [DJ96, DJ98, KJJ01]. Aujourd'hui toutes connues, elles sont au nombre de 17 [KJJ03].

Les relations convexes sont les relations ORD-Horn pouvant s'exprimer à l'aide d'un ensemble de clauses ORD unitaires tel que si $a \neq b$ est une clause présente alors la clause $a \leq b$ ou $b \leq a$ est également présente. La relation $\{p, m, o, s, d\}$ est une relation convexe car la satisfaction de cette relation par deux intervalles x et y peut s'exprimer par l'ensemble de clauses ORD suivant :

$$x^- \leq x^+, x^- \neq x^+, y^- \leq y^+, y^- \neq y^+, x^+ \leq y^+, x^+ \neq y^+.$$

La classe des relations convexes est constituée de 83 relations. De plus, tout RCQ $\diamond$-cohérent non trivialement incohérent défini à partir de cette classe est globalement cohérent. De manière générale, cette propriété n'est pas satisfaite par les RCQ $\diamond$-cohérents définis sur l'ensemble des relations ORD-Horn.

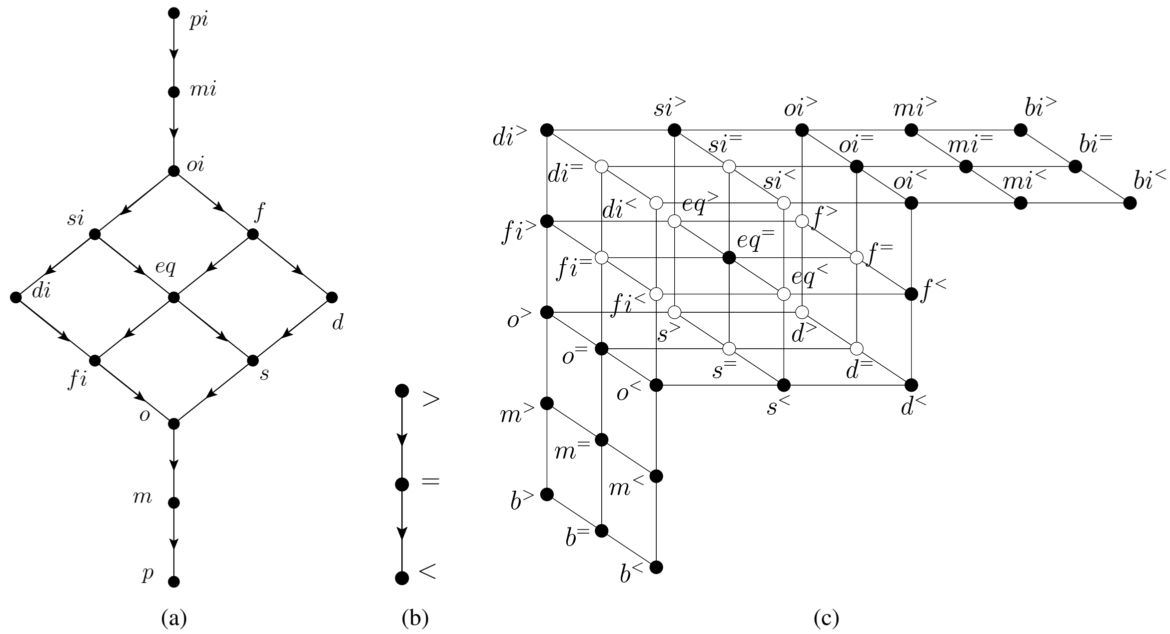


FIGURE 2.1 – (a) le treillis des intervalles, (b) le treillis du calcul des points, (c) le treillis du calcul INDU.

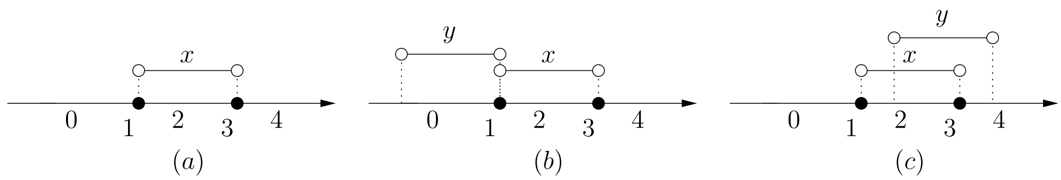


FIGURE 2.2 – Partionnement de la droite en 5 zones par un intervalle x (a), la relation $meets$ (m) satisfaite par y et x et la relation $overlappedBy$ (oi) satisfaite par y et x (c).

Les relations convexes peuvent être facilement caractérisées à l'aide du treillis des intervalles $(\mathsf{B}, \leq)$ [Nök91, Lig91] représenté par la figure 2.1(a). Ce treillis peut être défini de plusieurs manières, nous décrivons l'une d'entre elles proposée par LIGOZAT. Étant donné un intervalle de la droite x, nous pouvons définir 5 zones auxquelles nous attribuons un entier compris entre 0 et 4 (voir Figure 2.2). À chaque relation de base r est associé le couple d'entiers (i, j) avec i (resp. j) la référence de la zone dans laquelle doit se trouver la borne inférieure (resp. la borne supérieure) d'un intervalle y satisfaisant avec x la relation de base r. Nous avons, par exemple, le couple $(0, 1)$ qui est associé à la relation de base $meets$ (m), et le couple $(2, 4)$ qui est associé à la relation $overlappedBy$ (oi). La relation d'ordre $\leq$

définissant le treillis des intervalles se définit par $r \leq r'$ ssi $i_r \leq i_{r'}$ et $j_r \leq j_{r'}$, avec (i_r, j_r) et $(i_{r'}, j_{r'})$ les deux couples d'entiers attribués respectivement à r et r'. Nous avons par exemple, $m \leq oi$.

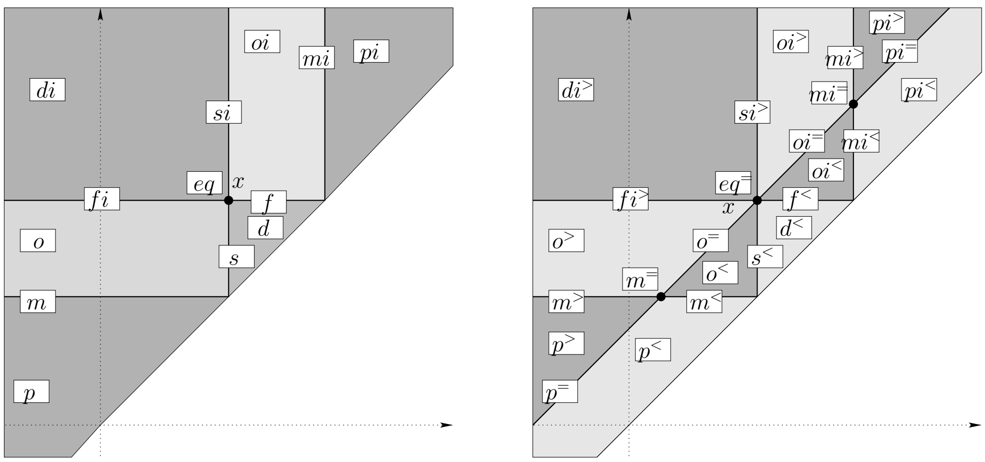
Les relations convexes correspondent aux intervalles de ce treillis. La relation $\{p, m, o, s, d\}$ est, par exemple, la relation convexe correspondant à l'intervalle $[p, d]$ de $(\mathsf{B}, \leq)$. À partir des relations convexes et de la notion de dimension d'une relation, LIGOZAT [Lig96] définit l'ensemble des relations préconvexes, ensemble qui coïncide avec l'ensemble des relations ORD-Horn. Il attribue une dimension (un entier compris entre 0 et 2) à chacune des relations de base du calcul des intervalles. Pour une relation de base r, cet entier correspond au nombre maximum d'égalités de bornes imposées par une relation de base (2) moins le nombre d'égalités de bornes imposées par r. Ainsi, la dimension des relations de base p, pi, o, oi, d, di est de 2, celle de m, mi, f, fi, s, si est de 1 et celle de eq est de 0. La dimension d'une relation $R \in 2^{\mathsf{B}}$ est le maximum des dimensions des relations de base la composant. Une relation R est préconvexe si et seulement si, afin d'obtenir la plus petite relation convexe contenant R, nous rajoutons uniquement des relations de base de dimension strictement inférieure à la dimension de R. Considérons par exemple, la relation $R = \{p, o, d\}$. La dimension maximale des relations de base appartenant à R est de 2. D'autre part, la plus petite des relations convexes incluant R est la relation $\{p, m, o, s, d\}$. Ainsi, pour obtenir cette relation à partir de R, nous devons ajouter les relations de base m et s de dimension 1, nombre strictement inférieur à 2. Nous pouvons en conclure que $R = \{p, o, d\}$ est une relation préconvexe.

Nous avons apporté différentes contributions concernant la caractérisation de classes ou d'ensembles traitables dans le contexte de certains formalismes qualitatifs : le calcul des intervalles généralisés [BCFO98a, BCFO98b, BCL00]**[Con04]**[p115], les calculs des rectangles et des n-pavés [BCF98, BCF99a, BCF99d], le calcul des n-points [BCF99c, BC02b], le calcul des points cycliques [BCL03b] et le calcul INDU [BCL03a]**[BCL06]**[p141].

Pour les formalismes qualitatifs dont les relations de base sont interprétées sur des intervalles ou des points de la droite, nous avons suivi la ligne de raisonnement proposée par LIGOZAT en définissant les relations convexes et préconvexes à partir de la notion de dimension et d'un treillis. Les différents treillis ont été définis à partir de produits du treillis des intervalles, de produits du calcul des points (voir Figure 2.1(b)) ou de produits des deux, en apportant éventuellement des restrictions sur la structure obtenue. À titre d'illustration, considérons le treillis du calcul INDU représenté par la figure 2.1(c). Nous avons défini ce treillis en réalisant le produit du treillis des intervalles avec celui des points. Une relation convexe du calcul INDU correspond à un intervalle de ce treillis dans lequel ont été supprimés les éléments ne correspondant pas à des relations du calcul (éléments appelés relations virtuelles et représentés par des cercles blancs sur la figure). Nous avons, par exemple, la relation convexe $\{si^>, oi^>, oi^=, oi^<\}$ qui correspond à l'intervalle $[si^<, oi^>]$ privé des relations virtuelles $si^=$ et $si^>$. Les notions de dimension et de relations préconvexes sont définies de la même manière que pour le calcul des intervalles. Nous avons montré que la $\diamond$-cohérence est complète pour les classes des relations convexes du calcul des n-pavés, du calcul des n-points et du calcul des intervalles généralisés. Concernant les relations préconvexes, nous avons des résultats différents de ceux établis dans le cadre du calcul des intervalles. Nous avons, par exemple, montré que pour l'ensemble des relations préconvexes du calcul des n-pavés (pour $n > 1$) et le calcul des n-points (pour $n > 2$), le problème de la cohérence est non polynomiale.

Suite à ce résultat, nous avons distingué deux notions de préconvexité : la préconvexité faible et la préconvexité forte. Les relations faiblement préconvexes correspondent à la notion de préconvexité initiale. La notion de préconvexité forte que nous avons introduite est définie de la manière suivante. Une relation fortement préconvexe est une relation R faiblement préconvexe telle que pour chaque relation convexe R', $R \cap R'$ est une relation faiblement préconvexe. Notons que l'ensemble des relations convexes est inclus dans l'ensemble des relations fortement préconvexes, lui-même inclus dans l'ensemble des relations faiblement préconvexes. Pour les formalismes précédemment cités (le calcul des n-pavés, le calcul des n-points et le calcul des intervalles généralisés), nous avons montré que pour les ensembles de relations fortement préconvexes, le problème de la cohérence peut se résoudre au moyen de la méthode de fermeture par faible composition. Pour montrer ce résultat, nous avons introduit une méthode plus faible que la fermeture par faible composition basée sur l'opération de triangulation suivante : $C_{ij} \leftarrow C_{ij} \cap I(C_{ik} \diamond C_{kj})$, avec $I(R)$ correspondant à la fermeture convexe de la relation $R \in 2^{\mathsf{B}}$ (la plus petite relation convexe contenant R). Cette méthode a été introduite pour palier le fait que pour certains formalismes, nous ne savons pas si l'ensemble des relations fortement préconvexes est stable pour l'opération par faible composition (contrairement à la stabilité de l'ensemble des relations fortement préconvexes pour l'intersection et l'inverse). Nous avons également établi que pour le calcul des 3-points, la classe des relations fortement préconvexes est une classe maximale traitable.

Dans le cadre du formalisme INDU, nous avons également étudié les relations convexes et préconvexes [BCL03a]**[BCL06]**[p141]. Lors de cette étude, nous avons obtenu des résultats différents. En effet pour les RCQ définis à partir de relations singletons et *a fortiori* à partir de relations fortement préconvexes, la $\diamond$-cohérence n'est pas complète. Malgré tout, le problème de la cohérence sur l'ensemble des relations fortement préconvexes est polynomial, contrairement à celui sur les ensembles des relations faiblement préconvexes. Nous avons montré que tout RCQ défini par des relations fortement préconvexes peut être traduit de manière équivalente en un ensemble de contraintes linéaires particulières pouvant être résolu en temps polynomial. Ces contraintes linéaires appelées contraintes de Horn sont des disjonctions d'inéquations de la forme $a_1.x_1 + \ldots + a_k.x_k \leq a$ et d'inégalités de la forme $a_1.x_1 + \ldots + a_k.x_k \neq a$, avec $a, a_1, \ldots, a_k$ des entiers et $x_1, \ldots, x_k$ des variables sur les réels, et contenant au plus une inéquation. Dans [Kou96], KOUBARAKIS établit que les contraintes de Horn peuvent être résolues en temps polynomial. Toujours concernant le calcul INDU, nous avons également caractérisé un ensemble de relations distinctes de l'ensemble des relations fortement préconvexes pour lequel la $\diamond$-cohérence est complète. Cette classe contient 11854 relations sur les 2^{25} relations du formalisme INDU. Chacune de ses relations est une relation faiblement préconvexe satisfaisant une propriété particulière avec les relations convexes de INDU pouvant se traduire dans le calcul des intervalles d'ALLEN.



(a) (b)

FIGURE 2.3 – Représentation des relations de base du calcul des intervalles dans le plan (a) et représentation de celles du calcul de INDU (b).

Des notions telles que convexité et la préconvexité peuvent être définies et étudiées dans le cadre d'une représentation géométrique des différentes relations de base (voir Figure 2.3). En prenant de nouveau comme exemple le calcul INDU, nous pouvons représenter chaque relation de base par une région particulière du plan muni d'un repère orthogonal. Pour se faire, chaque intervalle $x = (x^-, x^+)$ du domaine est représenté dans le plan par le point d'abscisse x^- et d'ordonnée x^+. En fixant un intervalle de référence $x = (x^-, x^+)$, à chaque relation r est associée la région de points correspondant aux intervalles $y = (y^-, y^+)$ satisfaisant avec x la relation de base r. À une relation $R \in 2^{\mathsf{B}}$ est associée l'union des régions des relations de base la composant, voir Figure 2.3(b). À partir de cette représentation, différentes notions et propriétés peuvent être caractérisées. Nous avons, par exemple, la notion de dimension d'une relation R qui peut être définie par la dimension de la région représentant la relation R. La notion de préconvexité peut également être définie à l'aide de la fermeture topologique : une relation R est préconvexe si et seulement si la région correspondant à sa fermeture convexe est incluse dans la fermeture topologique de la région la représentant. Dans [LC05], nous montrons à travers différents exemples les avantages certains que peut apporter une telle représentation géométrique des relations de base dans l'étude des formalismes qualitatifs pour le temps et l'espace.

Dans[BCL03b], nous étudions le problème de la cohérence des RCQ définis dans le calcul ternaire des points cycliques. Nous paramétrons ce problème avec deux ensembles B et T. L'ensemble B correspond aux relations permises pour les contraintes de la forme C_{iij} et l'ensemble T aux relations permises pour les contraintes de la forme C_{ijk} avec v_i,v_j et v_k trois variables distinctes. T est un ensemble fermé pour les opérations de permutation et de rotation. Pour différents couples (B,T), nous étudions la complexité du problème de la cohérence et caractérisons des cas polynomiaux ainsi que des cas où le problème reste NP-complet. Nous montrons également que pour certains cas polynomiaux, la méthode de fermeture par faible composition est complète pour le problème de la cohérence. Pour le calcul des points cycliques et de manière générale pour les calculs basés sur des relations d'arité a, cette méthode est basée sur l'opération suivante : $C(v_{i_1}, \ldots, v_{i_a}) \leftarrow C(v_{i_1}, \ldots, v_{i_a}) \cap \diamond(C(v_{i_1}, \ldots, v_{i_{a-1}}, v_j), C(v_{i_1}, \ldots, v_{i_{a-2}}, v_j, i_a), \ldots, C(v_j, v_{i_2}, \ldots, v_{i_a}))$ avec $v_{i_1}, \ldots, v_{i_a} \in V$. Cette opération généralise l'opération de triangulation utilisée dans le cadre binaire. À titre d'exemple, en considérant les ensembles $B = \{\emptyset, \{B_{aab}\}, \{B_{aba}\}, \{B_{baa}\}, \{B_{aab}, B_{aba}\}, \{B_{aab}, B_{baa}\}, \{B_{aba}, B_{baa}\}\}$ et $T = \{\emptyset, \{B_{abc}\}, \{B_{acb}\}\}$, nous avons montré que le problème de la cohérence sur (B, T) se résout en temps polynomial à l'aide de la méthode de fermeture par faible composition.

2.3 Contraintes éligibles et heuristiques de sélection de contraintes

L'algorithme Cohérence décrit en sous-section 2.1.2 nécessite de réaliser un choix concernant la future contrainte à traiter à chaque étape de la recherche (ligne 5). Cette sélection se fait parmi l'ensemble des contraintes non déjà traitées aux étapes précédentes. La notion d'éligibilité que nous avons introduite dans [CLS07] permet de réduire cet ensemble de choix en écartant des contraintes qui n'ont pas besoin d'être traitées par un découpage en sous-relations de la classe $\mathcal{C}$. Les contraintes qui devront être traitées sont appelées contraintes éligibles et celles qui ne le seront pas, les contraintes non éligibles. Étant donné un RCQ $\mathcal{N} = (V, C)$ sur lequel est appliqué l'algorithme de recherche Cohérence, ces deux types de contraintes sont définis de la manière suivante :

Définition 4 *Une contrainte C_{ij} entre les variables $v_i, v_j \in V$ d'un* RCQ *$\mathcal{N}$ est dite éligible si à aucun moment au cours du traitement de $\mathcal{N}$ la relation C_{ij} n'a appartenu à la classe $\mathcal{C}$. Dans le cas contraire, la contrainte entre C_{ij} est dite non éligible.*

Notons qu'une contrainte non éligible en cours de traitement ne pourra jamais devenir éligible par la suite. Nous avons montré qu'au cours de la recherche, il était suffisant de traiter uniquement les contraintes éligibles. Ainsi, à chaque étape de la recherche, la sélection de la contrainte à traiter se fera parmi les contraintes éligibles. L'instruction de la ligne 5 de la fonction Cohérence doit être substituée par une instruction de la forme : *Sélectionne* $(v_i, v_j) \in$ Eligibles, avec Eligibles la structure de données contenant les couples de variables correspondant aux contraintes éligibles. La prise en compte de la notion d'éligibilité nécessite d'autres modifications de l'algorithme de recherche et de la méthode de fermeture par composition que nous allons décrire. Tout d'abord, avant tout traitement, l'ensemble Eligibles doit être initialisé par les couples de variables dont les contraintes sont définies par une relation de la classe $\mathcal{C}$: Eligibles $\leftarrow \{(v_i, v_j) : i < j$ et $C_{ij} \notin \mathcal{C}\}$. Examinons, maintenant, les deux cas où une contrainte doit être retirée de l'ensemble des contraintes éligibles :

- à chaque étape de la recherche, après le découpage de la contrainte C_{ij} sélectionnée en sous-relations de l'ensemble (ligne 9 de Cohérence), la contrainte C_{ij} sera successivement réduite à une relation de $\mathcal{C}$. Cette contrainte doit donc devenir non éligible et être retirée de l'ensemble des contraintes éligibles.
- Lors du calcul de la fermeture par faible composition du RCQ $\mathcal{N}$, une opération de triangulation peut mener à ce qu'une contrainte éligible entre deux variables v_i et v_j soit définie par une relation de $\mathcal{C}$. Dans ce cas, la contrainte C_{ij} doit également être retirée de l'ensemble des contraintes éligibles.

Dans le cadre de travaux plus récents [Con11, CC11], nous avons défini des algorithmes de résolution de RCQ utilisant une notion d'éligibilité plus forte et plus facile à mettre en œuvre. Notons par $\mathcal{N}_{\mathsf{Init}} = (V, C^{\mathsf{Init}})$ le RCQ initial sur lequel doit se réaliser la résolution et notons par $R^{\mathcal{C}}$ la fermeture d'une relation $R \in 2^{\mathsf{B}}$ pour la classe $\mathcal{C}$ (la plus petite relation de $\mathcal{C}$ contenant R). Nous avons montré que la sélection des contraintes à traiter peut se faire uniquement parmi les contraintes correspondant à l'ensemble des couples de variables $\{(v_i, v_j) : (C_{ij})^{\mathcal{C}} \not\subseteq C_{ij}^{\mathsf{Init}}\}$. Nous pouvons ainsi définir une nouvelle notion d'éligibilité de la manière suivante. Une contrainte entre deux variables v_i et v_j est éligible si et seulement si $(C_{ij})^{\mathcal{C}} \not\subseteq C_{ij}^{\mathsf{Init}}$. La mise en œuvre de cette nouvelle notion d'éligibilité est plus simple puisque il suffit d'une part de sauvegarder le RCQ initial et d'autre part de sélectionner à chaque étape de la recherche une contrainte C_{ij} telle que $(C_{ij})^{\mathcal{C}} \not\subseteq C_{ij}^{\mathsf{Init}}$. Nous avons réalisé des travaux expérimentaux montrant l'intérêt pratique de la notion d'éligibilité. Pour clore sur la notion d'éligibilité, notons qu'elle peut être mise en œuvre dans le cadre d'un quelconque prétraitement (pourvu qu'il ne supprime que des relations de base incohérentes) et avec toute méthode de filtrage des contraintes permettant d'obtenir une cohérence locale plus forte que la $\diamond$-cohérence (nous décrirons de telles cohérences locales par la suite).

Nous avons également réalisé une étude expérimentale complète [Saa08] des différentes heuristiques concernant le choix de la prochaine contrainte à sélectionner (ligne 5 de la fonction Cohérence) et l'ordre utilisé lors du traitement des sous-relations issues du découpage de la contrainte sélectionnée (ligne 9 de la fonction Cohérence). Des heuristiques efficaces peuvent grandement réduire le parcours en profondeur et en largeur de l'arbre de recherche. De même, dans le cadre des implémentations de la méthode de fermeture par faible composition, les heuristiques sélectionnant les contraintes et les sous-relations les plus restrictives dans un premier temps sont les plus efficaces et permettent un élagage important de l'arbre de recherche. Parmi les heuristiques de choix de la prochaine contrainte à traiter, nous pouvons citer deux heuristiques performantes $AscDom$ et $DomTriangle$. L'heuristique $AscDom$ consiste à sélectionner une contrainte de plus petite cardinalité de manière similaire à l'heuristique Dom [HE80] utilisée dans le cadre des CSP discrets. L'heuristique $DomTriangle$ sélectionne une contrainte de plus petite cardinalité mais prend également en compte un deuxième critère correspondant à la minimisation du nombre de relations de base appartenant à l'ensemble des sous RCQ de trois variables contenant la contrainte candidate. Nous avons défini des heuristiques plus sophistiquées utilisant une pondération dynamique des contraintes à la manière de l'heuristique $WDeg$ utilisée dans le cadre général des CSP discrets [BHLS04].

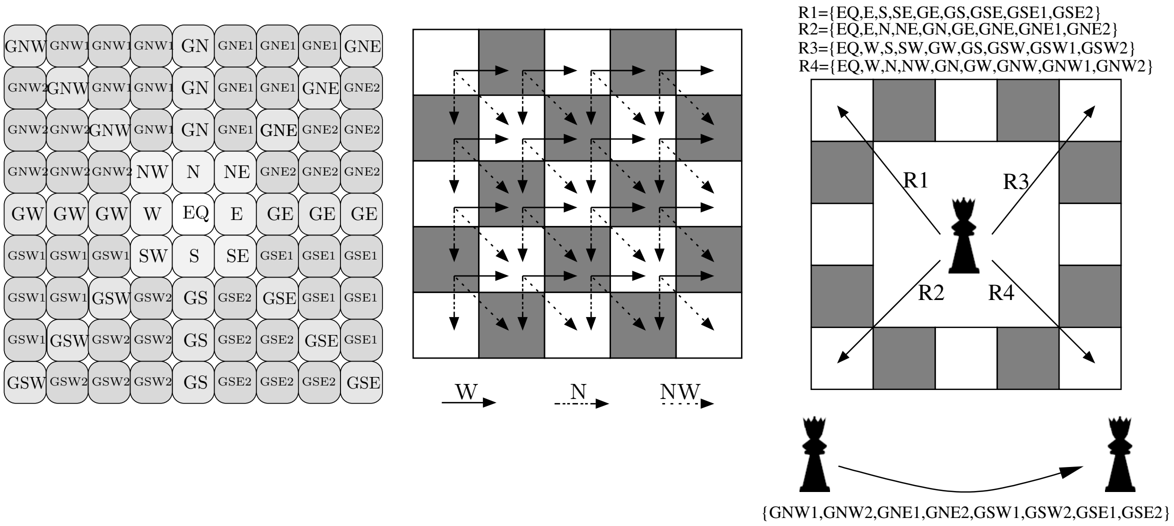
2.4 Les contraintes gelées

Lorsque nous modélisons un problème de positionnement spatial ou temporel à l'aide d'un RCQ pour certaines applications concrètes, nous pouvons parfois discerner deux types de contraintes. Le premier type correspond à des contraintes définies sur des entités du domaine afin de modéliser un objet temporel ou spatial complexe ne pouvant pas simplement être représenté par une seule entité du domaine. Le second type de contraintes correspond aux contraintes de positionnement proprement dites entre ces objets complexes. Dans [CLS07], nous appelons les contraintes du premier type, contraintes structurelles ou contraintes d'environnement et celles du second type, contraintes de position ou contraintes critiques. La distinction entre ces deux types de contraintes est très intuitive et se fait dans le cadre de la modélisation du problème concerné. *In fine*, ces deux types de contraintes sont définis par les contraintes d'un RCQ du formalisme qualitatif choisi. À des fins d'illustration, nous avons considéré dans [CLS07] le problème bien connu des n-reines. Ce problème consiste à placer n reines sur un échiquier de $n \times n$ sans qu'une reine puisse *prendre* une autre reine. Notre modélisation de ce problème utilise un formalisme qualitatif particulier appelé ADC25 (Algèbre des Directions Cardinales 25). Ce formalisme inspiré du calcul des directions cardinales de LIGOZAT est basé sur 25 relations de base dont le domaine est le plan des entiers. Chacune des 25 relations de base correspond à une direction cardinale et à une relation de distance (distance égale à 0, distance égale à 1, distance supérieure à 1) particulière entre deux points du plan (voir Figure 2.4(a)).

Le RCQ de ADC25 permettant de modéliser le problème des n-reines est défini sur un ensemble de $n \times n$ variables modélisant les cases de l'échiquier auquel sont ajoutées n variables modélisant les n reines. Ses contraintes sont définies par trois ensembles : un ensemble de contraintes entre les variables représentant les cases afin de modéliser l'échiquier (voir Figure 2.4(b)), un ensemble de contraintes forçant chaque point représentant un reine à être égal à un point représentant une case de l'échiquier et un ensemble de contraintes forçant deux points à ne pas avoir la même abscisse ou la même ordonnée. Ces deux derniers ensembles de contraintes sont illustrés par la figure 2.4(c). Les contraintes du premier ensemble sont utilisées pour spécifier l'objet complexe correspondant à l'échiquier et correspondent aux contraintes que nous avons nommées contraintes structurelles. Les deux derniers ensembles qui sont utilisées afin de positionner correctement les différentes reines correspondent aux contraintes critiques. Pour une autre illustration, le lecteur peut se reporter à [CLS09] où une modélisation du problème du *job shop scheduling* considéré par SADEH [Sad91] est proposée en utilisant un RCQ du calcul INDU et un RCQ du calcul des intervalles.

Pour la modélisation du problème des n-reines proposée et pour d'autres modélisations, les contraintes structurelles représentent intuitivement un ensemble de contraintes cohérentes auquel l'ajout des contraintes critiques peut introduire une incohérence. Afin de proposer une méthode de résolution efficace prenant en compte cette distinction entre les contraintes structurelles et les contraintes critiques, nous avons introduit la notion de contraintes gelées [CLS07]. Grossièrement, une contrainte gelée est simplement une contrainte qui ne pourra jamais être modifiée lors de la résolution du problème de la cohérence du RCQ. Les contraintes gelées du RCQ correspondront aux contraintes structurelles, les non gelées aux contraintes critiques. En considérant de nouveau la modélisation du problème des n-reines et en notant par G_R l'ensemble des couples de variables des contraintes gelées pour ce problème, G_R est défini par l'ensemble des couples (v_i, v_j) avec v_i et v_j deux variables représentant deux cases de l'échiquier.

Pour prendre en compte la notion de contraintes gelées lors de la résolution d'un RCQ, nous devons introduire deux modifications dans l'algorithme de recherche présenté dans la section 2.1. La première modification concerne la méthode de fermeture par faible composition utilisée pour le filtrage des contraintes, celle-ci ne devant pas modifier une contrainte gelée. Par conséquent, l'opération de triangulation $C_{ij} \leftarrow C_{ij} \cap (C_{ik} \diamond C_{kj})$ ne doit être réalisée que pour les couples de variables (v_i, v_j) n'appartenant pas à l'ensemble G, avec G l'ensemble des couples de variables correspondant aux contraintes gelées.



(a) (b) (c)

FIGURE 2.4 – Illustration des relations de base de ADC25 (a), contraintes entre les cases (b) et contraintes concernant les reines (c).

Le RCQ obtenu à travers cette nouvelle méthode n'est pas $\diamond$-cohérent de manière générale mais possède une cohérence locale plus faible que nous avons appelée $\langle\diamond, F\rangle$-cohérence. Cette cohérence locale est formellement définie de la manière suivante :

Définition 5 *Étant donnés un* RCQ $\mathcal{N} = (V, C)$ *et un ensemble* $G \subseteq V \times V$*, le* RCQ $\mathcal{N}$ *est* $\langle\diamond, G\rangle$*-cohérent si et seulement si pour tout triplet de variables* $v_i, v_j, v_k \in V$*, si* $C_{ij} \notin G$ *alors* $C_{ij} \subseteq C_{ik} \diamond C_{kj}$.

La seconde modification de l'algorithme de recherche concerne la sélection des contraintes à traiter à chaque étape de la recherche (ligne 5 de la fonction Cohérence). La sélection ne devra se faire que parmi les contraintes non gelées puisqu'une contrainte gelée ne doit pas être modifiée. La sélection de la prochaine contrainte à traiter se fera donc par l'instruction suivante :

Sélectionne $(v_i, v_j) \in V \times V$ avec $i < j$ tel que (v_i, v_j) non déjà sélectionné et $(v_i, v_j) \notin G$.

Une condition suffisante pour que l'algorithme CohérenceG résolve le problème de la cohérence est que tout RCQ $\mathcal{N}' = (V, C')$ satisfaisant les quatre propriétés suivantes soit cohérent :
(1) $\mathcal{N}' \subseteq \mathcal{N}$,
(2) pour tout $v_i, v_j \in V$, $C'_{ij} = C_{ij}$ si $(v_i, v_j) \in G$ sinon $C'_{ij} \in \mathcal{C}$,
(3) $\mathcal{N}$ et $\mathcal{N}'$ sont équivalents,
(4) $\mathcal{N}'$ est $\langle\diamond, G\rangle$-cohérent.

Notons que, dans le cas présent, l'ensemble $\mathcal{C}$ utilisé pour découper les contraintes lors de la recherche n'est pas forcément une classe mais un ensemble contenant les relations singletons. De plus, nous pouvons montrer que cette condition est satisfaite pour tout RCQ $\mathcal{N} = (V, C)$ issu de la modélisation des n-reines précédemment donnée en considérant l'ensemble des contraintes gelées G_R et comme ensemble de découpage l'ensemble $\mathcal{C}$ formé des 25 relations singletons du calcul ADC25. Ainsi, paramétrée de ces deux ensembles, la fonction CohérenceG peut être utilisée pour résoudre le problème des RCQ issus du problème des n-reines. En réalisant des expériences sur les RCQ issus du problème des n reines, nous avons pu constater que l'algorithme CohérenceG permet d'une part d'éviter un nombre conséquent d'opérations de triangulation et d'autre part de réduire l'arbre de recherche, comparativement à l'algorithme Cohérence. Il est tout de même clair que l'algorithme CohérenceG ne peut être utilisé que pour des problèmes structurés et sous des conditions particulières.

2.5 Un famille de cohérences locales : les $\overset{\diamond}{f}$-cohérences

Force est de constater que la méthode de la fermeture par faible composition est quasiment la seule méthode d'inférence utilisée comme prétraitement ou méthode de filtrage générique dans le cadre de la résolution d'un RCQ. D'une part, cette méthode offre un excellent équilibre entre le temps d'exécution et la force de filtrage des contraintes. D'autre part, de nombreuses classes traitables pour lesquelles la $\diamond$-cohérence est complète ont été caractérisées et utilisées efficacement pour la résolution de RCQ. Néanmoins, nous pouvons nous poser la question de savoir s'il n'existerait pas de méthodes basées sur une cohérence locale autre que la $\diamond$-cohérence qui pourrait être aussi utiles et efficaces. Afin d'apporter des éléments de réponses à cette question, nous avons proposé dans **[CL10]**[p203] une nouvelle famille de cohérences locales plus fortes que la $\diamond$-cohérence.

Avant de présenter cette famille de cohérences, au travers d'un exemple nous allons voir comment peut être détectée la non cohérence d'une relation de base à l'aide d'un découpage des contraintes et de la méthode de fermeture par faible composition alors que seule, *i.e.* sans découpage, cette méthode ne le peut pas.

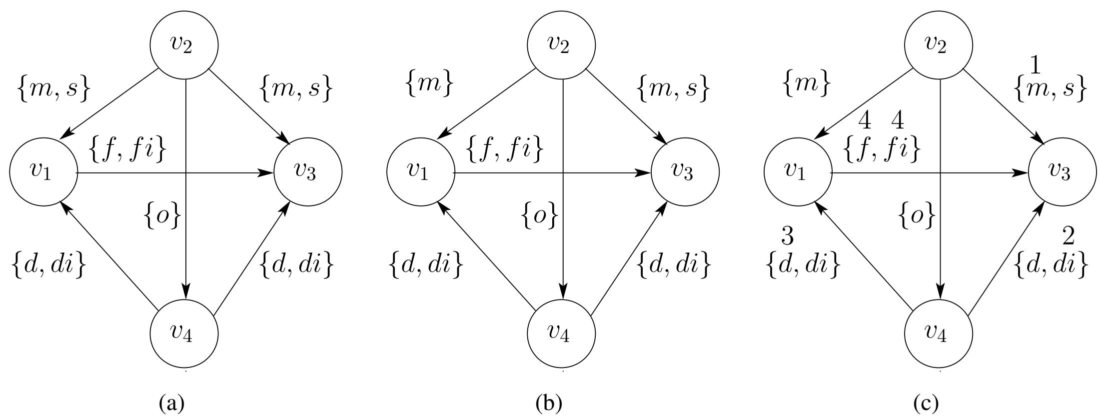


FIGURE 2.5 – Trois RCQ du calcul des intervalles.

Considérons le RCQ $= (V, C)$ du calcul des intervalles représenté par la figure 2.5(a). Ce RCQ correspond à un exemple de RCQ $\diamond$-cohérent et non cohérent donné par ALLEN dans [All83]. Découpons la contrainte C_{21} en deux sous-relations $\{m\}$ et $\{s\}$. Définissons maintenant C_{21} par la première de ces deux relations. Nous obtenons le RCQ décrit par la figure 2.5(b). Nous allons réaliser pas à pas une séquence d'opérations de triangulation sur ce nouveau RCQ. La figure 2.5(c) indique les relations après chacune de ces opérations de triangulation.

1. $C_{21} \diamond C_{13} = \{m\} \diamond \{f, fi\} = \{p, d, o, s\}$, ainsi la relation de base m peut être retirée de C_{23}.
2. $C_{42} \diamond C_{23} = \{oi\} \diamond \{s\} = \{d, oi, f\}$, la relation de base di peut donc être enlevée de C_{43}.
3. $C_{42} \diamond C_{21} = \{oi\} \diamond \{m\} = \{di, o, fi\}$, nous pouvons donc retirer d de C_{41}.
4. $C_{14} \diamond C_{43} = \{d\} \diamond \{d\} = \{d\}$, la contrainte C_{13} devient vide.

Après ces quatre opérations de triangulation, nous avons obtenu la contrainte vide. Nous pouvons donc retirer la relation de base m du RCQ initial puisque cette relation de base ne pourra jamais être satisfaite. En substituant la contrainte C_{21} par la deuxième sous-relation issue de son découpage, *i.e.* la relation

$\{s\}$, et en réalisant une séquence d'opérations de triangulation, nous pouvons également obtenir une contrainte vide. La relation de base s peut donc être supprimée de la contrainte C_{21}. La contrainte C_{21} devient vide et nous concluons que le RCQ initial est incohérent.

De manière générale, afin d'obtenir une méthode de filtrage plus forte que la méthode de fermeture par faible composition, nous pouvons imaginer une méthode qui pour chaque contrainte : (1) partage la contrainte en sous-relations, (2) substitue la contrainte par chacune des sous-relations issues du découpage et (3) supprime de la contrainte les relations de base détectées non cohérentes par le calcul de la fermeture par faible composition du RCQ obtenu après chaque substitution. Ce traitement doit être réalisé jusqu'à ce qu'un point fixe soit obtenu.

Afin d'étudier de telles méthodes de filtrage, nous avons introduit dans **[CL10]**[p203] une famille de cohérences locales où chacune des cohérences est définie à partir d'une application f associant à chaque relation $R \in 2^{\mathsf{B}}$ un ensemble de sous-relations de 2^{B} formant une couverture de R, *i.e.* $\bigcup f(R) = R$ pour tout $R \in 2^{\mathsf{B}}$. Dans la suite, $\mathcal{F}$ dénotera l'ensemble de toutes ces applications f. La cohérence locale associée à une application $f \in \mathcal{F}$ est appelée la $\overset{\diamond}{f}$-cohérence et correspond à la propriété suivante. Un RCQ est $\overset{\diamond}{f}$-cohérent si, et seulement si, pour chaque contrainte, en substituant la relation R définissant la contrainte par chacune des sous-relations de $f(R)$, aucune des relations de base de la sous-relation n'est supprimée par application de la fermeture par faible composition. Formellement, la $\overset{\diamond}{f}$-cohérence est définie de la manière suivante :

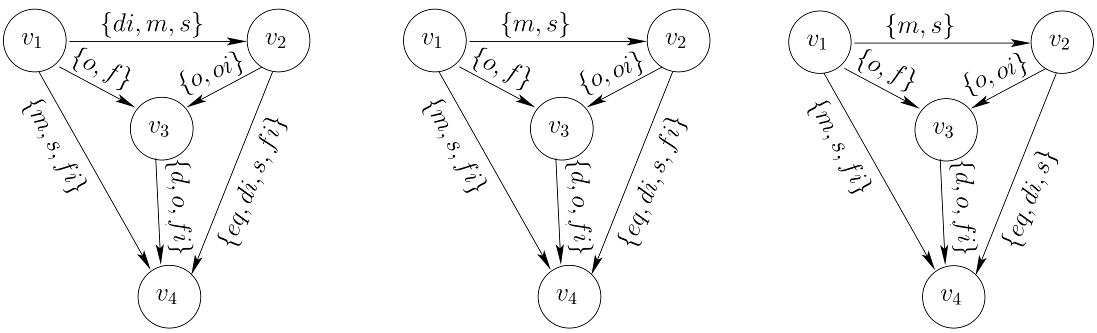
Définition 6 *Un* RCQ $\mathcal{N}$ *est* $\overset{\diamond}{f}$*-cohérent si, et seulement si, pour chaque couple* (v_i, v_j) *de variables de* $\mathcal{N}$ *et pour tout* $S \in f(\mathcal{N}[i,j])$, $\diamond(\mathcal{N}_{[i,j]/S})[i,j] = S$.

Dans **[CL10]**[p203], nous étudions les forces relatives des $\overset{\diamond}{f}$-cohérences correspondant aux applications $f \in \mathcal{F}$. Plus exactement, nous étudions la relation d'ordre $\trianglerighteq$ (est plus forte que) définie par : pour tout $f, f' \in \mathcal{F}$, $\overset{\diamond}{f}$-cohérence $\trianglerighteq$ $\overset{\diamond}{f'}$-cohérence si, et seulement si, tout RCQ $\overset{\diamond}{f}$-cohérent est également $\overset{\diamond}{f'}$-cohérent.

En considérant l'application $f_\diamond$ définie par $f_\diamond(R) = \{R\}$ pour tout $R \in 2^{\mathsf{B}}$, nous avons montré que la $f_\diamond$-cohérence correspond à la $\diamond$-cohérence et que de plus, la $f_\diamond$-cohérence est plus faible que toute autre $\overset{\diamond}{f}$-cohérence. Ainsi, tout RCQ $\overset{\diamond}{f}$-cohérent pour une application quelconque $f \in \mathcal{F}$ est forcément $f_\diamond$-cohérent et $\diamond$-cohérent. Nous avons également montré que la cohérence correspondant au cas où toutes les relations sont partagées de manière la plus fine possible, *i.e.* partagées par des relations singletons, est plus forte que tout autre cohérence. Formellement, cette cohérence correspond à l'application f_{B} définie par $f_{\mathsf{B}}(R) = \{\{b\} : b \in R\}$ et $f_{\mathsf{B}}(\emptyset) = \{\emptyset\}$. La f_{B}-cohérence peut être vue comme la cohérence SAC [DB97, BD08] (Singleton Arc Consistency) définie et étudiée dans le cadre des CSP discrets. Remarquons que le RCQ représenté par la figure 2.5(a) ne satisfait pas la $\overset{\diamond}{f_{\mathsf{B}}}$-cohérence. En effet, nous avons précédemment constaté qu'en définissant sa contrainte entre v_2 et v_1 par $\{m\}$ et en appliquant la fermeture par faible composition, la relation de base m est supprimée de cette contrainte.

Ainsi, chaque application f de $\mathcal{F}$ définit une cohérence plus forte que la $\overset{\diamond}{f}$-cohérence et plus faible que la $\overset{\diamond}{f_{\mathsf{B}}}$-cohérence. À titre d'illustration, nous pouvons également considérer l'application $f_{\neq}$ associant l'ensemble $f_{\neq}(R) = \{R \setminus \{b\} : b \in r\}$ pour tout R avec $|R| > 1$, $f_{\neq}(R) = \{R\}$ sinon. Par exemple, $f_{\neq}(\{di, m, s\})$ correspond à l'ensemble $\{\{m, s\}, \{di, s\}, \{di, m\}$. La figure 2.6 représente trois RCQ du calcul des intervalles $\mathcal{N}_1$, $\mathcal{N}_2$ et $\mathcal{N}_3$. Ces trois RCQ sont équivalents. $\mathcal{N}_1$ est $\diamond$-cohérent mais pas $\overset{\diamond}{f_{\neq}}$-cohérent. En effet, en remplaçant la relation définissant la contrainte entre v_1 et v_2 de $\mathcal{N}_1$ par $\{di, m\} \in f_{\neq}(\{di, m, s\})$ et en appliquant la fermeture par faible composition, la relation de base di est supprimée. Le RCQ $\mathcal{N}_2$ est $\overset{\diamond}{f_{\neq}}$-cohérent mais pas $\overset{\diamond}{f_{\mathsf{B}}}$-cohérent puisque $\diamond(\mathcal{N}_{2[1,3]/\{fi\}})[1,3] = \emptyset$. Le RCQ $\mathcal{N}_3$ est quant à lui $\overset{\diamond}{f_{\mathsf{B}}}$-cohérent.

Nous pouvons également définir des applications $\mathcal{F}$ permettant de partitionner chaque relation $R \in 2^{\mathsf{B}}$. Pour cela, nous pouvons par exemple utiliser une partition $P = \{R_1, \ldots, R_k\}$ de l'ensemble des



(a) (b) (c)

FIGURE 2.6 – Trois RCQ $\mathcal{N}_1$ (a), $\mathcal{N}_2$ (b) et $\mathcal{N}_3$ (c) tels que $\mathcal{N}_3 \subseteq \mathcal{N}_2 \subseteq \mathcal{N}_1$ et, $\mathcal{N}_1$ $\diamond$-cohérent, $\mathcal{N}_2$ $\overset{\diamond}{f_{\neq}}$-cohérent et $\mathcal{N}_3$ $\overset{\diamond}{f_{\mathsf{B}}}$-cohérent.

relations de base B et définir l'application f_P par $f_P(R) = \{R \cap R_i : i \in \{1, \dots, k\}\} \setminus \{\emptyset\}$ pour tout $R \neq \emptyset$ et $f_P(R) = \{\emptyset\}$ dans le cas où $R = \emptyset$. À titre d'illustration, considérons, dans le cadre du calcul des intervalles, les trois partitions suivantes de l'ensemble B :

- $P_1 = \{\{p, m, o, fi, s, d\}, \{pi, mi, oi, f, si, di, eq\}\}$,
- $P_2 = \{\{p, m, o\}, \{fi, s, d\}, \{pi, mi, oi\}, \{f, si, di, eq\}\}$,
- $P_3 = \{\{p\}, \{m, o\}, \{fi\}, \{s, d\}, \{pi\}, \{mi, oi\}, \{f, eq\}, \{si, di\}\}$.

En considérant par exemple la relation $\{m, o, pi, si, di\}$, son découpage sera différent en fonction de la partition considérée :

- $f_{\diamond}(\{m, o, pi, si, di, eq\}) = \{\{m, o, pi, si, di, eq\}\}$,
- $f_{P_1}(\{m, o, pi, si, di, eq\}) = \{\{m, o\}, \{pi, si, di\}\}$,
- $f_{P_2}(\{m, o, pi, si, di, eq\}) = \{\{m, o\}, \{pi\}, \{si, di, eq\}\}$,
- $f_{P_3}(\{m, o, pi, si, di, eq\}) = \{\{m, o\}, \{pi\}, \{eq\}, \{si, di\}\}$ et
- $f_{\mathsf{B}}(\{m, o, pi, si, di, eq\}) = \{\{m\}, \{o\}, \{pi\}, \{eq\}, \{si\}, \{di\}\}$.

Nous pouvons constater que les applications $f_{\diamond}$, f_{P_1}, f_{P_2}, f_{P_3} et f_{B} définissent des découpages de plus en plus fins. En effet, en considérant par exemple f_{P_1} et f_{P_2}, pour tout $R \in 2^{\mathsf{B}}$, il existe pour tout $R' \in f_{P_1}(R)$ un sous-ensemble $S \subseteq f_{P_2}(R)$ tel que $R' = \bigcup S$. À partir de ce constat et d'une propriété établie dans **[CL10]**[p203], nous pouvons affirmer que $\overset{\diamond}{f_{\mathsf{B}}}$-cohérence $\trianglerighteq$ $\overset{\diamond}{f_{P_3}}$-cohérence $\trianglerighteq$ $\overset{\diamond}{f_{P_2}}$-cohérence $\trianglerighteq$ $\overset{\diamond}{f_{P_1}}$-cohérence $\trianglerighteq$ $\overset{\diamond}{f_{\diamond}}$-cohérence. Remarquons que la définition d'une $\overset{\diamond}{f}$-cohérence partitionnant chacune des relations $R \in 2^{\mathsf{B}}$ peut être reliée à l'approche de BENNACEUR et AFFANE [BA01] qui proposent dans le cadre des CSP discrets une méthode de filtrage basée sur le partitionnement des domaines et sur la cohérence locale AC.

La relation d'ordre $\trianglerighteq$ (est plus forte que) est un ordre partiel ; ainsi la $\overset{\diamond}{f}$-cohérence et la $\overset{\diamond}{f'}$-cohérence, avec $f, f' \in \mathcal{F}$ ne sont pas forcément comparables. Par exemple, nous avons la $\overset{\diamond}{f_{P_2}}$-cohérence et la $f_{\neq}$-cohérence qui ne sont pas comparables, aucune des deux n'est plus forte que l'autre. Nous avons néanmoins prouvé que pour tout $f, f' \in \mathcal{F}$, il existe une application $f^{sup} \in \mathcal{F}$ (resp. $f^{inf} \in \mathcal{F}$) telle que la $\overset{\diamond}{f^{sup}}$-cohérence resp. la $\overset{\diamond}{f^{inf}}$-cohérence) est la plus faible (resp. la plus forte) des cohérences plus fortes (resp. plus faibles) que la $\overset{\diamond}{f}$-cohérence et la $\overset{\diamond}{f'}$-cohérence.

Par ailleurs, nous avons montré que pour certaines $\overset{\diamond}{f}$-cohérences, il n'existe pas forcément une fermeture pour tout RCQ $\mathcal{N}$, *i.e.* il n'existe pas forcément un unique plus grand (pour $\subseteq$) sous-RCQ $\overset{\diamond}{f}$-cohérent de $\mathcal{N}$. Afin d'obtenir des cohérences locales ayant un bon comportement, nous avons caractérisé un important sous ensemble de $\mathcal{F}$, dénoté par $\mathcal{F}^*$, pour lequel pour toute application f et pour tout RCQ $\mathcal{N}$, il existe un unique plus grand sous-RCQ de $\mathcal{N}$ $\overset{\diamond}{f}$-cohérent. L'ensemble $\mathcal{F}^*$ contient notamment les applications $f_\diamond$, $f_{\neq}$, f_{B} et f_P (avec P une partition de B) données comme exemples précédemment. Nous avons également proposé un algorithme générique permettant de calculer la fermeture de tout RCQ selon la $\overset{\diamond}{f}$-cohérence pour une application $f \in \mathcal{F}^*$. Intuitivement, cet algorithme examine dans une boucle principale chacune des contraintes du RCQ afin de supprimer les relations de base non possibles du fait de la $\overset{\diamond}{f}$-cohérence. Chaque contrainte C_{ij} est traitée en la définissant itérativement par chacune des relations $R \in f(C_{ij})$ et par application de la fermeture par faible composition afin de détecter parmi ces relations de base, des relations de base non cohérentes. Lorsqu'au moins une contrainte est modifiée, une nouvelle boucle principale est réalisée. Cet algorithme a une complexité en temps en $O(n^7)$ (avec n le nombre de variables du RCQ). Malgré cette complexité théorique élevée, l'algorithme n'exécute en pratique que quelques boucles principales. Nous avons réalisé des études expérimentales sur des RCQ du calcul des intervalles générés aléatoirement et démontré son intérêt. En effet, nous avons constaté que d'une part pour certaines $\overset{\diamond}{f}$-cohérences, la détection de RCQ incohérents par calcul de la fermeture selon la $\overset{\diamond}{f}$-cohérence subsume largement celle par le calcul de la fermeture par faible composition. Pour certaines, nous sommes très proches de la détection de l'ensemble des RCQ incohérents. Sans surprise, nous avons également remarqué que plus une $\overset{\diamond}{f}$-cohérence était forte, plus le coût du calcul de la fermeture selon cette $\overset{\diamond}{f}$-cohérence est élevé. D'autre part, nous avons constaté que l'incohérence de certains RCQ pouvait être détectée par calcul de la fermeture par $\overset{\diamond}{f}$-cohérence dans un temps largement inférieur que celui mis par le solveur GQR [GWW08] (qui est actuellement un des solveurs de contraintes qualitatives les plus rapides). Le calcul de la fermeture par $\overset{\diamond}{f}$-cohérence a donc clairement tout son intérêt dans le cadre d'une phase de prétraitement des RCQ à résoudre. Son utilisation dans le cadre d'un filtrage des contraintes lors d'une recherche n'est pas forcément viable du fait de son coût en temps trop élevé pour une application $f \in \mathcal{F}^*$ quelconque. Néanmoins, pour certaines applications particulières $\mathcal{F}$, il est envisageable de diminuer ce coût en particularisant l'algorithme générique de calcul de fermeture. Pour clore cette section, notons qu'avec une approche similaire, nous avons caractérisé dans **[CL11]**[p215] différentes familles de cohérences locales dans le cadre des CSP discrets.

2.6 Résolution du problème de la cohérence des RCQ par résolution de CSP discrets

Les RCQ sont des CSP binaires [Lec09] particuliers où les variables prennent leurs valeurs dans le domaine D qui est généralement infini et où chaque contrainte est définie de manière intensionnelle par l'ensemble des relations de base possibles entre chaque couple de variables. Une manière générique d'obtenir un CSP discret représentant un RCQ est de considérer sa représentation duale en prenant pour variables les contraintes du RCQ et d'associer comme domaine à chaque variable les symboles représentant les relations de base de la contrainte dont elle est issue. Les contraintes du CSP discret sont définies de manière extensionnelle en explicitant les triplets de relations de base possibles du fait de la table de composition. Ces contraintes sont donc ternaires dans le cas où les relations de base sont binaires. De manière formelle, nous pouvons définir ce CSP discret de la manière suivante [DCLS06] :

Définition 7 *Soit un* RCQ $\mathcal{N} = (V, C)$. CSP($\mathcal{N}$) *dénotera le CSP discret* $\mathcal{P} = (\mathcal{X}, \mathcal{D}, \mathcal{C})$ *défini de la manière suivante :*

- *pour chaque paire de variables* $v_i, v_j \in V$ *telles que* $i \leq j$ *est introduite une variable* x_{ij} *dans l'ensemble* $\mathcal{X}$ *;*

- *le domaine de chaque variable* $x_{ij} \in \mathcal{X}$ *est défini par* $\mathcal{D}(x_{ij}) = C_{ij}$ *;*
- *pour chaque triplet de variables* $v_i, v_j, v_k \in V$ *telles que* $i < k < j$*, la contrainte* c_{ijk} *portant sur les variables* $x_{ij}, x_{ik}, x_{kj} \in \mathcal{X}$ *et associée à l'ensemble de triplets* $\{(r_1, r_2, r_3) : r_1, r_2, r_3 \in$ B *et* $r_1 \in r_2 \diamond r_3\}$ *est ajoutée à l'ensemble* $\mathcal{C}$*.*

Remarquons que la taille d'une relation définissant une contrainte ternaire c_{ijk} peut être réduite en la définissant par l'ensemble de triplets $\{(r_1, r_2, r_3) : r_1 \in C_{ij}, r_2 \in C_{ik}, r_3 \in C_{kj} \in$ B et $r_1 \in r_2 \diamond r_3\}$ ou bien encore par $\{(r_1, r_2, r_3) : r_1 \in C_{ij} \cap (C_{ik} \diamond C_{kj}), r_2 \in C_{ik}, r_3 \in C_{kj} \in$ B et $r_1 \in r_2 \diamond r_3\}$.

Dans [DCLS06], nous avons étudié les liens entre un RCQ $\mathcal{N}$ et sa traduction en CSP discret CSP($\mathcal{N}$). Les principales propriétés sont les suivantes :

- Si le RCQ $\mathcal{N}$ est cohérent alors le CSP discret CSP($\mathcal{N}$) est cohérent.
- Le RCQ $\mathcal{N}$ est $\diamond$-cohérent si et seulement si le CSP discret CSP($\mathcal{N}$) est GAC (Generalized Arc Consistent).
- Si RCQ $\mathcal{N}$ est $\diamond$-cohérent alors $\mathcal{N}$ est (0,3)-cohérent et le CSP discret CSP($\mathcal{N}$) est fortement 3-cohérent.
- Si le CSP discret CSP($\mathcal{N}$) est GAC alors le RCQ $\mathcal{N}$ admet un scénario $\diamond$-cohérent.
- Si le RCQ $\mathcal{N}$ est défini dans un formalisme qualitatif pour lequel tout scénario $\diamond$-cohérent est cohérent et si le CSP discret CSP($\mathcal{N}$) est GAC alors $\mathcal{N}$ est cohérent.

Dans le cadre des formalismes qualitatifs pour lesquels tout scénario $\diamond$-cohérent est cohérent, le problème de la cohérence d'un RCQ peut être décidé par la résolution de sa traduction en CSP discret. Un inconvénient majeur de la traduction en CSP discret d'un RCQ est la taille de l'instance obtenue. En effet, un CSP discret obtenu par traduction d'un RCQ ayant n variables est défini sur n^2 variables avec des domaines d'au plus $|B|$ valeurs et $n.(n-1).(n-2)/6$ contraintes d'arité 3 définies par des relations contenant au plus $|B|^3$ triplets.

Pour palier à la taille importante des instances obtenues, nous avons étudié dans [DCLS07] une approche utilisant des relaxations du CSP discret obtenu par traduction du RCQ. Cette approche consiste simplement à ne pas traduire toutes les contraintes du RCQ à partir d'un ensemble donné de relations T. Pour un RCQ $\mathcal{N} = (V, C)$, une contrainte entre deux variables v_i et v_j est traduite si et seulement si $C_{ij} \notin T$. Le CSP discret $\mathcal{P}$ obtenu est une relaxation de CSP($\mathcal{N}$). Ainsi, la non cohérence de $\mathcal{P}$ permet d'affirmer la non cohérence du RCQ $\mathcal{N}$. Par contre, dans le cas général, caractériser une solution de $\mathcal{P}$ ne permet pas d'affirmer que $\mathcal{N}$ est cohérent. Pour rendre l'approche complète, de manière itérative, chaque solution s de $\mathcal{P}$ est traduite en sous-RCQ $\mathcal{N}'$ de $\mathcal{N}$ (en utilisant la traduction inverse de CSP). Puis un test de cohérence est réalisé sur $\mathcal{N}'$. Dans le cas où la cohérence de $\mathcal{N}'$ est détectée, $\mathcal{N}$ est cohérent. Dans le cas contraire, nous recherchons et examinons une nouvelle solution de $\mathcal{P}$. Remarquons que le test de cohérence de $\mathcal{N}'$ peut être réalisé par calcul de sa fermeture par faible composition dans le cas où T est un sous-ensemble d'une classe traitable contenant les relations singletons et pour laquelle la $\diamond$-cohérence est complète. Nous avons réalisé des tests expérimentaux sur des RCQ du calcul des intervalles générés aléatoirement en définissant notamment l'ensemble T par $\{\Psi\}$. Pour certains RCQ générés aléatoirement, nous avons constaté l'efficacité de cette approche.

2.7 Résolution du problème de la cohérence des RCQ par résolution de problèmes SAT

Pour de nombreux formalismes qualitatifs, l'étude du problème de la cohérence des RCQ a conduit à la définition de passerelles entre ce problème et le problème SAT [Saï08]. Des réductions polynomiales du problème SAT vers le problème de la cohérence des RCQ ont été définies afin de caractériser sa NP-complétude pour un fragment ou pour l'ensemble des relations du formalisme considéré. Inversement, une manière de caractériser une classe traitable d'un formalisme est de définir une réduction polynomiale du problème de la cohérence des RCQ de cette classe vers un fragment polynomial du problème SAT tel que le problème HORNSAT [DG84]. Nous pouvons, par exemple, citer les travaux de NEBEL et BÜRCKERT [NB94, NB95] dans lesquels est proposée une traduction polynomiale du problème de la cohérence des RCQ définis par des relations ORD-Horn du calcul des intervalles vers un problème de satisfiabilité de clauses de Horn propositionnelles (voir Section 2.2).

Plus récemment, des traductions du problème de la cohérence des RCQ en problème SAT ont été proposées [PTS06, PTS08, WW09] afin de profiter de l'efficacité des prouveurs SAT actuels. Une traduction générique en problème SAT d'un RCQ $\mathcal{N} = (V, C)$, appelée traduction *support*, consiste à définir pour chacune des relations de base r de chacune des contraintes C_{ij} (avec $i < j$) une variable propositionnelle r_{ij} et de définir l'ensemble des clauses suivantes :

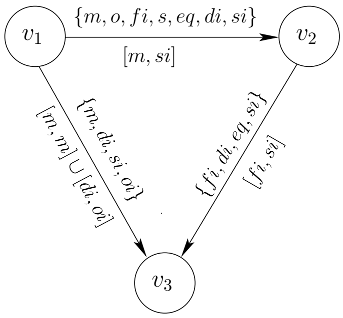
- clauses **AtLeastOne (ALO)** : la clause $\bigvee_{r \in C_{ij}} r_{ij}$ est introduite pour chaque contrainte C_{ij} (avec $i < j$) afin de stipuler qu'au moins une des relations de base de chaque contrainte doit être satisfaite ;
- clauses **AtMostOne (AMO)** : la clause $\neg r_{ij} \vee \neg r'_{ij}$ est ajoutée pour chaque paire distincte de relations de base r et r' de chaque contrainte C_{ij} (avec $i < j$) afin de contraindre qu'au plus une relation de base par contrainte doit être satisfaite ;
- clauses **Support** : la clause $\neg r_{ik} \vee \neg r'_{kj} \vee \bigvee_{r \in (r_{ik} \diamond r_{kj}) \cap C_{ij}} r$ est introduite pour chaque relation de base $r_{ik} \in C_{ik}$ et chaque relation de base $r_{kj} \in C_{kj}$ (avec $i < k < j$) pour indiquer les relations de base r pouvant éventuellement être satisfaites pour la contrainte C_{ij} compte tenu de la table de faible composition et des contraintes C_{ik} et C_{kj}.

La figure 2.7 illustre cette traduction à partir d'un RCQ du calcul des intervalles. L'ensemble de clauses SAT obtenues par la traduction *support* est satisfiable si, et seulement si, il existe un scénario $\diamond$-cohérent du RCQ $\mathcal{N}$. Pour tout formalisme qualitatif tel que le calcul des intervalles pour lequel tout scénario $\diamond$-cohérent est cohérent, la résolution de cet ensemble de clauses permet de déterminer la cohérence ou la non cohérence du RCQ $\mathcal{N}$. L'inconvénient majeur d'une telle traduction est le nombre très important de clauses obtenues (près d'un million de clauses et plus de dix mille variables pour un RCQ du calcul des intervalles avec 50 variables en utilisant la traduction *support* [PTS08]). Malgré tout, certaines expérimentations montrent que des solveurs SAT efficaces permettent de résoudre plus rapidement certains RCQ à travers certaines traductions [PTS08, LHR09]. Notons que les expériences concernant ces traductions ont été réalisées uniquement sur des RCQ du calcul des intervalles qui contient 13 relations de base. Il serait intéressant de réaliser des travaux expérimentaux sur des RCQ d'autres formalismes qualitatifs, en particulier des formalismes basés sur un nombre plus important de relations de base.

Dans [CD07, CD08], nous avons proposé une traduction des RCQ en SAT ayant la particularité de considérer la classe traitable correspondant à l'ensemble des relations convexes. Pour le calcul des intervalles et d'autres formalismes, les relations convexes peuvent être définies par les intervalles d'un treillis $(\mathsf{B}, \leq)$ (voir Section 2.2). Comme nous le verrons, notre traduction, que nous appellerons traduction *treillis*, utilise cette structure de treillis pour lequel nous supposons vérifiées des propriétés concernant l'opération de faible composition et l'opération d'inverse :

- pour tout $r_1, r_2 \in \mathsf{B}$, si $r_1 \leq r_2$ alors $r_2^{\smile} \leq r_1^{\smile}$;
- pour tout $r_1, r_2, r_3, r_4 \in \mathsf{B}$ $[r_1 \diamond r_2] \diamond [r_3, r_4] = [\mathsf{Inf}(r_1 \diamond r_3), \mathsf{Sup}(r_2 \diamond r_3)]$.

Pour de nombreux formalismes qualitatifs, un treillis possédant ces propriétés existe, en particulier pour le calcul des intervalles. Étant donné un RCQ $\mathcal{N} = (V, C)$, notre traduction en SAT de ce RCQ comprendra cinq ensembles de clauses. Dans un premier temps, nous supposons que $\mathcal{N}$ est un RCQ défini



Traduction *support* :
ALO : $m_{12} \vee o_{12} \vee fi_{12} \vee s_{12} \vee eq_{12} \vee di_{12} \vee si_{12}$, $m_{13} \vee di_{13} \vee si_{13} \vee oi_{13}$, $fi_{23} \vee di_{23} \vee eq_{23} \vee si_{23}$.
AMO : $\neg m_{12} \vee o_{12}$, $m_{12} \vee \neg o_{12}$, $\neg m_{12} \vee fi_{12}, m_{12} \vee \neg fi_{12}, \neg m_{12} \vee s_{12}$, $m_{12} \vee \neg s_{12}, \neg m_{12} \vee eq_{12}, m_{12} \vee \neg eq_{12}$, $\neg m_{12} \vee di_{12}, m_{12} \vee \neg di_{12}, \neg m_{12} \vee si_{12}$, $m_{12} \vee \neg si_{12}$,$\neg o_{12} \vee fi_{12}, \ldots$
Support :
$\neg m_{12} \vee \neg eq_{23} \vee m_{13}$, $\neg m_{12} \vee \neg si_{23} \vee m_{13}$, $\neg o_{12} \vee \neg fi_{23} \vee m_{13}$, $\neg o_{12} \vee \neg di_{23} \vee di_{13}$, $\neg o_{12} \vee \neg si_{23} \vee di_{13} \vee oi_{13}$, $\neg fi_{12} \vee \neg di_{23} \vee di_{13}$, $\neg fi_{12} \vee \neg si_{23} \vee di_{13}$, $\neg s_{12} \vee \neg fi_{23} \vee m_{13}$, $\neg s_{12} \vee \neg di_{23} \vee di_{13} \vee m_{13}, \ldots$

Traduction *treillis* :
I : $m \leq C_{12}$, $C_{12} \leq si$, $m \leq C_{13} \vee di \leq C_{13}$, $m \leq C_{13} \vee C_{13} \leq oi$, $C_{13} \leq m \vee di \leq C_{13}$, $C_{13} \leq m \vee C_{13} \leq oi$, $fi \leq C_{23}$, $C_{23} \leq si.$
II : $\neg(fi \leq C_{12}) \vee \neg(s \leq C_{12}) \vee eq \leq C_{12}$,$\neg(C_{12} \leq fi) \vee \neg(C_{12} \leq s) \vee C_{12} \leq p$, $\neg(di \leq C_{12}) \vee \neg(eq \leq C_{12}) \vee si \leq C_{12} \ldots$
III : $\neg(fi \leq C_{12}) \vee \neg(C_{12} \leq d)$, $\neg(d \leq C_{12}) \vee \neg(C_{12} \leq fi), \ldots$
IV : $\neg(p \leq C_{12}) \vee C_{12} \leq pi$, $\neg(C_{12} \leq p) \vee pi \leq C_{12}, \ldots$
V : $\neg(o \leq C_{12}) \vee \neg(di \leq C_{23}) \vee p \leq C_{ij}$, $\neg(C_{ik} \leq o) \vee \neg(C_{kj} \leq di) \vee C_{ij} \leq di, \ldots$

FIGURE 2.7 – Un RCQ $\mathcal{N}$ du calcul des intervalles et quelques clauses correspondant à sa traduction SAT par la traduction *support* et la traduction treillis.

par des relations convexes et que donc chaque contrainte C_{ij} est définie par un intervalle $[a_{ij}, sb_{ij}]$ du treillis $(\mathsf{B}, \leq)$. Les ensembles de clauses sont définis sur les variables propositionnelles correspondant à l'ensemble $\{a \leq C_{ij}, C_{ij} \leq a : \text{ pour tout } a \in \mathsf{B} \text{ et } v_i, v_j \in V\}$:

- pour chaque contrainte $C_{ij} = [a_{ij}, b_{ij}]$ avec $i, j \in \{0, \ldots, n-1\}$, deux clauses unitaires bornant les relations de base possibles sont introduites :

$$a_{ij} \leq C_{ij} \text{ et } C_{ij} \leq b_{ij} \quad \text{(I)}$$

- Des clauses modélisant une propriété concernant les infimums et les supremums du treillis sont introduites pour tout $a, b \in \mathsf{B}$:

$$\neg(a \leq C_{ij}) \vee \neg(b \leq C_{ij}) \vee \mathsf{Sup}\{a, b\} \leq C_{ij} \text{ et } \neg(C_{ij} \leq a) \vee \neg(C_{ij} \leq b) \vee C_{ij} \leq \mathsf{Inf}\{a, b\} \quad \text{(II)}$$

- Des clauses correspondant à la propriété de transitivité de l'ordre partiel $\leq$ sont introduites pour tout $a, b \in \mathsf{B}$ tels que $a \not\leq b$:

$$\neg(a \leq C_{ij}) \vee \neg(C_{ij} \leq b) \quad \text{(III)}$$

- Deux clauses concernant l'opération inverse et le treillis sont introduites pour tout $a \in \mathsf{B}$:

$$\neg(a \leq C_{ij}) \vee C_{ji} \leq a^{\smile} \text{ et } \neg(C_{ij} \leq a) \vee a^{\smile} \leq C_{ji} \quad \text{(IV)}$$

- Pour chaque triplet de contraintes C_{ik}, C_{kj}, C_{ij}, avec $i < j$, deux clauses concernant l'opération de faible composition et le treillis sont introduites pour tout $a, b \in \mathsf{B}$:

$$\neg(a \leq C_{ik}) \vee \neg(b \leq C_{kj}) \vee \mathsf{Inf}(a \circ b) \leq C_{ij} \text{ et } \neg(C_{ik} \leq a) \vee \neg(C_{kj} \leq b) \vee C_{ij} \leq \mathsf{Sup}(a \circ b) \quad \text{(V)}$$

Nous avons supposé que le RCQ $\mathcal{N}$ est défini par des relations convexes, *i.e.* par des relations correspondant aux intervalles du treillis $(\mathsf{B}, \leq)$. Dans le cas où $\mathcal{N}$ n'est pas convexe, lors de la définiton des clauses (I), chaque contrainte C_{ij} doit être découpée en sous-relations convexes $R_1^{ij}, \ldots, R_k^{ij}$ et nous devons introduire les clauses correspondant à la disjonction $\bigvee_{r=[a,b]\in\{R_1^{ij},\ldots,R_k^{ij}\}}(a \leq C_{ij} \wedge C_{ij} \leq b)$. L'intuition derrière la traduction *support* est de rechercher à travers SAT un scénario $\diamond$-cohérent à partir des combinaisons de relations possibles pour chaque triplet de variables. L'approche concernant la traduction *treillis* est différente. En effet, cette traduction permet de modéliser les sous-RCQ de trois variables convexes et cohérents à l'aide du treillis et de l'opération de faible composition. Ainsi, nous avons établi la propriété suivante : pour tout RCQ $\mathcal{N}$, l'ensemble des clauses issues de la traduction *treillis* est satisfiable si, et seulement si, $\mathcal{N}$ admet un sous-RCQ convexe $\diamond$-cohérent et ne contenant pas la relation vide. Dans le cas où la $\diamond$-cohérence est complète pour la classe des relations convexes, nous pouvons en conclure que l'ensemble des clauses issues de cette traduction est satisfiable si, et seulement si, $\mathcal{N}$ est cohérent. En pratique, sur des RCQ du calcul des intervalles, la traduction *treillis* est moins efficace que certaines autres traductions SAT actuelles mais sa définition a l'originalité d'utiliser une classe traitable. Récemment, une étude a été réalisée dans le cadre du stage Master Recherche *Systèmes Intelligents et Applications* d' ABDERRAHIM AITWAKRIME (co-encadré par DANIEL LE BERRE et moi même) afin de rendre plus efficace cette traduction. Une des améliorations attendues a consisté en la suppression de clauses inutiles en considérant certaines *symétries* et *dualités* concernant les opérations de composition et d'inverse et les supremums et infimums du treillis. Cette amélioration nous a permis d'économiser plus de la moitié des clauses de la traduction originale.

2.8 Décompositions arborescentes et triangulations appliquées aux RCQ

Un RCQ est un réseau de contraintes complet dans le sens où pour chaque couple de variables est définie une contrainte par une relation de 2^{B}. La définition d'une contrainte à l'aide de la relation Ψ (la relation composée de tous les éléments de B) permet de spécifier que localement, il n'y a pas de contrainte concernant la position relative des deux entités représentées par les deux variables concernées. Ainsi, de manière naturelle, nous pouvons définir le graphe de contraintes d'un RCQ $\mathcal{N} = (V, C)$, par le graphe non orienté $\mathsf{G}(\mathcal{N}) = (V, E)$ avec $(v_i, v_j) \in E$ si, et seulement si, $C_{ij} \neq \Psi$ et $v_i \neq v_j$. Nous pouvons constater que, contrairement aux études menées dans le cadre général des CSP, très peu d'études concernant la résolution de RCQ prennent en compte la structure d'un RCQ au travers de son graphe de contraintes. À notre connaissance, les seuls travaux de ce genre sont ceux de LI *et al.* [LHR09] qui ont récemment proposé une méthode permettant d'éviter la traduction de toutes les contraintes définies par la relation Ψ dans le cadre d'une traduction en problème SAT des RCQ du calcul des intervalles. À chaque étape, cette méthode récursive partage l'ensemble des variables V du RCQ $\mathcal{N}$ considéré en deux ensembles de variables V_1 et V_2 de telle manière que les contraintes sur V_1 et les contraintes sur V_2 soient équivalentes aux contraintes sur $V = V_1 \cup V_2$. Le processus est répété sur les RCQ $\mathcal{N}^1$ et $\mathcal{N}^2$ correspondant aux projections du RCQ $\mathcal{N}$ sur respectivement l'ensemble de variables V_1 et l'ensemble de variables V_2. Les contraintes définies par la relation Ψ dans le RCQ $\mathcal{N}$ non présentes dans $\mathcal{N}^1$ et $\mathcal{N}^2$ sont caractérisées comme non nécessaires à la recherche d'un scénario cohérent du RCQ global initial et donc non traduites en clauses SAT. L'avantage d'une telle traduction est que l'instance SAT obtenue est de plus petite taille qu'une instance issue d'une traduction complète de l'ensemble des contraintes.

Dans [Con11]**[CD11]**[p233], nous avons montré que la méthode utilisée par LI *et al.* revient à considérer des regroupements de variables issus d'une décomposition arborescente particulière du RCQ considéré. Comme dans le cadre des CSP discrets, une décomposition arborescente d'un RCQ peut se définir formellement de la manière suivante :

Définition 8 *Soit $\mathcal{N} = (V, C)$ un* RCQ *et* $\mathsf{G}(\mathcal{N}) = (V, E)$ *son graphe de contraintes. Une décomposition arborescente de $\mathcal{N}$ est un arbre $T = (X = \{X_0, \ldots, X_n\}, F)$ avec n un entier positif, où X est une famille de sous-ensembles de variables de V $(X_i \subseteq V)$, telle que :*

(1) $\bigcup\{X_i \in X\} = V$,

(2) $\forall (v, v') \in E$*, il existe* $X_i \in X$ *tel que* $v, v' \in X_i$*,*
(3) pour tout $X_i, X_j, X_k \in X$*, si* X_j *est sur l'unique chemin entre* X_i *et* X_k *alors* $X_i \cap X_k \subseteq X_j$*.*

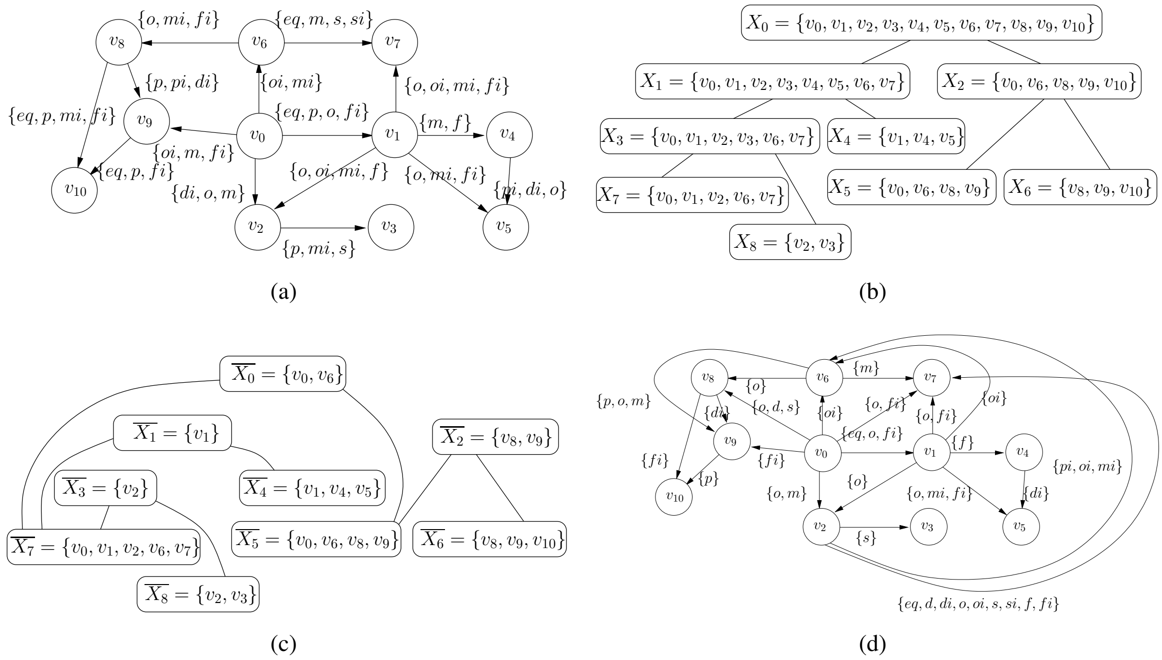


FIGURE 2.8 – (a) un RCQ $\mathcal{N}$ du calcul des intervalles, (b) une décomposition récursive de $\mathcal{N}$, (c) une décomposition arborescente de $\mathcal{N}$ et (d) un sous-RCQ de $\mathcal{N}$.

À titre d'illustration, la figure 2.8(b) représente une décomposition de l'ensemble des variables correspondant à la méthode de LI *et al.* appliquée au RCQ $\mathcal{N}$ de la figure 2.8(a). Seules les contraintes correspondant aux feuilles de cet arbre seront considérées lors d'une traduction en problème SAT. Dans la figure 2.8(c) est représentée une décomposition arborescente de $\mathcal{N}$ considérant les mêmes regroupements de paires ou triplets de variables que les nœuds terminaux de la décomposition arborescente.

Nous avons généralisé l'approche suivie par LI *et al.* en montrant que toute décomposition arborescente peut être utilisée afin de minimiser l'ensemble des contraintes à considérer lors d'une traduction en clauses SAT des RCQ du calcul des intervalles. Nous avons également montré que pour certaines classes traitables de relations, appliquer la méthode de la fermeture par faible composition uniquement sur les triplets de variables appartenant aux regroupements issus d'une décomposition arborescente est suffisant pour décider du problème de la cohérence du RCQ. Nous avons ainsi introduit une cohérence locale partielle basée sur la $\diamond$-cohérence proche de la cohérence locale PPC (Partial Path Consistency) proposée par BLIEK et SAM-HAROUD dans [BSH99] :

Définition 9 *Soit* $\mathcal{N} = (V, C)$ *un* RCQ *et* $X = \{X_0, \ldots, X_n\}$ *une famille de sous-ensembles de* V*.* $\mathcal{N}$ *est* $\overset{\diamond}{X}$*-cohérent si, et seulement si, pour chaque* $X_i \in X$*, le* RCQ $\mathcal{N}_{X_i}$ *(la projection de* $\mathcal{N}$ *sur l'ensemble de variables* X_i*) est un* RCQ $\diamond$*-cohérent.*

Pour certaines classes $\mathcal{C}$ traitables telles que l'ensemble des relations convexes de certains formalismes, la fermeture par faible composition d'un RCQ défini sur $\mathcal{C}$ permet d'obtenir un RCQ globalement cohérent (toute solution partielle peut être étendue en une solution). Nous avons montré que pour un RCQ $\mathcal{N}$ défini sur une telle classe $\mathcal{C}$ et pour un ensemble X de regroupements d'une décomposition arborescente de $\mathcal{N}$, si $\mathcal{N}$ est $\overset{\diamond}{X}$-cohérent et ne contient pas la relation vide alors $\mathcal{N}$ est cohérent. Nous avons également démontré ce résultat pour la classe des relations préconvexes du calcul des intervalles pour laquelle tout RCQ $\diamond$-cohérent possède des solutions partielles particulières, dites solutions maximales, pouvant être étendues en solutions [Lig94, Lig96]. Une solution partielle maximale est une solution pour laquelle toutes les relations de base satisfaites sont de dimension maximale par rapport à la dimension de la relation définissant la contrainte dans laquelle elles se trouvent (voir la section 2.2 pour la notion de dimension d'une relation).

Function Cohérence2($\mathcal{N}$) : Booléen

in : $\mathcal{N} = (V, C)$, un RCQ, X un ensemble d'ensembles de variables de V
output : $true$ si $\mathcal{N}$ est cohérent, $false$ sinon.
1 **begin**
2 $\quad \mathcal{N}^{Init} = (V, C^{Init}) \leftarrow \mathcal{N}$;
3 $\quad \mathcal{T} = (X, F) \leftarrow$ treeDecomposition($\mathcal{N}$);
4 $\quad \mathcal{N} \leftarrow$ preTreatment($\mathcal{N}$);
5 $\quad$ **if** $\mathcal{N} = \bot$ **then**
6 $\quad\quad$ **return** *false* ;
7 $\quad$ **return** CohérenceAux($\mathcal{N}$,X) ;

Function Cohérence2Aux($\mathcal{N}$,X) : Booléen

in : $\mathcal{N} = (V, C)$, un RCQ et X un ensemble d'ensembles de variables.
output : $true$ si $\mathcal{N}$ est cohérent, $false$ sinon.
begin
$\quad \mathcal{N} \leftarrow \underset{X}{\diamond}$–Cohérence($\mathcal{N}, X$);
$\quad$ **if** $\mathcal{N} = \bot$ **then**
$\quad\quad$ **return** *false* ;
$\quad$ Sélectionne $(v_i, v_j) \in \bigcup X$ tel que $(C_{ij})^{\mathcal{C}} \not\subseteq C_{ij}^{Init}$;
$\quad$ **if** *un tel couple n'existe pas* **then**
$\quad\quad$ **return** *true* ;
$\quad$ Partager C_{ij} en sous-relations $r_1, \ldots, r_k \in \mathcal{C}$;
$\quad$ **foreach** $k \in 1, \ldots, k$ **do**
$\quad\quad C_{ij} \leftarrow r_k$; $C_{ji} \leftarrow r_k^{\smile}$;
$\quad\quad$ **if** Cohérence2Aux($\mathcal{N}$,X) **then**
$\quad\quad\quad$ **return** *true* ;
$\quad$ **return** *false*

Ce résultat est particulièrement intéressant dans le cadre pratique de la résolution de RCQ du calcul des intervalles pour lequel la seule classe maximale traitable contenant les relations singletons est l'ensemble des relations des préconvexes. En effet, afin de rendre plus efficace l'algorithme de recherche utilisant le découpage des contraintes par les relations d'une classe traitable (voir Section 2.1), il suffit dans un premier temps de calculer une décomposition arborescente du RCQ $\mathcal{N}$ pour lequel on cherche à décider de la cohérence. Puis, il convient d'effectuer une recherche d'un sous-RCQ préconvexe cohérent non pas à partir du calcul de la fermeture par $\diamond$-cohérence mais à l'aide du calcul de la fermeture $\underset{X}{\diamond}$-cohérence comme méthode de filtrage, avec X l'ensemble des regroupements de la décomposition arborescente. Ce nouvel algorithme (voir la fonction Cohérence2) considère moins de contraintes lors de la recherche et du filtrage. Notons qu'à la ligne 4 de la fonction Cohérence2, l'appel d'un prétraitement est réalisé. Ce prétraitement peut être quelconque pourvu qu'il ne supprime que des relations de base incohérentes parmi les contraintes de $\mathcal{N}$. Pour certaines décompositions arborescentes, le nouvel algorithme proposé s'avère être expérimentalement plus rapide [CC11]. Notons que le temps du calcul d'une décomposition arborescente peut se faire en temps largement inférieur au temps de résolution à partir de certains algorithmes de triangulation [RTL76, BBHP04, KBvH01, BK10] appliqués aux graphe de contraintes du RCQ.

Remarquons que dans le cadre de cette étude, seuls les regroupements de variables issus d'une décomposition arborescente sont utilisés. Nous n'utilisons pas la structure arborescente de la décomposition. De ce fait, l'ensemble des résultats précédents peut être reformulé de manière équivalente en considérant les triangles de variables issus d'une triangulation du graphe de contraintes du RCQ [CC11]. Rappelons qu'un graphe triangulé est un graphe tel que chaque cycle de longueur strictement supérieure à 3 possède une corde (une arête joignant deux sommets non adjacents). Trianguler un graphe consiste à rajouter des arêtes au graphe afin d'obtenir un graphe triangulé. À partir de la triangulation d'un graphe de contraintes, nous pouvons définir une décomposition arborescente. Dans le cadre d'une étude expérimentale, nous avons constaté qu'un algorithme de triangulation basé sur l'algorithme *LexBFS* (Lexicographic Breadth First Search) de ROSE *et al.* [RTL76] ou basé sur l'heuristique *GreedyFillIn* [BK10] permet de réduire un nombre très important de couples ou de triangles de variables [CD11, CC11].

2.9 Implémentations

Durant les années 80 et 90, différentes implémentations ont été proposées afin de résoudre le problème de la cohérence des réseaux de contraintes qualitatives. Il s'agissait très souvent de programmes dédiés à des formalismes qualitatifs particuliers tels que le programme proposé par NEBEL et BÜRCKERT [NB94, NB95] afin de résoudre les RCQ définis sur le calcul des intervalles. Actuellement, il existe trois principaux programmes ouverts et génériques permettant de résoudre des RCQ de manière efficace dans lesquels sont implantées certaines des dernières avancées du domaine : SparQ (SPAtial Reasoning done Qualitatively) [WFW+06], GQR (Generic Qualitative Solver) [GWW08] et QAT (Qualitative Algebra Tools) [CLS06a]. SparQ est une boîte à outils permettant de résoudre des RCQ de formalismes qualitatifs dont les relations de base sont d'arité 2 ou 3. Il fournit, également, des outils permettant de transformer des descriptions quantitatives en descriptions qualitatives. SparQ est implémenté en langage LISP mais utilise également des librairies en langage C. GQR est un programme écrit en langage C++ ne traitant que des RCQ définis sur des relations de base d'arité 2. La version actuelle de GQR est certainement l'outil générique le plus efficace pour résoudre des RCQ binaires.

QAT [CLS06a] est une boîte à outils écrite en Java développée en collaboration avec MAHMOUD SAADE dans le cadre de sa thèse. Boîte à outils générique permettant de résoudre des RCQ définis sur des calculs dont les relations de base peuvent être d'une arité quelconque, elle contient trois principaux paquetages : le paquetage Algebra, le paquetage QCN et le paquetage Solver.

Le paquetage Algebra permet de traiter les aspects algébriques d'un formalisme qualitatif. Il est notamment possible de charger à partir d'un fichier XML suivant une DTD particulière la structure algébrique définissant le formalisme qualitatif : la définition des relations de base, les relations diagonales, la table de faible composition, la table de rotation et la table de permutation. Des fonctionnalités sont également présentes afin de construire une structure algébrique d'un formalisme qualitatif à partir de structures d'autres formalismes qualitatifs. Il est, par exemple, possible de créer une algèbre qualitative à partir d'une autre en regroupant des relations de base ou bien encore en réalisant le produit de deux autres algèbres qualitatives. Le paquetageAlgebra permet également la gestion des classes traitables.

Le paquetage QCN offre différentes fonctionnalités permettant de manipuler des RCQ : création d'un RCQ, ajout/suppression de variables, ajout/suppression/modification de contraintes, *etc.* Il permet également la génération de RCQ aléatoires selon les modèles proposés dans la littérature (voir [Neb96]).

Le paquetage RCQ contient aussi des classes correspondant à différentes heuristiques utilisables pour le parcours des variables ou des contraintes d'un RCQ et des sous-relations d'une relation définissant une contrainte.

Le paquetage Solver contient un ensemble de méthodes permettant la propagation de contraintes sur un RCQ ou la résolution d'un RCQ. Les différentes versions de la fermeture par faible composition ont été définies ainsi que différentes méthodes de résolution. Ces méthodes de résolution sont basées sur l'algorithme de recherche présenté en début de chapitre et sont paramétrables par différentes heuristiques et par l'utilisation de différentes classes traitables. Elles intègrent également des notions particulières telles que la notion de contraintes éligibles ou la notion de contraintes gelées. Ces différentes méthodes ont été implémentées pour le cas générique où l'arité des relations de base manipulées est quelconque. Pour des raisons d'efficacité, nous avons également réalisé des implémentations spécifiques pour le cas où l'arité des relations de base est 2.

Autour de ces trois paquetages de base ont été greffés, au cours du temps, de nouveaux paquetages consacrés à des applications particulières telles que la fusion des RCQ, la traduction des RCQ en problème SAT. QAT n'est pas l'outil le plus efficace en terme de temps pour la résolution de RCQ mais reste certainement l'outil le plus ouvert du domaine en terme d'utilisation.

Au cours de ces dernières années, nous avons également développé des programmes ad-hoc au calcul des intervalles permettant de valider expérimentalement certaines approches, en particulier celles présentées tout au long de ce chapitre. La plupart de ces programmes ont été développés en langage C afin d'obtenir des outils rapides.

2.10 Conclusion

Nous avons présenté dans ce chapitre un ensemble de travaux concernant directement la résolution du problème de la cohérence des réseaux de contraintes qualitatives. Comme nous avons pu le constater, ces travaux ont permis aussi bien d'apporter des résultats sur des formalismes qualitatifs particuliers (dans le cadre de recherche de classes traitables notamment) que des résultats généraux pouvant être mis en œuvre dans le cadre de nombreux formalismes qualitatifs. Comme exemples de contributions pouvant être employés dans le cadre de nombreux formalismes qualitatifs, nous pouvons citer les résultats obtenus concernant la notion de contraintes éligibles, ceux établis lors de l'étude de la famille des $\overset{\diamond}{f}$-cohérences, ainsi que ceux concernant l'utilisation de décompositions arborescentes des structures des RCQ dans le cadre de leur résolution.

Nos études ne se bornent pas exclusivement à des travaux sur la résolution du problème de la cohérence des RCQ par des méthodes de résolution de contraintes qualitatives. Nous avons également étudié ce problème au travers des CSP discrets et du problème SAT. La maturité des recherches concernant ces deux formalismes ont permis de définir des méthodes très efficaces de résolution et offrent des alternatives très intéressantes à la résolution du problème de la cohérence des RCQ par des méthodes de résolution considérant uniquement des contraintes qualitatives. Dans [WW09], WESTPHAL et WÖLFL ont mené une étude expérimentale complète dans laquelle sont comparées l'utilisation d'un solveur de contraintes qualitatives (GQR [GWW08]) avec celle d'un solveur de CSP discret (Mistral) et celle d'un solveur SAT (MiniSat [ES03]) afin de résoudre des RCQ définis dans différents formalismes qualitatifs. Dans l'ensemble, le solveur de contraintes qualitatives s'est avéré être plus efficace que les deux autres solveurs dans le cadre de formalismes qualitatifs basés sur peu de relations base (le calcul des intervalles et le calcul RCC8). Par contre, pour les RCQ définis sur des formalismes qualitatifs basés sur un nombre important de relations de base (le calcul OPRA_2 avec 72 relations de base et le calcul OPRA_4 avec 272 relations de base [Mor06]), les deux autres solveurs prennent l'avantage sur le solveur de contraintes qualitatives. Ces résultats démontrent clairement qu'aucune des trois approches, résolution directe des

RCQ, résolution par CSP discrets, résolution par problème SAT, ne doit être négligée.

Comme nous le verrons dans la conclusion de ce rapport, les différents travaux exposés dans ce chapitre ouvrent des perspectives de recherche très intéressantes dans le cadre de la définition de méthodes efficaces pour la résolution du problème de cohérence de RCQ.

Chapitre 3

Au delà du problème de la cohérence des RCQ : logiques spatio-temporelles et fusion de RCQ

Des logiques combinant des logiques temporelles et les contraintes qualitatives ont été proposées [WZ00] afin de pouvoir représenter et raisonner sur des informations spatio-temporelles. Avec ces langages, nous pouvons par exemple représenter le fait qu'une région a été à un moment donné au cours du temps contenue par une autre région ou bien encore qu'une entité ponctuelle se déplace toujours vers sa droite au cours du temps. En première partie de ce chapitre, nous présentons nos travaux **[BC02a]**[p85] concernant un langage logique basé sur celui de la logique propositionnelle temporelle linéaire dans lequel les propositions sont définies par des contraintes qualitatives. Nous décrirons également une étude concernant des réseaux de contraintes qualitatives permettant d'exprimer des contraintes cycliques au cours du temps. Ces réseaux de contraintes appelés UPQCN (Ultimately Periodic Qualitative Constraint Networks) [CLT05] peuvent être vus comme des formules particulières du langage logique spatio-temporel introduit précédemment.

La deuxième partie de ce chapitre est dévolue à des travaux récents concernant la problématique de la fusion d'informations temporelles ou spatiales représentées par des RCQ [CKS08, CKMS09b, CKMS09c]**[CKMS10b]**[p185]**[CKMS09a]**[p165]. Cette problématique est notamment importante dans le cadre d'applications concernant le domaines des SIG ou bien encore dans le cadre d'applications devant gérer des préférences de différents utilisateurs sur des informations temporelles. Dans le cadre de la thèse de Nicolas Schwind co-encadré par Souhila Kaci, Pierre Marquis et moi-même, nous avons défini différentes familles d'opérateurs de fusion de RCQ que nous décrivons en fin de chapitre.

3.1 Raisonnement spatio-temporel à partir de contraintes qualitatives

3.1.1 Combinaisons de formalismes qualitatifs à base de contraintes.

Afin de pouvoir raisonner sur des entités spatiales évoluant au cours du temps, plusieurs approches ont été proposées. L'une d'entre elles consiste à combiner deux formalismes qualitatifs à base de contraintes, l'un permettant de raisonner sur le temps, l'autre sur l'espace. Cette approche a, par exemple, été suivie par Gerivini et Nebel [GN02] qui combinent les contraintes du calcul RCC8 avec celles du calcul des intervalles. Les problèmes de contraintes considérées dans ces travaux, appelés problèmes de contraintes STCC (Spatio-Temporal Constraint Calculus), sont définis par des contraintes du calcul RCC8 annotées par des variables représentant des intervalles et par des contraintes du calcul d'ALLEN sur ces variables. À titre d'illustration, considérons le problème STCC défini par les contraintes suivantes :

$$x : (X\{DC, EC, TPP\}Y);\ x : (Y\{TPP, EC\}Z);\ y : (X\{EC, EQ, PO\}Y);\ y : (Y\{DC, EC\}Z);$$
$$x\ \{m, o, eq\}\ y.$$

Dans cet exemple, nous avons deux variables temporelles x et y, ainsi que trois variables spatiales X, Y et Z. La première des contraintes données spécifie que les régions X et Y doivent satisfaire une des relations de base de RCC8 de l'ensemble $\{DC, EC, TPP\}$ sur l'intervalle x. La dernière de ces contraintes indique que les intervalles x et y doivent satisfaire une des relations du calcul des intervalles appartenant à l'ensemble $\{m, o, eq\}$.

Une interprétation d'un problème de contraintes STCC est définie, par une application α qui associe à chaque variable temporelle un intervalle de la droite des rationnels et une application associant à chaque nombre rationnel et à chaque variable spatiale un sous-ensemble fermé régulier et non vide d'un espace topologique. Une interprétation sera cohérente si toutes les contraintes temporelles sont satisfaites par les intervalles et si toutes les contraintes spatiales sont satisfaites en chacun des points strictement compris entre les bornes de l'intervalle affecté à la variable temporelle l'annotant. GERIVINI et NEBEL montrent que le problème de la cohérence des problèmes de contraintes STCC est un problème NP-complet même dans le cas où chacune des relations temporelles et spatiales est définie par une seule relation de base ou la relation universelle (toutes les relations sont permises). Ils étudient également des interprétations dans lesquelles les rapports entre les tailles doivent persister au cours du temps et des interprétations dans lesquelles le changement des positions relatives spatiales doit se réaliser de manière continue dans le temps. Ils montrent que le problème de la cohérence reste NP-complet dans les deux cas.

3.1.2 Encapsulation de contraintes spatiales dans une logique temporelle.

Une autre approche permettant de raisonner sur l'évolution d'entités spatiales au cours du temps est d'encapsuler les contraintes spatiales dans une logique temporelle.

Dans [WZ00], WOLTER and ZAKHARYASCHEV proposent plusieurs langages logiques basés sur celui de la logique propositionnelle temporelle linéaire [SC85] dans lequel intuitivement les propositions sont remplacées par des contraintes du calcul de RCC8. Dans **[BC02a]**[p85], nous étudions un langage logique similaire, à la différence que nous considérons de manière générique un ensemble de relations de base B d'un formalisme qualitatif. Ces études ont permis de caractériser la complexité du problème de la satisfiabilité pour ces logiques. Avant de présenter ces résultats, nous décrirons le langage étudié dans **[BC02a]**[p85], que nous appellerons dans la suite $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$. Pour RCC8, ce langage correspond au langage $\mathcal{S}T_1$ étudié dans [WZ00].

Étant donné un ensemble de symboles représentant des relations de base B et un ensemble de variables V $x, y, z, \ldots$ représentant des élements du domaine D, les formules du langage $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ sont définies inductivement de la manière suivante :

$$f ::= \top \mid r(\bigcirc^m x, \bigcirc^n y) \mid \neg f \mid (f \vee g) \mid (f\ \mathbf{U}\ g);$$

où r appartient à B, n, m sont deux entiers positifs et x, y deux variables de V. Le modèle de temps considéré est linéaire, discret, borné dans le passé, non borné dans le futur et peut donc être représenté par $(\mathbb{N}, <)$, avec $<$ la relation d'ordre usuelle sur les entiers. Un modèle associé au langage $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ est défini par une application ϵ qui associe à chaque variable $x \in V$ et à chaque instant $i \in \mathbb{N}$, une valeur de D notée $\epsilon(x, i)$. La satisfiabilité d'une formule f de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ à un instant $i \in \mathbb{N}$ par un modèle ϵ, notée $\epsilon, i \models f$, est définie inductivement de la manière suivante :

- $\epsilon, i \models \top$;
- $\epsilon, i \models r(\bigcirc^m x, \bigcirc^n y)$ ssi $\epsilon(x, i+m)\ r\ \epsilon(y, i+n)$;
- $\epsilon, i \models \neg f$ ssi non $\epsilon, i \models f$;
- $\epsilon, i \models f \vee g$ ssi $\epsilon, i \models f$ ou $\epsilon, i \models f$;
- $\epsilon, i \models f\ \mathbf{U}\ g$ ssi il existe $k \geq i$ tel que $\epsilon, i \models g$ et pour tout $j \in \{i, \ldots, k-1\}$ $\epsilon, j \models f$.

Une formule f est satisfiable si et seulement s'il existe un modèle ϵ tel que $\epsilon, 0 \models f$. Intuitivement, $r(\bigcirc^m x, \bigcirc^n y)$ exprime que la valeur de x à l'instant $i+m$ satisfait la relation de base r avec la valeur de y à l'instant $i+n$. $f\ \mathbf{U}\ g$ exprime à l'instant i que dans le futur f sera satisfaite jusqu'à ce que g le soit. Les opérateurs temporels $\mathbf{F}$ (pour un instant dans le futur) et $\mathbf{G}$ (pour tous les instants futurs) peuvent s'exprimer à partir de l'opérateur temporel $\mathbf{U}$ par $\mathbf{F}\ f \equiv \top\ \mathbf{U}\ f$ et $\mathbf{G}\ f \equiv \neg(\top\ \mathbf{U}\ \neg f)$. L'opérateur temporel $\bigcirc$ sur une formule f (à l'instant suivant la formule f est vraie) peut s'exprimer à l'aide de l'opérateur $\bigcirc$ sur les variables puisque x et y satisfont la relation de base r à l'instant suivant si et seulement si $r(\bigcirc^1 x, \bigcirc^1 y)$ est satisfaite à l'instant courant.

Dans les exemples suivants, pour une variable x, x et $\bigcirc x$ correspondent respectivement à $\bigcirc^0 x$ et $\bigcirc^1 x$. Supposons que l'ensemble B correspond à l'ensemble des relations de base du calcul des directions cardinales de LIGOZAT et considérons trois variables x, y, z représentant trois entités spatiales ponctuelles du plan. Soit la formule $f = NW(y,x) \wedge NE(z,x) \wedge \mathbf{G}(E(\bigcirc y, y) \wedge W(\bigcirc y, y) \wedge EQ(\bigcirc x, x)) \wedge \mathbf{F}(N(y,x) \wedge N(z,x))$. Cette formule exprime les informations suivantes, y est au nord-ouest de x et se déplacera vers l'est au cours du temps, z est au nord-est de x et se déplacera vers l'ouest au cours du temps, x ne bouge pas, à un moment donné y et z seront au nord de x. La figure 3.1 illustre un modèle ϵ de cette formule f de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$. En utilisant les relations de base de RCC8, la formule $\mathbf{G}(NTPP(\bigcirc x, x) \wedge DC(x,y))$, stipule que la région x décroîtra tout au long du temps et les régions x et y seront toujours déconnectées.

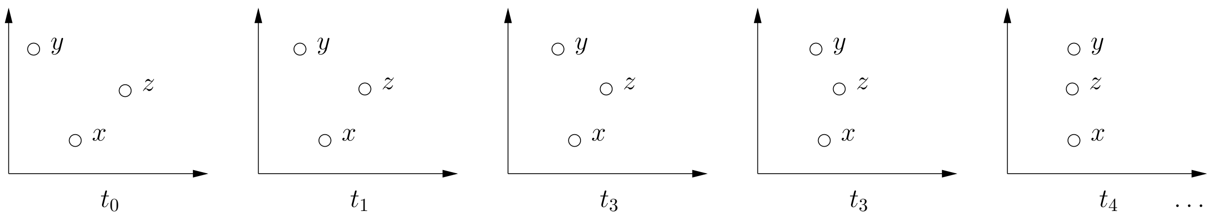


FIGURE 3.1 – Un modèle ϵ de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ avec B l'ensemble des relations de base du calcul des directions cardinales.

Dans **[BC02a]**[p85], nous avons établi un résultat de complexité concernant les relations de base B pour lesquelles tout scénario cohérent est globalement cohérent (toute instanciation partielle peut-être étendue en une solution). Cette propriété est possédée par l'ensemble de relations de base de nombreux formalismes qualitatifs parmi lesquels le calcul des intervalles, le calcul des points, le calcul des rectangles, le calcul des directions cardinales sur les points, ... Pour ces formalismes, nous avons démontré que le problème de déterminer si une formule de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ est satisfiable ou non est un problème PSPACE-complet. Intuitivement, ce résultat est basé sur le fait qu'un modèle satisfaisant une formule f a un scénario (en considérant les relations de base satisfaites entre les variables au cours du temps) périodique à partir d'un instant t borné polynomialement par rapport à la taille de f. À partir de ce résultat, nous avons proposé un algorithme non déterministe consistant à deviner un scénario cohérent d'un modèle fini. Ce modèle fini pouvant être étendu à l'infini du fait de la propriété de globale cohérence imposée sur les scénarios cohérents définis sur B.

Notre résultat ne peut pas s'appliquer dans le cas où le langage $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ est basé sur les relations du calcul de RCC8 puisque les scénarios cohérents de ce calcul ne sont pas globalement cohérents. Néanmoins, WOLTER and ZAKHARYASCHEV [WZ00] établissent que le problème de la satisfiabilité du langage $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ est EXPSPACE lorsque les relations de base de RCC8 sont considérées. Pour établir ce résultat, ils étudient des modèles particuliers de la logique modale définie par le produit de PLTL et de la logique modale propositionnelle $S4_u$. Les contraintes du calcul RCC8 peuvent être en effet encodées en formules de la logique modale propositionnelle $S4_u$ [Eme90b, BdRV01] (système modale $S4$ avec la modalité universelle). Pour cette logique, peut être utilisé comme interprétation un espace topologique où chaque variable est interprétée par un sous-ensemble de cette topologie, les opérateurs booléens $\neg$, $\vee$ et $\wedge$ par respectivement le complémentaire, l'union et l'intersection, les opérateurs modaux de nécessité et de possibilité $\Box$ et $\Diamond$ par respectivement l'intérieur et la fermeture topologique. Ainsi, de manière naturelle, chaque relation de base de RCC8 peut être encodée en formule de cette logique bimodale. Nous avons, par exemple, $DC(v_1, v_2)$ qui se traduira par $\neg\Diamond_u(v_1 \wedge v_2)$ (il n'existe pas de point appartenant à v_1 et v_2) et $EC(v_1, v_2)$ par $\Diamond_u(v_1 \wedge v_2) \wedge \neg\Diamond_u(\Box v_1 \wedge \Box v_2)$ (l'intersection entre v_1 et v_2 est non vide et l'intersection de leurs intérieurs est vide). Le fait que les variables v_1 et v_2 soient interprétées par des réguliers fermés non vides s'exprimera également par $\Diamond_u v_1 \wedge \Diamond_u v_2$ (v_1 et v_2 non vides) et $\Box_u((\Box\neg v_1 \vee \Diamond\Box v_1) \wedge (\Box\neg v_2 \vee \Diamond\Box v_2))$ (v_1 et v_2 fermés réguliers, v_1 et v_2 coïncide avec la fermeture de leur intérieur). Les modèles considérés pour ces formules correspondront à une assignation de chacune des variables par un sous-ensemble non vide régulier et fermé d'un espace topologique ou un modèle fini de KRIPKE satisfaisant $S4$. La transformation des relations de base de RCC8 en logique modale a également été utilisée par RENZ [Ren98] afin de caractériser des modèles particuliers de RCC8 lors de son étude du problème de la cohérence des RCQ de RCC8. Remarquons également que l'utilisation de logiques modales à plusieurs dimensions afin de temporiser les relations de base de RCC8 a été initialement proposée par BENNETT *et al.* dans [BBC99].

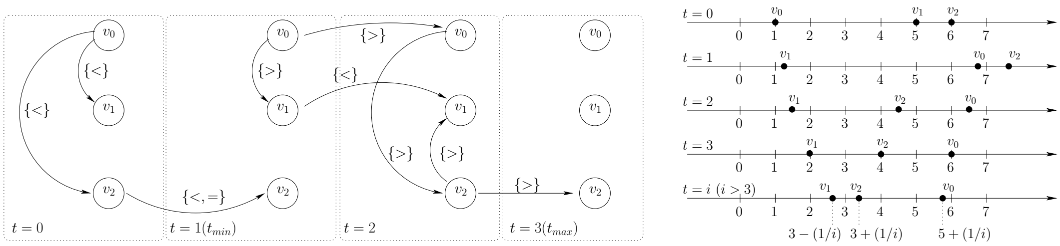
3.1.3 Réseaux de contraintes qualitatives périodiques

Dans [CLT05], nous avons proposé des réseaux de contraintes qualitatives particuliers permettant de décrire des contraintes évoluant au cours du temps. Au bout d'un temps fini, ces contraintes évoluent de manière périodique (les mêmes contraintes sont reproduites à l'infini). Ces réseaux de contraintes qualitatives particuliers sont appelés UPQCN (Ultimately Periodic Qualitative Constraint Networks). Comme pour le langage précédent, la structure temporelle retenue est représentée par $(\mathbb{N}, <)$. Ainsi, un instant sera représenté par un entier t. Un UPQCN se définit formellement de la manière suivante :

Définition 10 *Un* UPQCN $\mathcal{R}$ *est un quadruplet* (V, C, t_{min}, t_{max})*, avec :*

- $V = \{v_0, \ldots, v_{n-1}\}$ *est un ensemble fini de* n *variables ;*
- t_{min}, t_{max} *sont deux entiers tels que* $0 \leq t_{min} \leq t_{max}$ *;*
- C *est une application de* $V \times V \times \{0, \ldots, t_{max}\} \times \{0, \ldots, t_{max}\}$ *vers* 2^{B} *telle que pour tout* $v_i, v_j \in V$ *et* $t_i, t_j \in \{0, \ldots, t_{max}\}$*,* $C(v_i, v_i, t_i, t_i) \subseteq \{\mathsf{Id}\}$ *et* $C(v_i, v_j, t_i, t_j) = C(v_j, v_i, t_j, t_i)^{\smile}$*.*

Dans un contexte de représentation d'informations spatio-temporelles, l'ensemble des variables V d'un UPQCN représente un ensemble d'entités spatiales évoluant au cours du temps. L'ensemble de relations de base $C(v_i, v_i, t_i, t_i)$ contraint la position relative de l'entité spatiale à l'instant t_i représentée par v_i et celle à l'instant t_j représentée par v_j. Les contraintes de positionnement relatif concernant les entités spatiales représentées par V sont données explicitement par l'application C pour la période de temps comprise entre l'instant t_0 et l'instant t_{max}. De plus, les contraintes correspondant à la période de temps comprise entre les instants t_{min} et t_{max} doivent être également être satisfaites pour toutes les périodes de temps débutant après l'instant t_{min}. Plus formellement, l'ensemble de relations de base $C(v_i, v_i, t_i, t_i)$,



(a) (b)

FIGURE 3.2 – Un UPQCN $\mathcal{R}$ (a) et une solution de ses solutions (b).

avec $t_{min} \leq t_i, t_j t_{max}$ correspondent aux relations de base possibles entre v_i à l'instant $t_i + k$ et v_j à l'instant $t_j + k$ pour tout entier $k \geq 0$.

La figure 3.2(a) représente un UPQCN $\mathcal{R}$ défini à partir des relations de base du calcul des instants interprétées spatialement. Les trois variables v_0, v_1, v_2 représentent trois points de la droite se déplaçant au cours du temps. La relation $C(v_1, v_1, t_1, t_2)$ indique que le point représenté par v_1 se trouvera à l'instant 1 sur la droite strictement avant sa position à l'instant 2. De plus, comme t_{min} correspond à 1, la relation $\{<\}$ devra être également satisfaite par la position de ce point à l'instant $1 + k$ avec celle à l'instant $2 + k$ pour tout $k \geq 0$. Intuitivement, nous avons le point représenté par v_1 qui se déplacera sur sa droite à partir de l'instant 1. La figure 3.2(b) représente une solution du UPQCN $\mathcal{R}$.

Dans un contexte temporel, chaque variable $v_i \in V$ représente une activité ou un événement récurrent. La couple $(v_i, t_i) \in V \times \mathbb{N}$ représentera sa $(t_i+1)^{\text{ème}}$ occurrence. $C(v_i, v_j, t_i, t_j)$ est un ensemble de relations de base qui contraint la relation temporelle entre la $(t_i + 1)^{\text{ème}}$ occurrence de v_i et la $(t_j + 1)^{\text{ème}}$ occurrence de v_j.

Les UPQCN peuvent être considérés comme sous-langage de la logique $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ présentée dans la section précédente. En effet, chaque contrainte d'un UPQCN $\mathcal{R}$ (V, C, t_{min}, t_{max}) peut être représentée par une formule utilisant les opérateurs **G** (pour tous les instants futurs) et $\bigcirc$ (à l'instant suivant). Nous ne donnerons pas de traduction formelle mais juste deux exemples concernant le UPQCN $\mathcal{R}$ illustré par la figure 3.2(a). La contrainte stipulant que v_2 à l'instant 0 doit satisfaire avec v_2 à l'instant 1 la relation $\{<, =\}$ sera représentée par la formule $(< (v_2, \bigcirc v_2)) \vee (= (v_2, \bigcirc v_2))$ de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$. Le fait que la relation $\{<\}$ devra être satisfaite par la position du point v_1 à l'instant $1 + k$ avec celle à l'instant $2 + k$ pour tout $k \geq 0$ peut être exprimée par la formule $\mathbf{G}(< (\bigcirc v_1, \bigcirc \bigcirc v_1))$.

Dans [CLT05], nous décrivons un algorithme polynomial permettant de résoudre le problème de la cohérence des UPQCN définis par des relations appartenant à une classe traitable pour laquelle tout RCQ $\diamond$-cohérent est globalement cohérent. Notons que la classe des relations convexes de certains formalismes qualitatifs possède une telle propriété.

3.2 Fusion de réseaux de contraintes qualitatives

Dans certaines applications, notamment dans le cadre de systèmes multi-agents ou de systèmes d'informations géographiques distribués, plusieurs sources peuvent chacune fournir des informations temporelles ou spatiales. Ces informations pouvant être contradictoires, il est nécessaire de définir et de mettre en œuvre des méthodes de fusion pour résoudre les conflits éventuels.

Dans le cadre des stages de Master Recherche de ALI ZITOUNI et NICOLAS SCHWIND et de la thèse de NICOLAS SCHWIND, nous avons étudié la problématique de la fusion de réseaux de contraintes qualitatives.

De manière générale, un opérateur de fusion considère un multi-ensemble de RCQ et retourne également un multi-ensemble de RCQ. Nous parlerons d'opérateur homogène lorsque l'opérateur de fusion ne traite que des RCQ définis sur le même ensemble de relations de base B. Dans le cas contraire, nous parlerons d'opérateurs hétérogènes. Les opérateurs hétérogènes que nous avons définis et étudiés prennent en entrée des RCQ dont les relations sont définies sur un même domaine D. Nous réalisons également la distinction entre les opérateurs de fusion sémantiques et les opérateurs de fusion syntaxiques. Intuitivement, les premiers considèrent pour chacun des RCQ en entrée les scénarios cohérents qu'ils représentent. Ainsi, substituer un RCQ donné en entrée par un RCQ équivalent ne modifiera par le résultat produit par un opérateur sémantique. Les opérateurs de fusion syntaxiques sont quant à eux sensibles à une telle substitution. Pour ces derniers, une contrainte d'un RCQ est considérée comme une information à part entière, par conséquent la modification d'une contrainte peut entraîner des résultats différents. Pour l'ensemble des opérateurs proposés, nous avons réalisé une étude algorithmique et également une étude concernant leurs propriétés logiques.

Fortement inspirés par les études réalisées dans le cadre de la fusion de bases de croyances en logique propositionnelle [Rev93, Lin96, LM98, KP99, KP02], nous avons, ces dernières années, proposé différentes familles d'opérateurs de fusion de RCQ : des opérateurs sémantiques homogènes [CKS08, CKMS09c], des opérateurs sémantiques hétérogènes **[CKMS09a]**[p165], des opérateurs syntaxiques homogènes **[CKMS10b]**[p185].

3.2.1 Opérateurs de fusion de RCQ sémantiques et homogènes

Dans [CKS08, CKMS09c], nous avons défini une famille d'opérateurs de fusion prenant en entrée un multi-ensemble fini $\mathcal{K} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ de RCQ défini sur le même ensemble de relations de base B et sur le même ensemble de variables V. Dans la suite, un scénario S sur l'ensemble V (sans aucune précision concernant le RCQ pour lequel S est sous-RCQ) est implicitement un scénario du RCQ (V, C) défini par $C(v, v') = \Psi$ pour tout $v, v' \in V$ tels que $v \neq v'$. Le processus de fusion proposé se décline en différentes étapes dans lesquelles des distances sont calculées et agrégées à l'aide de fonctions d'agrégations [Lin96, LM98, KLM04] :

Étape 1. La première étape consiste à calculer pour chaque scénario cohérent S défini sur V et chaque RCQ $\mathcal{N}$ appartenant à $\mathcal{K}$ une distance locale $d(S, \mathcal{N})$. Le calcul de cette distance locale s'effectue par agrégation des distances entre S et chacun des scénarios cohérents S' de $\mathcal{N}$ de la manière suivante :

$$d(S, \mathcal{N}) = min\{d(S, S') \text{ tel que } S' \text{ scénario cohérent de } \mathcal{N}\} \text{ si } \mathcal{N} \text{ est cohérent, 0 sinon.}$$

Des distances utilisées dans le cadre de la fusion en logique propositionnelle peuvent être facilement adaptées aux scénarios. La distance drastique et la distance de Hamming entre scénarios peuvent par exemple être définies de la manière suivante :

$$d_D(S, S') = 0 \text{ si } S = S', 1 \text{ sinon (distance drastique),}$$

$$d_H(S, S') = |\{(v_i, v_j) \in V \times V \text{ tel que } i < j \text{ et } S[v_i, v_j] \neq S'[v_i, v_j]\}| \text{ (distance de Hamming).}$$

Des distances spécifiques au cadre des formalismes qualitatifs ont également été proposées. Ces distances utilisent des graphes de voisinage conceptuel [Fre92, EM95, CG96, GN02, DM04] dans lesquels sont arrangées les différentes relations de base en fonction de leur voisinage. Deux relations de base r et r' sont voisines lorsqu'en considérant une configuration de deux entités x et y satisfaisant r et une certaine transformation continue des entités, nous pouvons obtenir une configuration de x et y satisfaisant r' sans passer par une configuration satisfaisant une relation r'' distincte de r et r'. En considérant par exemple, les relations de base du calcul des intervalles et une transformation consistant à déplacer de manière continue une des bornes d'un intervalle, nous obtenons le graphe de voisinage conceptuel G_1 représenté par la figure 3.3(a). En considérant maintenant la transformation qui consiste à déplacer un intervalle sans que sa durée puisse être modifiée, nous obtenons le graphe de voisinage conceptuel G_2 illustré par la figure 3.3(b).

Étant donné un graphe de voisinage conceptuel G, en notant $d_G(r, r')$ la distance dans G entre deux relations de base $r, r' \in \mathsf{B}$, nous pouvons définir une distance entre deux scénarios S et S' de la manière suivante :

$$d_G(S, S') = \sum\{d_G(r, r') \text{ tel que } i < j \text{ et } S[v_i, v_j] = \{r\} \text{ et } S'[v_i, v_j] = \{r'\}\} \text{ (distance de voisinage).}$$

Notons que d'autres fonctions d'agrégation que la somme ($\sum$) peuvent être utilisées (la fonction Max par exemple). À titre d'illustration, considérons les deux scénarios S et S' représentés par les figures 3.3(c) et 3.3(d). En considérant les différentes distances entre scénarios présentés précédemment nous avons : $d_D(S, S') = 1, d_H(S, S') = 5, d_{G_1}(S, S') = d_{G_1}(s, eq) + d_{G_1}(p, m) + d_{G_1}(m, m) + d_{G_1}(o, d) + d_{G_1}(o, d) + d_{G_1}(oi, d) = 1+1+0+2+2+2 = 8, d_{G_2}(S, S') = d_{G_2}(s, eq) + d_{G_2}(p, m) + d_{G_2}(m, m) + d_{G_2}(o, d) + d_{G_2}(o, d) + d_{G_2}(oi, d) = 2+1+0+2+2+2 = 9$.

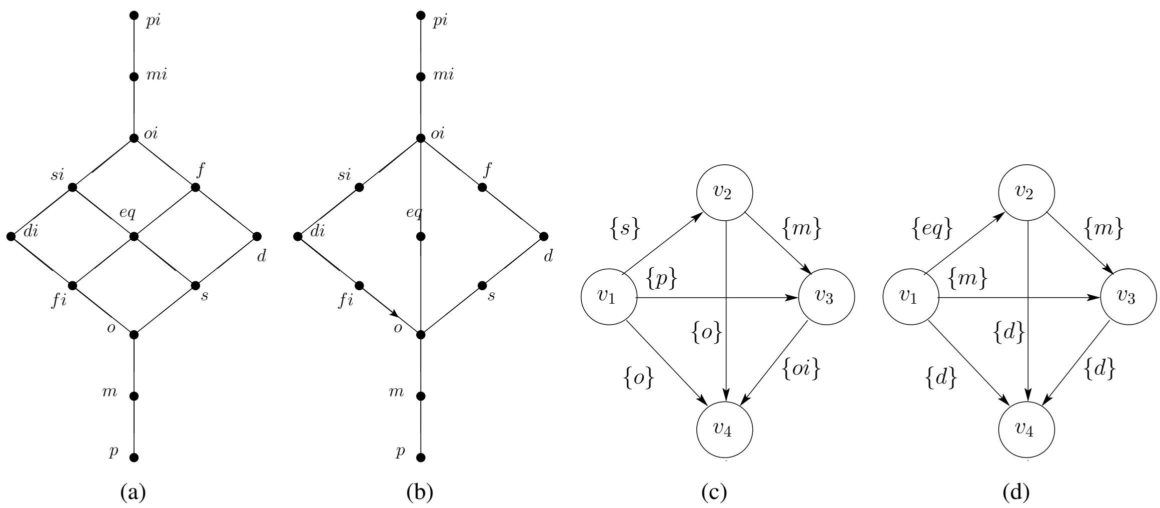


FIGURE 3.3 – Les graphes de voisinage conceptuel G_1 (a) et G_2 (b), deux scénarios S (c) et S' (d) du calcul des intervalles.

Étape 2. La deuxième étape consiste à agréger pour chaque scénario cohérent S défini sur V les distances locales obtenues pour chacune des RCQ de $\mathcal{K}$, afin d'obtenir une distance globale entre S et $\mathcal{K}$, que nous noterons par $d(S, \mathcal{K})$. Une fonction d'agrégation telle que Max ou $\sum$ est utilisée pour calculer cette distance globale. Supposons que $\mathcal{K}$ contienne trois RCQ $\mathcal{N}_1$, $\mathcal{N}_2$ et $\mathcal{N}_3$ et que les distances locales obtenues à l'étape précédente entre le scénario S et ces trois RCQ soient les suivantes : $d(S, \mathcal{N}_1) = 4$, $d(S, \mathcal{N}_2) = 2$ et $d(S, \mathcal{N}_3) = 8$. En utilisant la fonction Max, nous obtenons comme distance globale $d(S, \mathcal{K}) = 8$, tandis qu'avec la fonction $\sum$, nous avons $d(S, \mathcal{K}) = 14$.

Étape 3. La troisième étape réalise la sélection des scénarios cohérents sur V les plus proches de $\mathcal{K}$, *i.e.* ceux ayant la distance globale minimale, afin de définir le résultat de la fusion.

Dans ce processus de fusion, en plus des différentes distances et fonction d'agrégation paramétrant l'opérateur de fusion considéré, un RCQ $\mathcal{N}_{IC}$ appelé RCQ coercitif peut être pris en compte. $\mathcal{N}_{IC}$ représente des contraintes d'intégrité du système considéré et délimite les scénarios à retenir lors du processus de fusion. Seuls les scénarios cohérents de $\mathcal{N}_{IC}$ peuvent appartenir au résultat de fusion.

3.2.2 Opérateurs de fusion de RCQ sémantiques et hétérogènes

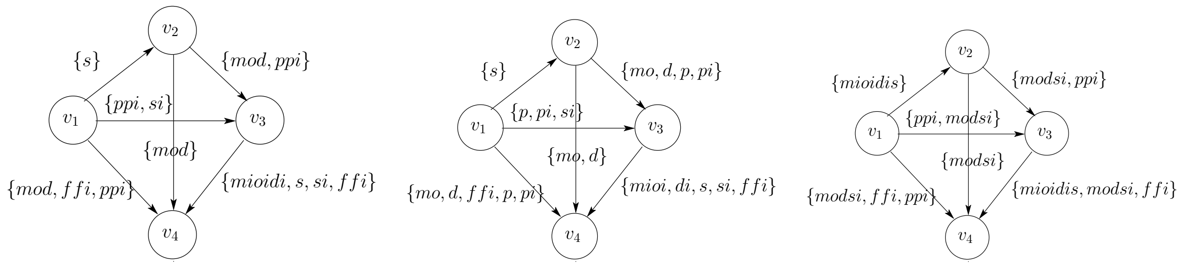
Dans **[CKMS09a]**[p165], nous avons défini un opérateur de fusion permettant de traiter un multi-ensemble $\mathcal{K} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ de RCQ non forcément définis sur le même ensemble de relations de base. Néanmoins, nous avons effectué l'hypothèse que tous les ensembles de relations de base sont définis sur le même domaine D. Les notions de raffinements et d'abstractions d'ensembles de relations de base sont au centre de notre approche. Formellement, ces deux notions se définissent de la manière suivante :

Définition 11 *Soient* B *et* B′ *deux ensembles de relations de base définies sur le même domaine* D. B *est un raffinement de* B′ *si et seulement si, pour toute relation de base* $r' \in$ B′ *il existe un sous-ensemble* $R \subseteq$ B *tel que* $\bigcup\{r \in R\} = r'$. B *est une abstraction de* B′ *si et seulement si* B′ *est un raffinement de* B

Par exemple, l'ensemble des relations de base du calcul d'INDU est un raffinement de l'ensemble des relations de base du calcul des intervalles. Et inversement, l'ensemble des relations de base du calcul des intervalles est une abstraction de l'ensemble des relations de base du calcul d'INDU. Dans la suite, nous considérerons à titre d'illustration deux ensembles de relations de base construits par l'union de relations de base du calcul des intervalles : $\mathsf{B}_A = \{eq, ppi, mod, mioidi, s, si, ffi\}$ et $\mathsf{B}_B = \{eq, p, pi, mosi, miois, d, di, ffi\}$. Le nom de chaque nouvelle relation de base contient les noms des relations de base du calcul des intervalles à partir desquelles elles ont été définies. La relation de base $mioi$ correspond par exemple à l'union de la relation de base mi et la relations de base oi. B_1 et B_2 sont deux abstractions des relations de base du calcul des intervalles. Étant donné un ensemble d'ensembles de relations de base $\mathcal{B} = \{\mathsf{B}_1, \ldots, \mathsf{B}_k\}$ définies sur un même domaine D, nous dirons que B est un raffinement (resp. une abstraction) de $\mathcal{B}$ si et seulement si B est un raffinement (resp. une abstraction) de tout $\mathsf{B}_i \in \mathcal{B}$.

Dans **[CKMS09a]**[p165], nous avons montré que pour tout ensemble d'ensembles de relations de base $\mathcal{B} = \{\mathsf{B}_1, \ldots, \mathsf{B}_k\}$ définies sur un même domaine D, il existe un raffinement de $\mathcal{B}$, noté $\mathsf{Raf}(\mathcal{B})$, qui est une abstraction de tout raffinement de $\mathcal{B}$. De manière duale, il existe une abstraction de $\mathcal{B}$, notée $\mathsf{Abs}(\mathcal{B})$, qui est un raffinement de toute abstraction de $\mathcal{B}$. En considérant les deux ensembles de relations de base B_A et B_B définis précédemment, nous avons : $\mathsf{Raf}(\{\mathsf{B}_A, \mathsf{B}_B\}) = \{eq, p, pi, mo, mioi, d, di, s, si, ffi\}$ et $\mathsf{Abs}(\{\mathsf{B}_A, \mathsf{B}_B\}) = \{eq, ppi, modsi, mioidis, ffi\}$.

Soit un multi-ensemble $\mathcal{K} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ où chacun des RCQ $\mathcal{N}_i$ est défini sur un ensemble de relations de base B_i (définies sur le domaine D). L'approche que nous avons proposée dans **[CKMS09a]**[p165] afin de fusionner les informations représentées par $\mathcal{K}$ consiste à définir dans un premier temps un multi-ensemble $\mathcal{K}'$ contenant la traduction de chacun des RCQ $\mathcal{N}_i$ de $\mathcal{K}$ dans le formalisme défini par l'ensemble des relations de base $\mathsf{Raf}(\{\mathsf{B}_1, \ldots, \mathsf{B}_k\})$. Dans un deuxième temps, un opérateur de fusion homogène présenté dans la section précédente est appliqué avec en entrée le multi-ensemble $\mathcal{K}'$. La traduction de chaque RCQ $\mathcal{N}_i$ est réalisée en raffinant chacune des relations de base définissant ses contraintes par des relations de base de $\mathsf{Raf}(\{\mathsf{B}_1, \ldots, \mathsf{B}_k\})$. À titre d'illustration, supposons que $\mathcal{K}$ comprenne deux RCQ ; le premier défini sur l'ensemble de relations de base B_A et le second défini sur l'ensemble de relations de base B_B. En suivant l'approche de fusion présentée, ces deux RCQ doivent être traduits dans le formalisme défini par $\mathsf{Raf}(\{\mathsf{B}_A, \mathsf{B}_B\})$. Supposons que le premier de ces deux RCQ soit le RCQ $\mathcal{N}$ représenté par la figure 3.4(a). La traduction de $\mathcal{N}$ sur $\mathsf{Raf}(\{\mathsf{B}_A, \mathsf{B}_B\})$ est le RCQ représenté par la figure 3.4(b). Notons que le RCQ obtenu et $\mathcal{N}$ ont exactement le même ensemble de solutions.



(a) (b) (c)

FIGURE 3.4 – (a) Un RCQ $\mathcal{N}$ défini sur B_A, (b) la traduction de $\mathcal{N}$ sur $\mathsf{Raf}(\{\mathsf{B}_A, \mathsf{B}_B\})$ et (c) l'approximation de $\mathcal{N}$ sur $\mathsf{Abs}(\{\mathsf{B}_A, \mathsf{B}_B\})$.

Étant donné un multi-ensemble $\mathcal{K} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ où chacun des RCQ $\mathcal{N}_i$ est défini sur un ensemble de relations de base B_i (définies sur le domaine D), une autre approche de fusion consiste à traduire les différents RCQ de $\mathcal{K}$ dans le formalisme correspondant aux relations de base $\mathsf{Abs}(\{\mathsf{B}_1, \ldots, \mathsf{B}_k\})$ puis à réaliser une fusion sur le multi-ensemble $\mathcal{K}'$ obtenu à l'aide d'un opérateur de fusion homogène. La traduction de chaque RCQ $\mathcal{N}_i$ est réalisée en substituant chacune des relations de base définissant ses contraintes par la relation de base de $\mathsf{Abs}(\{\mathsf{B}_1, \ldots, \mathsf{B}_k\})$ l'incluant. Le RCQ obtenu n'est plus équivalent. Néanmoins il a pour solution toute solution du RCQ traduit. La figure 3.4(c) représente la traduction sur $\mathsf{Abs}(\{\mathsf{B}_A, \mathsf{B}_B\})$ du RCQ $\mathcal{N}$ de la figure 3.4(a).

3.2.3 Opérateurs de fusion de RCQ syntaxiques et homogènes

Dans **[CKMS10b]**[p185], nous proposons et étudions deux classes d'opérateurs syntaxiques et homogènes, la première notée Δ_1 et la seconde notée Δ_2.

Le processus de fusion correspondant à la classe d'opérateurs de fusion Δ_1 diffère de la famille des opérateurs homogènes sémantiques décrite en sous-section 3.2.1 par le calcul des distances locales entre chaque scénario cohérent S défini sur V et chaque RCQ $\mathcal{N}$ appartenant à $\mathcal{K}$. En effet, pour les opérateurs de fusions Δ_1, cette distance locale prend en compte tous les scénarios de $\mathcal{N}$ et non pas uniquement les scénarios cohérents de $\mathcal{N}$. Formellement, la distance locale entre un scénario cohérent S défini sur V et un RCQ $\mathcal{N} \in \mathcal{K}$ est définie pour la famille Δ_1 de la manière suivante :

$$d(S, \mathcal{N}) = min\{d(S, S') \text{ tel que } S' \text{ scénario de } \mathcal{N}\}.$$

Comme pour les opérateurs homogènes sémantiques décrits en sous-section 3.2.1, ces distances locales sont agrégées afin d'obtenir une distance globale $d(S, \mathcal{K})$ entre chaque scénario cohérent S défini sur V et le multi-ensemble $\mathcal{K}$. Les scénarios cohérents pour lesquels la distance globale est la plus petite sont retenus afin de définir le résultat de fusion.

L'approche concernant la famille d'opérateurs de fusion Δ_2 est relativement différente dans le sens où elle est basée sur des calculs de distances locales entre relations. Ces distances locales sont agrégées afin de définir une distance globale entre chaque scénario cohérent S défini sur V et le multi-ensemble $\mathcal{K} = \{\mathcal{N}_1, \ldots, \mathcal{N}_k\}$ de RCQ à fusionner. Le calcul de ces distances globales se réalise grâce aux deux étapes suivantes :

Étape 1. Nous supposons donnée une fonction de distance permettant de calculer une distance $d(r, R)$ entre une relation de base $r \in \mathsf{B}$ et une relation $R \in 2^{\mathsf{B}}$. Notons qu'une telle fonction peut être définie à l'aide d'une fonction d'agrégation et une fonction de distance sur les relations de base B (une distance issue d'un graphe de voisinage conceptuel par exemple). À partir des distances entre relations de base de

B et les relations de 2^B, est calculée, pour chaque paire de variables $v_i, v_j \in V$, une distance locale entre chaque relation de base $r \in 2^{\mathsf{B}}$ et le multi-ensemble composé des relations définissant les contraintes entre v_i et v_j des différents RCQ de $\mathcal{K}$, noté $\mathcal{K}[i,j]$ dans la suite. Le calcul de cette distance locale, notée $d(r, \mathcal{K}[i,j])$, nécessite également une fonction d'agrégation ; en utilisation la fonction d'agrégation $\sum$, il est défini de la manière suivante :

$$d(r, \mathcal{K}[i,j]) = \sum\{d(r, \mathcal{N}[i,j]) \text{ tel que } \mathcal{N} \in \mathcal{K}\}.$$

Étape 2. Les distances locales calculées précédemment sont agrégées afin de définir une distance globale entre chaque scénario cohérent S défini sur V et le multi-ensemble $\mathcal{K}$. En utilisant par exemple la fonction d'agrégation Max, cette distance globale est définie de la manière suivante :

$$d(S, \mathcal{K}) = \mathsf{Max}\{d(S[i,j], \mathcal{K}[i,j]) \text{ tel que } v_i, v_j \in V \text{ et } i < j\}.$$

De manière similaire aux opérateurs précédents, les scénarios cohérents définis sur V de plus petite distance globale sont retenus afin de définir le résultat de fusion.

3.3 Conclusion

Nous avons présenté en première partie de ce chapitre, un ensemble de travaux concernant l'étude d'une logique spatio-temporelle ($\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$) définie à partir de $PLTL$ pour laquelle les propositions ont été substituées par des contraintes qualitatives définies par l'ensemble de relations de base B. Pour certains ensemble de relations de base, déterminer si une formule de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ est satisfiable ou non est un problème PSPACE-complet. Nous avons également décrit nos travaux concernant les réseaux de contraintes périodiques nommées UPQCN. Ces réseaux de contraintes peuvent être considérés comme un sous langage de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$. Pour certaines classes de relations, nous avons vu que le problème de la cohérence d'un UPQCN est polynomial. Dans [CLST06], nous avons étudié des réseaux de contraintes périodiques définies à partir d'intervalles numériques. Nous avons montré que le problème de la cohérence de ces réseaux de contraintes est polynomial.

Dans la deuxième partie de ce chapitre, nous avons décrit nos travaux concernant le problème de la fusion des RCQ. Cette étude a été largement inspirée des travaux concernant la fusion de bases propositionnelles. Néanmoins, nous avons caractérisé différentes familles d'opérateurs utilisant des notions propres au cadre des formalismes qualitatifs pour le temps et l'espace. Nous avons par exemple utilisé des distances entre relations de base calculées à partir de graphes de voisinage conceptuel. Comme autre exemple propre au cadre qualitatif, nous pouvons citer les concepts de raffinements et d'abstractions d'ensembles de relations de base qui nous ont permis de définir des opérateurs hétérogènes. Ces travaux offrent des perspectives de recherche qui seront décrites dans la chapitre suivant.

Conclusion et perspectives de recherche

Nos travaux de recherche s'articulent autour d'études théoriques et pratiques de formalismes qualitatifs pour la représentation et le raisonnement sur le temps et l'espace. Ces études nous ont conduit à traiter ces formalismes sous différents aspects : résolution de contraintes, logiques, fusion d'information. Nos travaux concernant l'axiomatisation des relations de base de certains formalismes qualitatifs sont décrits dans le premier chapitre. Nos contributions concernant la résolution des réseaux de contraintes qualitatives sont présentées dans le deuxième chapitre. Enfin, dans le dernier chapitre, nous avons décrit d'une part nos travaux sur une logique spatio-temporelle et d'autre part ceux concernant la problématique de fusion des réseaux de contraintes qualitatives.
Ces travaux ouvrent de nombreuses perspectives de recherche autour du problème de la cohérence des RCQ et plus largement autour de problématiques liées à la représentation des connaissances par des contraintes qualitatives. Ils nous permettent également d'envisager des applications, notamment dans les domaines de l'apprentissage à distance et de la santé.

Autour du problème de la cohérence des RCQ

Six thématiques principales se dégagent de nos perspectives concernant la résolution du problème de la cohérence des RCQ.

Recherche de classes traitables. Étant donné un formalisme qualitatif, une étude primordiale consiste à déterminer précisément la frontière entre les ensembles de relations pour lesquels le problème de la cohérence est polynomial et ceux pour lesquels il ne l'est pas. Aujourd'hui, la totalité des classes traitables est connue pour certains formalismes qualitatifs. Néanmoins, des classes traitables restent à déterminer pour de nombreux autres formalismes, en particulier des classes pour lesquelles la $\diamond$-cohérence est complète.

Cohérences locales. Une première étude concernant notre nouvelle famille de cohérences locales, nommément la famille des $\overset{\diamond}{f}$-cohérences, a montré l'intérêt de considérer dans le cadre qualitatif des cohérences plus fortes que la $\diamond$-cohérence. Ces cohérences peuvent être utilisées pour caractériser des classes traitables. Nous pouvons en effet imaginer qu'une $\overset{\diamond}{f}$-cohérence soit complète pour le problème de la cohérence d'un ensemble de relations alors que la $\diamond$-cohérence ne l'est pas. Obtenir une cohérence locale plus forte que la $\diamond$-cohérence au travers d'une $\overset{\diamond}{f}$-cohérence peut également avoir un intérêt pratique, en particulier lors d'une phase de prétraitement du RCQ à résoudre. Une étude actuelle consiste à la mise en œuvre d'algorithmes efficaces de calcul de la fermeture par $\overset{\diamond}{f}$-cohérence d'un RCQ. Les algorithmes proposés pourront s'inspirer des études sur les cohérences locales SAC [BCDL11] et DisSAC [SV04] menées dans le cadre des CSP discrets. Associer une application de découpage f propre à chacune des contraintes d'un RCQ permet d'obtenir une nouvelle cohérence locale. Le cas particulier où la même application f est associée à toutes les contraintes nous ramènerait au cas d'une $\overset{\diamond}{f}$-cohérence. Cette nouvelle cohérence et d'autres pourront être étudiées dans le futur.

Propriétés structurelles des RCQ. Nos travaux concernant l'utilisation des décompositions arborescentes permettent d'obtenir des méthodes de résolution efficaces pour certains formalismes qualitatifs. Nous avons en effet montré que pour certaines classes traitables, l'algorithme de recherche habituellement utilisé dans le cadre de la résolution de RCQ reste complet lorsque sont uniquement considérés les triplets de variables appartenant à des regroupements d'une décomposition arborescente. Une économie de traitement est notamment réalisée pendant le filtrage des contraintes. Lors de ces études, nous n'avons pas pris en compte la structure des décompositions arborescentes mais seulement les regroupements de variables induits. Définir des algorithmes de recherche considérant cette structure est une de nos perspectives de travail. Comme pour l'algorithme BTD [JT03] utilisé dans le cadre des CSP discrets, nous pouvons utiliser d'une part les notions de *good* et de *nogood* afin de caractériser des instanciations de contraintes conduisant à une cohérence partielle ou une incohérence du RCQ lors de la recherche d'un scénario cohérent. D'autre part, l'ordre des groupements issu de la structure de la décomposition arborescente peut être employé afin de définir l'ordre dans lequel seront instanciées les différentes contraintes.

Résolution par SAT et CSP discrets. Malgré la très grande efficacité des méthodes actuelles de résolution du problème SAT et des CSP discrets, leur utilisation pour décider de la cohérence des RCQ se trouve affectée par la taille importante des instances obtenues par traduction. Une réponse récente à ce problème est l'utilisation des décompositions arborescentes afin d'éviter de traduire toutes les contraintes explicites d'un RCQ et toutes celles implicitement spécifiées par l'opération de faible composition. Les instances engendrées sont alors de moindre taille et plus rapidement résolues. Des techniques permettant de réduire les instances traduites restent à découvrir.

Différentes traductions des RCQ en problème SAT ont été définies dans la littérature. Dans ce cadre, nous avons proposé une traduction originale utilisant le découpage des contraintes en sous-relations convexes. La complétude de cette traduction nécessite que les relations de base soient arrangées dans un treillis possédant des propriétés particulières concernant l'opération de faible composition. Cette traduction est une première approche de traduction en problème SAT utilisant une classe traitable et mérite d'être étudiée plus en profondeur. Notamment, il serait intéressant de voir les liens entre une telle traduction et des encodages SAT des CSP discrets basés sur des domaines ordonnés linéairement [TTKB06, TTB11]. Des investigations doivent également être menées afin de définir des traductions à partir d'autres classes traitables telle que la classe des relations préconvexes. Définir une traduction générique basée sur toute classe traitable pour laquelle la $\diamond$-cohérence est complète est un objectif à atteindre.

Combinaisons de formalismes qualitatifs. Il n'est pas inhabituel de définir un formalisme qualitatif en combinant deux autres formalismes afin d'obtenir un langage plus riche. Le calcul INDU combine par exemple les relations de base du calcul des intervalles et celles du calcul des instants afin de pouvoir raisonner sur les rapports entre les durées des intervalles de la droite et leurs positions relatives. Les formalismes issus de telles combinaisons sont en règle générale basés sur un nombre relativement important de relations de base (25 relations de base pour INDU, 13^n pour l'algèbre des n-pavés, $\ldots$). Les techniques de résolution habituellement utilisées se trouvent affectées par ce nombre important de relations de base. Quelques méthodes spécifiques ont été proposées afin de résoudre des RCQ définis sur des combinaisons de formalismes qualitatifs. Dans [LLR09] par exemple, une méthode de calcul de fermeture par $\diamond$-cohérence est réalisée par un traitement itératif. Chaque étape de ce dernier consiste à calculer la fermeture par faible composition dans chacun des formalismes qualitatifs dont est issue la combinaison et à éliminer dans chacune des fermetures les relations de base non permises par les autres fermetures. Notons que cette méthode est similaire à celle que nous avions proposée dans le cadre de l'étude du calcul des intervalles enrichi de contraintes numériques [Con00]. Dans [LR10], une traduction en problème SAT des RCQ du calcul des rectangles est proposée et tire profit du fait que l'algèbre des rectangles est un double produit du calcul des intervalles. De manière générale, la résolution de RCQ issus de formalismes basés sur un nombre important de relations de base est un challenge important méritant une étude spécifique.

La très grande partie de nos travaux concerne des formalismes tenant uniquement compte de l'aspect qualitatif des informations temporelles ou spatiales. Néanmoins, certains concernent des formalismes considérant partiellement ou totalement des aspects numériques (l'algèbre des rectangles augmentés de contraintes quantitatives [Con00], les réseaux de contraintes temporelles périodiques et numériques [CLST06]). Dans certains cadres applicatifs, la prise en compte d'informations numériques est indispensable. Afin de répondre à cette exigence, nous souhaitons étudier des formalismes qualitatifs enrichis par des contraintes quantitatives (sur la taille des entités, les distances entre les entités, ...).

Implémentation. De nombreuses notions et techniques proposées lors de nos études ont été implémentées dans la librairie java QAT et dans des programmes *ad hoc* écrits en langage C/C++. La librairie QAT est une librairie complète et offre des fonctionnalités utilisables pour tout formalisme, quelque soit l'arité de ses relations de base. Afin de proposer des outils plus performants en terme de temps de traitement, une librairie en langage C++ est en cours de réalisation. Elle intègre l'ensemble des techniques et notions étudiées lors de nos recherches. Pour des raisons d'efficacité, elle sera principalement dédiée aux formalismes dont les relations de base sont d'arité 2.

Au delà du problème de la cohérence des RCQ

Nos perspectives liées à la représentation des connaissances par des contraintes qualitatives et ne concernant pas directement la résolution du problème de la cohérences des RCQ se déclinent au sein de quatre thématiques principales.

Problème de la minimalité des RCQ. Outre le problème de la cohérence, un autre problème important concernant les RCQ est celui de la minimalité. Étant donné un RCQ $\mathcal{N} = (V, C)$, ce dernier consiste à déterminer le sous-RCQ $\mathcal{N}'$ de $\mathcal{N}$, équivalent à $\mathcal{N}$ et tel que chacune de ses relations de base de ses contraintes appartienne à au moins un scénario cohérent de $\mathcal{N}$. Ce sous-RCQ $\mathcal{N}'$ peut également être vu comme l'union des scénarios cohérents de $\mathcal{N}$ et est appelé le RCQ minimal de $\mathcal{N}$. Le problème de la cohérence a donné lieu a bien plus de travaux que celui de la minimalité. Ceci s'explique par le fait que le problème de la minimalité peut se résoudre à l'aide d'un nombre d'appels polynomialement borné d'un algorithme résolvant celui de la cohérence. En effet, étant donné un RCQ $\mathcal{N} = (V, C)$, pour savoir si une relation de base r d'une contrainte C_{ij} appartient au RCQ minimal de $\mathcal{N}$, il suffit de substituer la relation définissant la contrainte C_{ij} par la relation singleton $\{r\}$ et de tester la cohérence du RCQ obtenu. Ce RCQ est cohérent si et seulement si r appartient au RCQ minimal. Une de nos perspectives de recherche est d'étudier des méthodes plus efficaces afin de résoudre le problème de la minimalité des RCQ. Des cohérences locales telles que les $\overset{\diamond}{f}$-cohérences, ainsi que les décompositions arborescentes, peuvent certainement être utilisées dans ce cadre.

Fusion de RCQ. La problématique de la fusion d'informations, en particulier d'informations temporelles et spatiales, est une problématique importante dans certaines applications. Pour résoudre le problème de la fusion de RCQ, nous avons défini différents opérateurs de fusion basés sur des calculs et des agrégations de distances entre scénarios et RCQ. Ces opérateurs sémantiques ou syntaxiques, homogènes ou hétérogènes peuvent être employés en fonction du type d'informations à gérer ou des propriétés attendues concernant le processus de fusion à réaliser. De nombreuses perspectives de recherche concernant

nos études sur la problématique de la fusion de RCQ sont à mener. L'une d'entre elles concerne la gestion d'informations hétérogènes. Nous avons déjà proposé des opérateurs de fusion permettant de traiter de telles informations. Ils opèrent sur des RCQ définis dans des formalismes qualitatifs dont les relations de base ont le même domaine D. Une perspective de recherche consiste à définir et à étudier des opérateurs considérant des RCQ dont les contraintes sont définies par des relations de base non forcément définies sur le même domaine. Cette étude peut s'inspirer des travaux sur la résolution des RCQ définis dans des formalismes qualitatifs issus de combinaisons de plusieurs formalismes qualitatifs. Les méthodes mises en œuvre dans ce cadre utilisent des passerelles permettant le transfert d'informations entre les différents formalismes.
Une autre perspective de recherche concerne l'étude de RCQ pondérés. Un RCQ pondéré est dans le cadre qualitatif l'équivalent d'un CSP pondéré [LS04]. Il est un RCQ particulier auquel est associé à chaque relation de base de chacune de ses contraintes un poids exprimé par un réel positif. Intuitivement, pour une contrainte donnée, plus le poids d'une relation de base est faible, moins la relation de base est préférée. En outre, un RCQ pondéré considère un opérateur permettant d'agréger les différents poids des relations de base constituant ses scénarios cohérents. La résolution d'un RCQ pondéré consiste à caractériser un ou plusieurs de ses scénarios cohérents de plus faible poids. Dans [CKMS10c], nous avons montré que le résultat de certains opérateurs syntaxiques de fusion de RCQ peut se calculer au travers de la résolution de RCQ pondérés. La résolution de RCQ pondérés est donc une alternative à certaines de nos approches de fusion de RCQ. Définir et étudier des méthodes efficaces de résolution de RCQ pondérés font partie de nos perspectives de recherche. Cette étude pourra être menée en s'inspirant des approches suivies en Intelligence Artificielle dans le cadre des CSP pondérés et des approches utilisées dans le cadre de la résolution de problèmes d'optimisation en Recherche Opérationnelle.

Axiomatisations en logique du premier ordre. Des axiomatisations en logique du premier ordre des relations de base permettent d'obtenir des langages plus expressifs que les RCQ. Des méthodes de raisonnement telles que des méthodes d'élimination des quantificateurs peuvent être associées à ces axiomatisations. Des investigations concernant ce domaine font partie de nos perspectives de recherche. En premier lieu, nous souhaitons implémenter les différentes méthodes d'élimination de quantificateurs que nous avons proposées dans le cadre du calcul des points cycliques et du calcul des intervalles cycliques. Il serait également intéressant de caractériser des axiomatisations de relations de base de formalismes qualitatifs non encore considérés dans ce cadre, comme le calcul INDU. Une axiomatisation concernant ce calcul permettrait de caractériser des propriétés utiles afin de définir des méthodes de résolution efficaces pour le problème de la cohérence (rappelons que pour ce formalismes la méthode de la $\diamond$-cohérence n'est pas complète pour le problème de la cohérence des scénarios).

Logiques spatio-temporelles. La logique $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ correspondant à la logique PLTL dont les propositions sont définies par des contraintes qualitatives offre un langage extrêmement riche. De nombreuses études la concernant sont à envisager. La première concerne la caractérisation de sous-langages polynomiaux ou NP-complets de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$. La définition des UPQCN était une première contribution concernant cette perspective. Des algorithmes permettant de caractériser un modèle ou de décider de la satisfiabilité d'une formule pour ces fragments ou pour tout le langage doivent être également définis et étudiés. Ils pourront par exemple être basés sur des méthodes des tableaux combinées avec des techniques de résolution de contraintes qualitatives.
Différentes problématiques méritant une étude à part entière peuvent facilement s'exprimer à l'aide de la logique $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$. Comme exemple, nous pouvons citer une problématique de planification qui consiste, étant donné un ensemble d'entités spatiales et une configuration sur ces entités, à déterminer un ensemble de configurations spatiales permettant d'y mener. Des contraintes régissant les déplacements ou expri-

mant des propriétés de ces entités peuvent être définies à l'aide de formules de $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ afin de délimiter les configurations possibles à un instant donné à partir d'une configuration satisfaite à l'instant précédent. Une autre perspective de recherche consiste en l'étude de logiques spatio-temporelles basées sur d'autres logiques temporelles que PLTL comme CTL [CES86] et CTL* [EH86].

Applications

Actuellement, nous nous intéressons à deux domaines applicatifs : l'apprentissage à distance et le domaine de la santé.

Apprentissage à distance. Un système d'apprentissage à distance concerne différents acteurs (apprentis, tuteurs, auteurs, administrateurs,...) accédant à différentes ressources pédagogiques ou matérielles selon certaines contraintes parmi lesquelles, des contraintes de ressources et des contraintes temporelles définissant l'enchaînement des différentes activités à réaliser. Par exemple, l'activation d'une activité particulière peut dépendre des résultats de l'étudiant obtenus à une activité précédemment réalisée. Des activités peuvent également être conduites de manière concurrente, d'autres peuvent exiger d'être réalisées simultanément par plusieurs étudiants. Certaines de ces contraintes peuvent être dans un premier temps définies avant l'apprentissage à l'aide d'un modèle formel. Elles sont dans un deuxième temps implémentées dans la plateforme d'apprentissage en ligne pour être prises en compte lors de l'apprentissage. En considérant des modèles très utilisés tels que le standard *IMS Simple Sequencing* ou bien encore le standard plus récent *IMS Learning Design*, nous pouvons constater que les aspects temporels ne sont pas pris en compte explicitement ou que leur spécification s'effectue de manière trop contraignante. Par exemple, dans le cas du standard *IMS Simple Sequencing*, les scénarios pouvant être spécifiés correspondent à des enchaînements séquentiels d'activités. Notre objectif est de proposer un modèle général permettant de définir explicitement des contraintes temporelles devant être vérifiées lors d'un apprentissage à distance. Nous souhaitons développer des outils logiciels permettant de valider des spécifications temporelles concernant un apprentissage à distance, ainsi que des outils permettant à l'environnement d'exécution de contraindre les utilisateurs du système d'apprentissage à satisfaire ces différentes spécifications. Dans ce cadre, une étude préliminaire [NC10] nous a conduit à étudier des réseaux de contraintes qualitatives dynamiques permettant de gérer des contraintes temporelles activables ou non en fonction de certains états du système.

Domaine de la santé. Le deuxième cadre applicatif pour lequel nous souhaitons mettre en œuvre nos résultats théoriques concerne le domaine de la santé et plus particulièrement celui de la prise en charge de personnes âgées, à domicile ou au sein de structures spécialisées de type EHPAD (Établissement d'Hébergement pour Personnes Âgées Dépendantes). Pour cette application, l'environnement est équipé de capteurs capables de détecter certaines actions ou activités effectuées par les personnes âgées. L'un de nos objectifs principaux est de fournir des outils permettant de détecter, à partir des différentes observations réalisées, des comportements anormaux ou déviants afin de prendre des décisions appropriées. L'étude consisterait d'une part à définir un modèle permettant de spécifier les différents scénarios normaux ou anormaux. D'autre part, des méthodes d'analyse combinant ces spécifications et les observations doivent être mises en œuvre afin de détecter les comportements atypiques des personnes âgées. Des méthodes de raisonnement associées à la logique $\mathcal{L}^{\mathsf{B}}_{\mathsf{PLTL}}$ peuvent certainement avoir toute leur utilité dans ce contexte.

La synthèse de nos activités s'achève donc par nos perspectives de recherche. Certaines s'inscrivent dans le continuité de nos travaux, d'autres orientations sont plus originales et laissent envisager à long terme l'enrichissement du raisonnement sur le temps et l'espace.

Deuxième partie

Curriculum vitae

Jean-François CONDOTTA

Maître de conférences en Informatique

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Statut actuel

Maître de conférences (section CNU 27 - Informatique), CRIL-CNRS, Université d'Artois
IUT de Lens (Département Informatique), PES.

État civil

Né le 04 septembre 1972 à Toulouse, nationalité française, célibataire, un enfant.

Domaines de recherche

Algorithmique pour l'inférence et la prise de décision, représentations et raisonnements pour le temps et l'espace, résolution de contraintes, logiques.

Formations

14 janvier 2000	**Doctorat en Informatique**, Université Paul Sabatier (Toulouse III).
juin 1997	**D.E.A. Représentation de la Connaissance et Formalisation du Raisonnement**, mention Bien (major de promotion), Université Paul Sabatier (Toulouse III).
juin 1996	**Maîtrise d'informatique mention Systèmes Informatiques Multimédia**, mention Bien, Université Paul Sabatier (Toulouse III).
Autres	Licence d'informatique (1995), D.E.U.G. Mathématiques Mécanique Physique Informatique (1994), Université Paul Sabatier (Toulouse III).

Stages de recherche et activités professionnelles

De sept. 2003 à maintenant	**Maître de conférences en Informatique à l'université d'Artois**, lieu d'enseignement : IUT de Lens, département GEA (jusqu'en août 2009), département Informatique (depuis septembre 2009), laboratoire de recherche : Centre de Recherche en Informatique de Lens (CRIL), PEDR (2004-2008), PES (depuis 2009).
De sept. 2001 à août 2003	**Maître de conférences en Informatique à l'université Paris-Sud (Paris XI)**, lieu d'enseignement : Faculté des sciences d'Orsay, laboratoire de recherche : Laboratoire d'Informatique pour la Mécanique et les Sciences de l'Ingénieur (LIMSI).
De sept. 2000 à août 2001	**ATER à l'université Paul Sabatier (Toulouse III)**, au sein de l'équipe de logique appliquée de l'IRIT (Institut de Recherche en Informatique de Toulouse).

D'octobre 1997 à août 2000 — **Moniteur et allocataire de recherche MENRT à l'université Paul Sabatier (Toulouse III)**, au sein de l'équipe de logique appliquée de l'IRIT. Durant cette période j'ai effectué, sous la direction de Luis Farinas del Cerro et Phillipe Balbiani, ma thèse qui consistait principalement en l'étude de problèmes de satisfaction de contraintes spatiales.

D'octobre 1996 à septembre 1997 — **Stage de DEA**, effectué au sein de l'équipe de logique appliquée de l'IRIT. Mon stage était dirigé par Vincent Dugat et Luis Farinas del Cerro et consistait en l'étude de formalismes temporels et spatiaux proposés en Intelligence Artificielle.

Responsabilités administratives

Commissions de choix et de sélection

2010 — Membre nommé de la commission de choix de l'université d'Artois pour le poste d'enseignant du second degré n°0171 Mathématiques pour les TICE.

2009 — Membre élu de la commission de choix de l'université d'Artois pour le poste de MCF en Informatique n°0267.

2008 — Membre nommé de la commission de choix de l'université d'Artois pour le poste d'enseignant du second degré n°0400 en économie-droit.

2005 à 2008 — Membre élu de la CSE (sections 25-26-27) de l'université d'Artois.

Conseils d'établissement

2002 à 2003 — Membre élu de la commission Communication Homme-Machine (CHM) du LIMSI.

2006 à 2009 — Membre élu du conseil de l'IUT de Lens.

Responsabilités liées à des formations et l'établissement

2010 à maintenant — Responsable de la Licence Professionnelle SIL Santé, porteur du projet de création de cette licence en 2009, co-responsable avec Maryse Roger depuis septembre 2011.

2008 à 2009 — Responsable de la Cellule de Transfert et Technologie (informatique) de l'IUT de Lens.

2006 à 2009 — Chef du département Gestion des Entreprises et des Administrations de l'IUT de Lens (le département GEA comprend trois formations principales : la licence pro. collaborateur comptable-TIC, la licence pro. Ressource Humaine - TIC et le DUT GEA, pour un total de plus de 300 étudiants encadrés par une douzaine d'enseignants permanents).

Responsabilités liées aux TICE

2008 à maintenant — Membre du comité TICE de l'université d'Artois et correspondant TICE de l'IUT de Lens auprès de l'université d'Artois.

2007 à 2011 — Responsable C2i du département GEA (jusqu'en 2009) et du DUT1 du département Informatique (jusqu'en 2011).

2010 — Participation au comité de pilotage du lancement du réseau social de l'université d'Artois.

Activités d'enseignements

Les enseignements réalisés concernent essentiellement les domaines de la programmation et des systèmes d'information. Voici de manière synthétique la liste des enseignements que j'ai pu réalisés lors de mon parcours :

- En tant que MCF à l'IUT de Lens au Département Informatique (depuis septembre 2009)
  - Algorithmique et Programmation (Langage C) - DUT Informatique
  - Projet d'Algorithmique (Langage C) - DUT Informatique
  - Algorithmique et Programmation (C++/Java) - Licence Pro. SIL Sécurité
  - Tableur Avancé - Licence Pro. SIL Santé
  - Algorithmique et Programmation (Java) - DUT SRC
  - Programmation WEB (JavaScript) - DUT SRC
- En tant que MCF à l'IUT de Lens au Département GEA (entre septembre 2003-2009)
  - Systèmes d'Informations de Gestion (SGBDR, Merise, Access) - DUT GEA
  - Tableurs, Traitements de Texte (Excel/Word) et C2I - DUT GEA
  - Tableur Avancé - Licence Pro. Collaborateur Comptable – TIC
  - E-Learning - Licence Pro. RH-TIC
- En tant que MCF à l'université Paris XI (Orsay) (entre septembre 2001-2003)
  - Algorithme et Programmation (Pascal) - L1
  - Bases de données relationnelles (MCD, AR, SQL) - L3
  - Bases de données mutli-dimensionnelles - MIAGE
  - Programmation Objet (Java) – ENSTA
- En tant que Moniteur et ATER à l'université Toulouse III (Paul Sabatier)
  - Algorithme et Programmation (Pascal) - L1

J'ai également effectué des cours sur le raisonnement spatial et temporel qualitatif dans le cadre des DEA "Information, Interaction, Intelligence" et "Sciences Cognitives" de Paris XI à Orsay en 2002-2003 (8 heures), et dans le cadre du DEA "Systèmes intelligents et applications" de l'université d'Artois, faculté de Lens en 2003 (4 heures). Depuis 2003, j'encadre chaque année des étudiants lors de leur stage en entreprise (DUT Info/GEA, Licence Pro. SIL Santé/RH-TIC).

Co-encadrement de thèses et de master recherche

Co-encadrement de master recherche

2002 **Abdelslam Nasri (LIMSI)**, co-encadrement à 50%, avec Gérard Ligozat.

2005 **Ali Zitouni**, co-encadrement à 50%, avec Souhila Kaci et Pierre Marquis.

2006 **Dominique D'Almeida**, co-encadrement à 33%, avec Christophe Lecoutre et Lakhdar Saïs.

2007 **Nicolas Schwind**, co-encadrement à 33%, avec Souhila Kaci et Pierre Marquis.

Co-encadrement de thèses

2005-2008 **Mahmoud Saade**, «Étude du raisonnement temporel basé sur la résolution de contraintes», soutenue le 15 décembre 2008, co-encadrement à 90%, avec Pierre Marquis.

2006-2010 **Dominiqe D'Almeida**, «Étude de systèmes de contraintes pour le raisonnement qualitatif temporel et spatial», soutenue le 3 décembre 2010, co-encadrement à 33%, avec Christophe Lecoutre et Lakhdar Saïs.

2007-2010 **Nicolas Schwind**, «Fusion de réseaux de contraintes qualitatives», soutenue le 10 décembre 2010, co-encadrement à 33%, avec Souhila Kaci et Pierre Marquis.

Projets scientifiques

2001 à 2005 **Projet Sémantiques de la transmodalité : application aux systèmes d'information géographique**, dans le cadre du programme interdisciplinaire "société de l'information", projet 2001-085 et sous la responsabilité de Gérard Ligozat (LIMSI).

2005 à 2008 **Projet ANR – PLANEVO**, objectif : réalisation d'un système de planification capable de prendre en compte une représentation complexe de l'univers, des actions et des buts à atteindre dans un cadre temporel, sous la responsabilité de Vincent Vidal (CRIL).

Comités de programme, d'organisation et de lecture

Comités de programme

2011 Workshop Benchmarks and Applications of Spatial Reasoning (IJCAI'11).

2009 AAAI Spring Symposium - Benchmarking of Qualitative Spatial and Temporal Reasoning Systems, Stanford University.

2006 à maintenant Atelier Représentation et Raisonnement sur le Temps et l'Espace (RTE).

2004 Symposium on Temporal Representation and Reasoning (TIME'04).

2003 Rencontres de Jeunes Chercheurs en Intelligence Artificielle (RJCIA'03).

Comités d'organisation

2005 (Journées Francophones par Programmation par Contraintes (JFPC'05)

Relectures

Conférences. RJCIA'03, RTE (07,09,11), RFIA (02,04),ICTAI'10, IJCAI (03,05,07,09), TIME'04, ECAI'04, CP (05,11), KR'08, etc.

Revues. Artificial Intelligence Journal (AIJ), Journal of Artificial Intelligence Research (JAIR)

Publications

Chapitre dans un ouvrage

[1] Jean-François Condotta. *Raisonnement sur l'espace et le temps*, chapter 7, pages 181–223. IGAT. Lavoisier, Hermes.

Revues internationales

[1] Philippe Balbiani, Jean-François Condotta, and Gérard Ligozat. On the consistency problem for the INDU calculus. *Journal of Applied Logic*, 4(2):119–140, 2006.

[2] Gérard Ligozat and Jean-François Condotta. On the relevance of conceptual spaces for spatial

and temporal reasoning. *Spatial Cognition and Computation*, 5(1):1–27, 2005.

[3] Gérard Ligozat, Debasis Mitra, and Jean-François Condotta. Spatial and temporal reasoning: Beyond allen's calculus. *AI Communications on Spatial and Temporal Reasoning*, 17(4):223–233, 2004.

[4] Jean-François Condotta. A general qualitative framework for temporal and spatial reasoning. *Constraints, Kluwer*, 9(2):99–121, 2004.

[5] Philippe Balbiani and Jean-François Condotta. Spatial reasoning about points in a multidimensional setting. *Applied Intelligence*, 17(3):221–238, 2002.

[6] Philippe Balbiani, Jean-François Condotta, and Luis Farinas del Cerro. Tractability results in the block algebra. *Journal of Logic and Computation, Oxford University Press*, 12(5):885–909, 2002.

Conférences internationales avec comité de lecture

[1] Jean-François Condotta and Christophe Lecoutre. A framework for decision-based consistencies. In Jimmy Ho-Man Lee, editor, *Proceedings of the 17th International Conference Principles and Practice of Constraint Programming, (CP'11)*, volume 6876 of *Lecture Notes in Computer Science*, pages 172–186. Springer, 2011.

[2] Jean-François Condotta and Dominique D'Almeida. Consistency of qualitative constraint networks from tree decompositions. In Carlo Combi, Martin Leucker, and Franck Wolter, editors, *Proceedings of the 18th International Symposium on Temporal Representation an Reasoning (TIME'11), Lübeck, Germany*, pages 149–156, 2011.

[3] Assef Chmeiss and Jean-François Condotta. Consistency of triangulated temporal qualitative constraint networks (to appear). In *Proceedings of the 23th IEEE International Conference on Tools with Artificial Intelligence (ICTAI'11), à paraître*, 2011.

[4] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. Majority merging: from boolean spaces to affine spaces. In Helder Coelho, Rudi Studer, and Michael Wooldridge, editors, *Proceedings of the 19th European Conference on Artificial Intelligence (ECAI'10), Lisbon, Portugal*, pages 627–632, 2010.

[5] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. A syntactical approach to qualitative constraint networks merging. In Christian G. Fermüller and Andrei Voronkov, editors, *Proceedings of the 17th International Conference Logic for Programming, Artificial Intelligence, and Reasoning (LPAR'10), Indonesia*, volume 6397 of *Lecture Notes in Computer Science*, pages 233–247. Springer, 2010.

[6] Jean-François Condotta and Christophe Lecoutre. A class of df-consistencies for qualitative constraint networks. In Fangzhen Lin, Ulrike Sattler, and Miroslaw Truszczynski, editors, *Proceedings of the Twelfth International Conference Principles of Knowledge Representation and Reasoning (KR'10), Toronto, Canada*, pages 319–328. AAAI Press, 2010.

[7] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. Merging qualitative constraint networks defined on different qualitative formalisms. In Kathleen Stewart Hornsby, Christophe Claramunt, Michel Denis, and Gérard Ligozat, editors, *Proceedings of the 9th International Conference Spatial Information Theory (COSIT'09)*, volume 5756 of *Lecture Notes in Computer Science*, pages 106–123, 2009.

[8] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. Merging qualitative constraints networks using propositional logic. In Claudio Sossai and Gaetano Chemello, editors, *Proceedings of 10th European Conference Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU'09)*, volume 5590 of *Lecture Notes in Computer Science*, pages 347–358, 2009.

[9] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. Merging qualitative constraint networks in a piecewise fashion. In *Proceedings of the 21st International Conference on Tools with Artificial Intelligence (ICTAI'09), Newark, USA*, pages 605–608. IEEE Computer Society, 2009.

[10] Jean-François Condotta, Souhila Kaci, and Nicolas Schwind. A framework for merging qualitative constraints networks. In David Wilson and H. Chad Lane, editors, *Proceedings of the Twenty-First International Florida Artificial Intelligence Research Society Conference (FLAIRS'08)*, pages 586–591. AAAI Press, 2008.

[11] Dominique D'Almeida, Jean-François Condotta, Christophe Lecoutre, and Lakhdar Saïs. Relaxation of qualitative constraint networks. In Ian Miguel and Wheeler Ruml, editors, *Proceedings of the 7th Symposium on Abstraction, Reformulation and Approximation (SARA'07)*, pages 93–108, Canada, 2007. LNCS 4612, Springer.

[12] Jean-François Condotta and Dominique D'Almeida. Qualitative constraints representation for the time and space in sat. In *Proceedings of the 19th IEEE International Conference on Tools with Artificial Intelligence (ICTAI'07)*, pages 74–77. IEEE Computer Society, 2007.

[13] Jean-François Condotta, Gérard Ligozat, and Mahmoud Saade. Eligible and frozen constraints for solving temporal qualitative constraint networks. In Christian Bessiere, editor, *Proceedings of the 13th International Conference Principles and Practice of Constraint Programming (CP'07), Providence,USA*, volume 4741 of *Lecture Notes in Computer Science*, pages 806–814. Springer, 2007.

[14] Jean-François Condotta, G. Ligozat, and Mahmoud Saade. A generic toolkit for n-ary qualitative temporal and spatial calculi. In *Proceedings of the 13th International Symposium on Temporal Representation and Reasoning (TIME'06)*, pages 78–86, Budapest, Hungary, 2006.

[15] Jean-François Condotta, Gérard Ligozat, Mahmoud Saade, and Stavros Tripakis. Ultimately periodic simple temporal problems (UPSTPs). In *Proceedings of the 13th International Symposium on Temporal Representation and Reasoning (TIME'06)*, pages 69–77. IEEE Computer Society, 2006.

[16] Jean-François Condotta, Gérard Ligozat, and Stavros Tripakis. Ultimately periodic qualitative constraint networks for spatial and temporal reasoning. In *Proceedings of the 17th IEEE International Conference on Tools with Artificial Intelligence (ICTAI'05), Hong Kong, China*, pages 584–588, 2005.

[17] Jean-François Condotta and Gérard Ligozat. Axiomatizing the cyclic interval calculus. In Didier Dubois, Christopher A. Welty, and Mary-Anne Williams, editors, *Proceedings of the Ninth International Conference Principles of Knowledge Representation and Reasoning (KR'04), Whistler,Canada*, pages 348–371. AAAI Press, 2004.

[18] Philippe Balbiani, Jean-François Condotta, and Gérard Ligozat. Reasoning about cyclic space: Axiomatic and computational aspects. In Christian Freksa, Wilfried Brauer, Christopher Habel, and Karl Friedrich Wender, editors, *Proceedings of Spatial Cognition 2003*, volume 2685 of *Lecture Notes in Computer Science*, pages 348–371. Springer, 2003.

[19] Philippe Balbiani, Jean-François Condotta, and Gérard Ligozat. On the consistency problem for the indu calculus. In *Proceedings of the 10th International Symposium on Temporal Representation and Reasoning (TIME'03), Cairns, Australia*, pages 203–211, 2003.

[20] Philippe Balbiani and Jean-François Condotta. Computational complexity of propositional linear temporal logics based on qualitative spatial or temporal reasoning. In Alessandro Armando, editor, *Proceedings of the 4th International Workshop on Frontiers of Combining Systems (FroCoS'02)*, volume 2309 of *Lecture Notes in Computer Science*, pages 162–176. Springer, 2002.

[21] Jean-François Condotta. The augmented interval and rectangle networks. In A.G. Cohn, F. Giunchiglia, and B. Selman, editors, *Proceedings of the Seventh International Conference on Principles of Knowledge (KR'00), Brekenbridge, USA*. Morgan Kaufmann Publishers, 2000.

[22] Jean-François Condotta. Tractable sets of the generalized interval algebra. In Werner Horn, editor, *Proceedings of the 14th European Conference on Artificial Intelligence (ECAI'00), Berlin*, 2000.

[23] Philippe Balbiani, Jean-François Condotta, and Gérard Ligozat. Reasoning about generalized intervals: Horn representability and tractability. In *Proceedings of the seventh international workshop on Temporal Representation an Reasoning (TIME'2000), Canada*, pages 23–30, 2000.

[24] Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. A new tractable subclass of the rectangle algebra. In Thomas Dean, editor, *Proceedings of the Sixteenth International Joint Conference on Artificial Intelligence (IJCAI'99)*, pages 442–447. Morgan Kaufmann, 1999.

[25] Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. A tractable subclass of the block algebra: constraint propagation and preconvex relations. In Pedro Barahona and José Júlio Alferes, editors, *The Ninth Portuguese Conf. on Artificial Intelligence (EPIA'99), Evora, Portugal*, volume 1695 of *Lecture Notes in Computer Science*, pages 75–89. Springer, 1999.

[26] Philippe Balbiani, Jean-François Condotta, Luis Fariñas del Cerro, and Aomar Osmani. A model for reasoning about generalized intervals. In F. Giunchiglia, editor, *Proceedings of the Eighth International Conference on Artificial Intelligence : Methods, Systems, Applications (AIMSA'98), LNAI 1480*, pages 50–61, 1998.

[27] Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. A model for reasoning about bidimensional temporal relations. In A. G. Cohn, L. Schubert, and S. C. Shapiro, editors, *Proceedings of the Sixth International Conference on Principles of Knowledge Representation and Reasoning (KR'98)*, pages 124–130. Morgan Kaufmann, 1998.

Workshops internationaux avec comité de lecture

[1] Jean-François Condotta, Gérard Ligozat, and Mahmoud Saade. A qualitative constraints for job shop scheduling. Technical Report AAAI Technical Report SS-09-02, AAAI Spring Symposium on Benchmarking of Qualitative Spatial and Temporal Reasoning Systems. 2009, 2009.

[2] Jean-François Condotta, Gérard Ligozat, and M. Saade. An empirical study of algorithms for qualitative temporal or spatial networks. In *Proceedings of the workshop on spatial reasoning (ECAI'06)*, pages 34–43, 2006.

[3] Jean-François Condotta Condotta, G. Ligozat, and Mahmoud Saade. A qualitative algebra toolkit. In *2nd IEEE International Conference on Information Technologies: from Theory to Applications (ICTTA'06)*, pages 1251–1252, Damascus, Syria, apr 2006.

[4] Dominique D'Almeida, Jean-François Condotta, Christophe Lecoutre, and Lakhdar Saïs. Relaxation of qualitative constraint networks. In Wölfl Stefan and Mossakowski Till, editors, *Proceedings of Workshop Qualitative Constraint Calculi Application and Integration (KI'06)*, pages 54–64, 2006.

[5] Gérard Ligozat, Debasis Mitra, and Jean-François Condotta. Spatial and temporal reasoning: Beyond allen's calculus. In *Proceedings of AAAI Spring Symposium on Foundations and Applications of Spatio-Temporal Reasoning (FASTR), AAAI Technical Report SS-03-03*, pages 46–53, 2003.

[6] Philippe Balbiani, Jean-François Condotta, and Luis Fariñas del Cerro. Spatial reasoning about points in a multidimensional setting. In *Proceedings of the workshop on temporal and spatial reasoning (IJCAI'99)*, pages 105–113, 1999.

[7] Philippe Balbiani, Jean-François Condotta, Luis Fariñas del Cerro, and Aomar Osmani. A model for reasoning about generalized intervals. In *Proceedings of the workshop on spatial reasoning (ECAI'98)*, 1998.

Conférences nationales avec comité de lecture

[1] Jean-François Condotta. Cohérence de réseaux de contraintes qualitatives spatio-temporelles à partir de décompositions arborescentes. In *Actes des Journées d'Intelligence Artificielle Fondamentale (IAF'11),Lyon, France*, 2011.

[2] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. Une approche syntaxique pour le problème de la fusion de réseaux de contraintes qualitatives. In *Sixièmes Journées Francophones de Programmation par Contraintes(JFPC'10)*, pages 103–112, Caen, France, 2010.

[3] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. Fusion majoritaire : des espaces booléens aux espaces affines. In *Actes des Journées Nationales de l'Intelligence Artificielle Fondamentale (IAF'10)*, Strasbourg, France, jun 2010.

[4] Amaneddine Nouhad and Jean-François Condotta. Scénarisation de processus d'apprentissage en ligne à l'aide de réseaux de contraintes qualitatives temporelles dynamiques. In *Actes du 17ème congrès francophone Reconnaissance des Formes et Intelligence Artificielle (RFIA'08)*, Caen, France, 2010.

[5] Jean-François Condotta and Dominique D'Almeida. Représentation de contraintes qualitative pour le temps et l'espace en sat. In *Actes du 16 ème congrès francophone Reconnaissance des Formes et Intelligence Artificielle (RFIA'08)*, pages 268–275, Amiens, jan 2008.

[6] Jean-François Condotta, Souhila Kaci, Pierre Marquis, and Nicolas Schwind. Utiliser la logique propositionnelle pour la fusion de réseaux de contraintes qualitatives. In *Représentation et Raisonnement sur le Temps et l'Espace (RTE'08)*, Montpellier, France, jun 2008.

[7] Jean-François Condotta, Gérard Ligozat, and Mahmoud Saade. Eligibilité et gel de contraintes pour la résolution de réseaux de contraintes qualitatives temporelles et spatiales. In *Actes du 16ème congrès francophone Reconnaissance des Formes et Intelligence Artificielle (RFIA'08)*, Amiens, France, jan 2008.

[8] Jean-François Condotta, G. Ligozat, and Mahmoud Saade. Qat : une boîte à outils dédiée aux algèbres qualitatives. In *Semaine de la Connaissance (SdC'06)*, volume 4, pages 149–155, Nantes, jun 2006.

[9] Jean-François Condotta, G. Ligozat, and Stavros Tripakis. Réseaux de contraintes quantitatives périodiques. In *Premières Journées Francophones de la Programmation par Contraintes(JFPC'05)*, pages 287–296, 2005.

[10] Jean-François Condotta and Gérard Ligozat. Une axiomatisation des relations qualitatives du calcul des intervalles cycliques. In *Actes du 14 ème congrès francophone Reconnaissance des Formes et Intelligence Artificielle (RFIA'04),Toulouse (France)*, volume 2, pages 1071–1080, 2004.

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[12] Philippe Balbiani and Jean-François Condotta. Une logique temporelle pour décrire l'évolution des positions relatives des agents dans l'espace. In *Actes des Journées Francophones Modèles Formels de l'Interaction (MFI'01)*, Arras, 2001.

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Troisième partie

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- Computational complexity of propositional linear temporal logics based on qualitative spatial or temporal reasoning. Philippe Balbiani and Jean-François Condotta. In Alessandro Armando, editor, *Proceedings of the 4th International Workshop on Frontiers of Combining Systems (FroCoS'02)*, volume 2309 of *Lecture Notes in Computer Science*, pages 162–176. Springer, 2002. (Page 85)

- Axiomatizing the cyclic interval calculus. Jean-François Condotta and Gérard Ligozat. In Didier Dubois, Christopher A. Welty, and Mary-Anne Williams, editors, *Proceedings of the Ninth International Conference Principles of Knowledge Representation and Reasoning (KR'04), Whistler,Canada*, pages 348–371. AAAI Press, 2004. (Page 103)

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Computational complexity of propositional linear temporal logics based on qualitative spatial or temporal reasoning

Philippe Balbiani and Jean-François Condotta.
In Alessandro Armando, editor, *Proceedings of the 4th International Workshop on Frontiers of Combining Systems (FroCoS'02)*, volume 2309 of *Lecture Notes in Computer Science*, pages 162–176. Springer, 2002.

Computational Complexity of Propositional Linear Temporal Logics Based on Qualitative Spatial or Temporal Reasoning

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Abstract. We consider the language obtained by mixing the model of the regions and the propositional linear temporal logic. In particular, we propose alternative languages where the model of the regions is replaced by different forms of qualitative spatial or temporal reasoning. In these languages, qualitative formulas describe the movement and the relative positions of spatial or temporal entities in some spatial or temporal universe. This paper addresses the issue of the formal proof that for all forms of qualitative spatial and temporal reasoning such that consistent atomic constraint satisfaction problems are globally consistent, determining of any given qualitative formula whether it is satisfiable or not is PSPACE-complete.

1 Introduction

Many real-world problems involve qualitative reasoning about space or time. Accordingly, the development of intelligent systems that relate to spatial or temporal information is gaining in importance. In the majority of cases, these intelligent systems provide specialized tools for knowledge representation and reasoning about qualitative relations between spatial or temporal entities. To illustrate the truth of this, one has only to mention the model of the regions designed by Randell, Cui and Cohn [15] and the model of the intervals elaborated by Allen [1]. In actual fact, there are many more models based on alternative qualitative relations between other spatial or temporal entities, see Balbiani and Condotta [2], Balbiani, Condotta and Fariñas del Cerro [3], Balbiani and Osmani [4], Cristani [7], Gerevini and Renz [9], Isli and Cohn [10], Ligozat [11], Ligozat [12], Moratz, Renz and Wolter [13] and Vilain and Kautz [19]. For instance, Ligozat [12] shows how to formulate our knowledge of the relative positions of objects represented by pairs of real numbers. For this purpose, he considers the 9 jointly exhaustive and pairwise distinct atomic relations obtained by comparing the relative positions of points in the real plane: south-west, south, south-east, west, east, north-west, north, north-east and equality. Take another example: Vilain and Kautz [19] demonstrate how to express our knowledge of

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the relative moments of events represented by real numbers. In this respect, they consider the 3 jointly exhaustive and pairwise distinct atomic relations obtained by comparing the relative moments of points on the real line: before, after and equality. In these models, knowledge representation and reasoning is performed through networks of constraints between spatial or temporal entities and the main issue consists in deciding consistency of such networks. This brings several researchers to the following question: what is the computational complexity of determining of any given spatial or temporal network of constraints between spatial or temporal entities whether it is consistent or not ?

Numerous applications require support for knowledge representation and reasoning about spatial and temporal relationships between moving objects. We should consider the following example: breaking up the map into its component parts, a geographer sees 2-dimensional objects and has to reason about the issue of the links between portions of space that continuously evolve as time goes by. That is the reason why several researchers made a resolution to devote themselves to the integration of spatial and temporal concepts into a single hybrid formalism. Among the hybrid formalisms for reasoning about space and time considered in computer science, there is nothing to compare with the language introduced by Wolter and Zakharyaschev [20] and obtained by mixing the model of the regions and the propositional linear temporal logic. The propositional linear temporal logic is asserting itself as one of the better known model for reasoning about program properties within the framework of the research carried out into the subject of specification and verification of reactive systems. Its combination with the model of the regions gives rise to a language of very great expressivity. Since determining of any given spatial network of constraints between regions whether it is consistent or not is in NP, see Nebel [14] and Renz and Nebel [16], whereas determining of any given formula of propositional linear temporal logic whether it is satisfiable or not is in PSPACE, see Sistla and Clarke [18], there is reason to believe that determining of any given formula of this language whether it is satisfiable or not is in PSPACE.

The one drawback is that the EXPSPACE upper bound for the complexity of the satisfiability problem for the formulas of the language introduced by Wolter and Zakharyaschev [20] does not coincide with the PSPACE-hardness lower bound. This induces us to extend the results obtained by Wolter and Zakharyaschev [20] to different forms of qualitative spatial or temporal reasoning. We aim to propose alternative languages where the model of the regions is replaced by different forms of qualitative spatial or temporal reasoning such that consistent atomic constraint satisfaction problems are globally consistent. In these languages, qualitative formulas describe the movement and the relative positions of spatial or temporal entities in some spatial or temporal universe. The requirement that consistent atomic constraint satisfaction problems are globally consistent is a sufficient condition for the fulfilment of our objective: the formal proof that determining of any given qualitative formula whether it is satisfiable or not is PSPACE-complete. Numerous forms of qualitative spatial or temporal reasoning fit this requirement, one has only to mention the models introduced by Allen [1], Balbiani and Condotta [2], Balbiani, Condotta and Fariñas del Cerro [3], Cristani [7], Ligozat [11], Ligozat [12] and Vilain and Kautz [19], and so it is reasonable to assume it. The paper is organized as follows. Before we extend the results obtained by Wolter and Zakharyaschev [20] to different forms of qualitative spatial or temporal reasoning, basic concepts relating to constraint satisfaction problems are introduced in section 2. These are the notions of networks of qualitative spatial or temporal constraints as well as solutions, consistency, partial solutions and global consistency. Section 3 deals with the basic concepts regarding the syntax and the semantics of our hybrid formalisms for reasoning about space and time. These are the notions of qualitative formulas, qualitative models as well as satisfiability. The main topic of section 4 is the proof that the question of determining of any given qualitative formula whether it is satisfiable or not requires polynomial space. Section 5 presents the concept of state to prove in section 6 that the question of determining of any given qualitative formula whether it is satisfiable or not is decidable in polynomial space.

2 Constraint Satisfaction Problems

Networks of constraints between spatial or temporal entities have been shown to be useful in formulating our knowledge of the relative positions of the objects that occupy space or in formulating our knowledge of the relative moments of the events that fill time. Within the framework of the research carried out in the domain of spatial or temporal reasoning, the main issue consists in deciding consistency of such networks. For our purposes we may only consider atomic constraint satisfaction problems, i.e., structures of the form $(\mathcal{X}, \mathcal{R})$ where $\mathcal{X}$ is a finite set of variables and $\mathcal{R}$ is a function with domain $\mathcal{X} \times \mathcal{X}$ and range a finite set ATO of atomic relations. The finite set ATO constitutes a list of jointly exhaustive and pairwise distinct atomic relations between positions or moments in some spatial or temporal universe VAL.

Example 1. Within the context of qualitative spatial reasoning in terms of points, see Ligozat [12], ATO consists of 9 atomic relations, sw, s, se, w, e, nw, n, ne and $=$. In this model of reasoning, VAL is the set of all pairs of real numbers.

Example 2. Within the context of qualitative temporal reasoning in terms of points, see Vilain and Kautz [19], ATO consists of 3 atomic relations, $<$, $>$ and $=$. In this model of reasoning, VAL is the set of all real numbers.

A solution of the atomic constraint satisfaction problem $(\mathcal{X}, \mathcal{R})$ is a function $\imath$ with domain $\mathcal{X}$ and range VAL such that for all $X, Y \in \mathcal{X}$, $\imath(X)$ and $\imath(Y)$ satisfy the atomic relation $\mathcal{R}(X, Y)$ in VAL. We shall say that the network $(\mathcal{X}, \mathcal{R})$ is consistent iff it possesses a solution. Deciding consistency of networks of atomic constraints between spatial or temporal entities constitutes the source of several problems in computer science. The thing is that those who tackled these problems proposed numerous algorithms for reasoning about space and time. For the most part, the proof that these algorithms are sound and complete is based on the notion of global consistency. A partial solution of the network $(\mathcal{X}, \mathcal{R})$ with respect to a subset $\mathcal{X}'$ of $\mathcal{X}$ is a function $\imath$ with domain $\mathcal{X}'$ and range VAL such that for all $X, Y \in \mathcal{X}'$, $\imath(X)$ and $\imath(Y)$ satisfy the atomic relation $\mathcal{R}(X, Y)$ in VAL. The atomic constraint satisfaction problem $(\mathcal{X}, \mathcal{R})$ is globally consistent if any partial solution can be extended to a solution. In the majority of cases, including the models introduced by Allen [1], Balbiani and Condotta [2], Balbiani, Condotta and Fariñas del Cerro [3], Cristani [7], Ligozat [11], Ligozat [12] and Vilain and Kautz [19]:

- Consistent networks of atomic constraints are globally consistent.

However this rule allows for a few exceptions, like the models introduced by Balbiani and Osmani [4], Gerevini and Renz [9], Isli and Cohn [10], Moratz, Renz and Wolter [13] and Randell, Cui and Cohn [15]. Our aim is to propose alternative languages to the language developed by Wolter and Zakharyaschev [20] where the model of the regions is replaced by different forms of qualitative reasoning about space and time which satisfy the rule.

3 Syntax and Semantics

Adapted from Wolter and Zakharyaschev [20], we define the set of all qualitative formulas as follows:

$$f ::= P(\bigcirc^m x, \bigcirc^n y) \mid \neg f \mid (f \vee g) \mid (f \mathbf{U} g);$$

where P ranges over the set ATO, m, n range over the set $\mathbb{N}$ of all integers and x, y range over the set VAR. Our intended interpretation of $(f\mathbf{U}g)$ is that "f holds at all following time points up to a time at which g holds". The other standard connectives are defined by the usual abbreviations. In particular, $\mathbf{F}f$ is $(\top\mathbf{U}f)$ and $\mathbf{G}f$ is $\neg(\top\mathbf{U}\neg f)$. The informal meaning of $\mathbf{F}f$ is that "there is a time point after the reference point at which f holds" whereas the informal meaning of $\mathbf{G}f$ is "f holds at all time points after the reference point". We follow the standard rules for omission of the parentheses. Our intended interpretation of atomic formula $P(\bigcirc^m x, \bigcirc^n y)$ is that "atomic relation P holds between the value of entity x in m units of time and the value of entity y in n units of time".

Example 3. Within the context of qualitative spatial reasoning in terms of points, see Ligozat [12], describing the movement and the relative positions of points x, y which move in a plane, qualitative formulas $\mathbf{G}(s(x, \bigcirc x) \vee w(x, \bigcirc x))$, $\mathbf{G}(e(y, \bigcirc y) \vee n(y, \bigcirc y))$ and $\mathbf{F}(s(x, y) \vee w(x, y))$ mean that x will always move to the north or to the east, y will always move to the west or to the south and a moment of time will come when x is to the south of y or x is to the west of y.

Example 4. Within the context of qualitative temporal reasoning in terms of points, see Vilain and Kautz [19], describing the movement and the relative positions of points x, y which move in a straight line, qualitative formulas $\mathbf{G}(x < \bigcirc x)$, $\mathbf{G}(y > \bigcirc y)$ and $\mathbf{F}(x = y)$ mean that x will always move to its right, y will always move to its left and a moment of time will come when x and y are in the same place.

Let f be a qualitative formula. The set of all individual variables in f will be denoted $var(f)$. The set of all subformulas of f will be denoted $SF(f)$. Let us be clear that there is strictly less than $Card(SF(f))$ $\mathbf{U}$-formulas in $SF(f)$. We define the size $\mid f \mid$ of f as follows:

- $\mid P(\bigcirc^m x, \bigcirc^n y) \mid = \max\{m, n\}$;
- $\mid \neg f \mid = \mid f \mid$;
- $\mid f \vee g \mid = \max\{\mid f \mid, \mid g \mid\}$;
- $\mid f\mathbf{U}g \mid = \max\{\mid f \mid, \mid g \mid\}$.

The number of occurrences of symbols in f will be denoted $length(f)$. It is well worth noting that $Card(var(f)) < length(f)$, $Card(SF(f)) < length(f)$ and $\mid f \mid < length(f)$. The set of all atomic formulas which individual variables are in $var(f)$ and which sizes are less than or equal to $\mid f \mid$ will be denoted $AF(f)$. The proof of the following lemma is simple and we do not provide it here.

Lemma 1. *Let f be a qualitative formula. Then $Card(AF(f)) = Card(ATO) \times Card(var(f))^2 \times (\mid f \mid +1)^2$.*

A function ϵ with domain $VAR \times \mathbb{N}$ and range the set VAL will be defined to be a qualitative model. The set VAL is the spatial or temporal universe in which the spatial or temporal entities of our language move. We define the relation "qualitative formula f is true at integer i in qualitative model ϵ", denoted $\epsilon, i \models f$, as follows:

- $\epsilon, i \models P(\bigcirc^m x, \bigcirc^n y)$ iff $P(\epsilon(x, i+m), \epsilon(y, i+n))$;
- $\epsilon, i \models \neg f$ iff $\epsilon, i \not\models f$;
- $\epsilon, i \models f \vee g$ iff $\epsilon, i \models f$ or $\epsilon, i \models g$;
- $\epsilon, i \models f\mathbf{U}g$ iff there is an integer k such that $i \leq k$, $\epsilon, k \models g$ and for all integers j, if $i \leq j$ and $j < k$ then $\epsilon, j \models f$.

An alternative formulation is "qualitative model ϵ satisfies qualitative formula f at integer i".

Example 5. Within the context of qualitative spatial reasoning in terms of points, see Ligozat [12], qualitative model ϵ of figure 1 satisfies qualitative formula $((s(x, \bigcirc x) \vee w(x, \bigcirc x)) \wedge (e(y, \bigcirc y) \vee n(y, \bigcirc y)))\mathbf{U}(s(x, y) \vee w(x, y))$ at integer 0.

Example 6. Within the context of qualitative temporal reasoning in terms of points, see Vilain and Kautz [19], qualitative model ϵ of figure 2 satisfies qualitative formula $((x < \bigcirc x) \wedge (y \bigcirc y))\mathbf{U}(x = y)$ at integer 0.

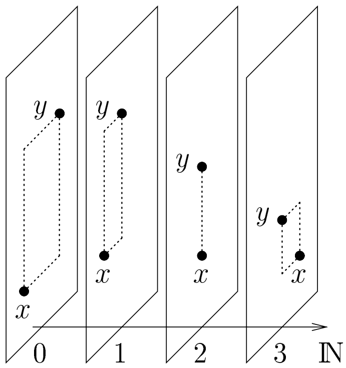

Fig. 1. A qualitative model based on qualitative spatial reasoning in terms of points.

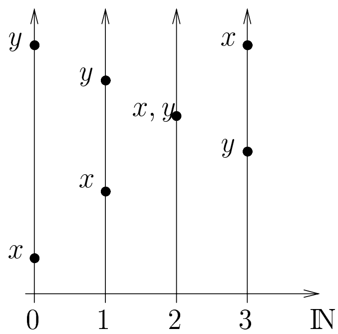

Fig. 2. A qualitative model based on qualitative temporal reasoning in terms of points.

Qualitative **U**-formula $f\mathbf{U}g$ will be defined to be fulfilled between integers i and j in qualitative model ϵ if $i \leq j$, $\epsilon, i \models f\mathbf{U}g$ and there is an integer k such that $i \leq k$, $k \leq j$ and $\epsilon, k \models g$. Qualitative formula f will be defined to be satisfiable if there is a qualitative model ϵ such that $\epsilon, 0 \models f$. The following equations specify which qualitative formulas are to count as formulas of, respectively, $\mathcal{L}_0(\mathbf{U})$ and $\mathcal{L}_1(\mathbf{U})$:

$$f ::= P(x, y) \mid \neg f \mid (f \vee g) \mid (f\mathbf{U}g);$$

$$f ::= P(\bigcirc^m x, \bigcirc^n y) \mid \neg f \mid (f \vee g) \mid (f\mathbf{U}g).$$

For the time being, let us mention outcomes of the results obtained by Wolter and Zakharyaschev [20] as regards the problem of determining of any given qualitative formula whether it is satisfiable or not: determining of any given formula of $\mathcal{L}_0(\mathbf{U})$ whether it is satisfiable or not is in $PSPACE$ whereas determining of any given formula of $\mathcal{L}_1(\mathbf{U})$ whether it is satisfiable or not is in $EXPSPACE$. Incidentally, we must not forget that the results obtained by Wolter and Zakharyaschev [20] regard the complexity of propositional linear temporal logics based on qualitative spatial reasoning in terms of regions. This brings us to the question of whether the results obtained by Wolter and Zakharyaschev [20] can be extended to different forms of qualitative spatial or temporal reasoning. What we have in mind is to prove the following important theorem for all forms of qualitative spatial and temporal reasoning in which consistent atomic constraint satisfaction problems are globally consistent.

Theorem 1. *Determining of any given qualitative formula whether it is satisfiable or not is $PSPACE$-complete.*

We outline how theorem 1 will be proved, but leave the details to the following sections. Firstly, we show how the question of determining of any given formula of propositional linear temporal logic whether it is satisfiable or not can be linearly reduced to the question of determining of any given formula of $\mathcal{L}_0(\mathbf{U})$ whether it is satisfiable or not. Secondly, we show how the question of determining of any given formula of $\mathcal{L}_1(\mathbf{U})$ whether it is satisfiable or not can be solved by means of a polynomial-space bounded nondeterministic algorithm.

4 Lower Bound

We first prove a simple theorem.

Theorem 2. *Determining of any given formula of $\mathcal{L}_0(\mathbf{U})$ whether it is satisfiable or not is $PSPACE$-hard.*

Proof. We define the set of all formulas of propositional linear temporal logic as follows:

$$f ::= p \mid \neg f \mid (f \vee g) \mid (f\mathbf{U}g);$$

where p ranges over a countable set of atomic formulas. Assuming that the set of all atomic formulas in propositional linear temporal logic is arranged in some determinate order $p_1, \ldots, p_N, \ldots$, assuming that the set of all individual variables in $\mathcal{L}_0(\mathbf{U})$ is arranged in some determinate order $x_1, y_1, \ldots, x_N, y_N, \ldots$, we define a linear function t that assigns to each formula f of propositional linear temporal logic the formula $t(f)$ of $\mathcal{L}_0(\mathbf{U})$ as follows:

- $t(p_N) = (x_N = y_N)$;
- $t(\neg f) = \neg t(f)$;
- $t(f \vee g) = t(f) \vee t(g)$;
- $t(f\mathbf{U}g) = t(f)\mathbf{U}t(g)$.

The reader may easily verify that a formula f of propositional linear temporal logic is satisfiable iff the formula $t(f)$ of $\mathcal{L}_0(\mathbf{U})$ is satisfiable. Seeing that determining of any given formula of propositional linear temporal logic whether it is satisfiable or not is $PSPACE$-hard, see Sistla and Clarke [18], we therefore conclude that determining of any given formula of $\mathcal{L}_0(\mathbf{U})$ whether it is satisfiable or not is $PSPACE$-hard.

We still have to prove that determining of any given formula of $\mathcal{L}_1(\mathbf{U})$ whether it is satisfiable or not is in $PSPACE$. In this respect, the concept of state will be of use to us.

5 *f*-States

Let ϵ be a qualitative model and i be an integer. Let $\widehat{\epsilon}_i$ be the function that assigns to each formula f of $\mathcal{L}_1(\mathbf{U})$ the set $\widehat{\epsilon}_i(f)$ of all atomic formulas true at i in ϵ which individual variables are in $var(f)$ and which sizes are less than or equal to $\mid f \mid$. Let $\widetilde{\epsilon}_i$ be the function that assigns to each formula f of $\mathcal{L}_1(\mathbf{U})$ the set $\widetilde{\epsilon}_i(f)$ of all subformulas of f true at i in ϵ. Let $\overline{\epsilon}_i$ be the function that assigns to each formula f of $\mathcal{L}_1(\mathbf{U})$ the structure $(\widehat{\epsilon}_{i-|f|}(f), \ldots, \widehat{\epsilon}_i(f), \widetilde{\epsilon}_i(f))$. In the case that $i <\mid f \mid$, we put:

$$\overline{\epsilon}_i(f) = (\underbrace{\emptyset, \ldots, \emptyset}_{|f|-i \text{ times}}, \widehat{\epsilon}_0(f), \ldots, \widehat{\epsilon}_i(f), \widetilde{\epsilon}_i(f)).$$

We first observe a simple lemma.

Lemma 2. *Let ϵ be a qualitative model, i be an integer and f be a formula of $\mathcal{L}_1(\mathbf{U})$. Assuming that the set of all variables in $var(f)$ is arranged in some determinate order $x_1, \ldots, x_N$, let n_1, n_2 be integers and l_1, l_2 be integers such that $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{i- \mid f \mid, \ldots, i\}$ and $l_2 \in \{i- \mid f \mid, \ldots, i\}$. Then there is exactly one atomic relation P such that for some integer k, $k \in \{i- \mid f \mid, \ldots, i\}$, $l_1 - k \in \{0, \ldots, \mid f \mid\}$, $l_2 - k \in \{0, \ldots, \mid f \mid\}$ and $P(\bigcirc^{l_1-k} x_{n_1}, \bigcirc^{l_2-k} x_{n_2}) \in \widehat{\epsilon}_k$.*

Proof. It is certain that there is an atomic relation P such that $P(\epsilon(x_{n_1}, l_1), \epsilon(x_{n_2}, l_2))$. Let k be $\min\{l_1, l_2\}$. The reader may easily verify that $k \in \{i- \mid f \mid, \ldots, i\}$, $l_1 - k \in \{0, \ldots, \mid f \mid\}$, $l_2 - k \in \{0, \ldots, \mid f \mid\}$ and $P(\bigcirc^{l_1-k} x_{n_1}, \bigcirc^{l_2-k} x_{n_2}) \in \widehat{\epsilon}_k$. If there is an atomic relation Q such that for some integer l, $l \in \{i- \mid f \mid, \ldots, i\}$, $l_1 - l \in \{0, \ldots, \mid f \mid\}$, $l_2 - l \in \{0, \ldots, \mid f \mid\}$ and $Q(\bigcirc^{l_1-l} x_{n_1}, \bigcirc^{l_2-l} x_{n_2}) \in \widehat{\epsilon}_l$ then $Q(\epsilon(x_{n_1}, l_1), \epsilon(x_{n_2}, l_2))$ and $P = Q$.

Let ω be the function that assigns to each formula f of $\mathcal{L}_1(\mathbf{U})$ the integer:

$$Card(ATO)^{Card(var(f))^2 \times (|f|+1)^3} \times 2^{Card(SF(f))}.$$

It is well worth noting that for all formulas f of $\mathcal{L}_1(\mathbf{U})$ and for all qualitative models ϵ, the range of the function that assigns to each integer i the structure $\overline{\epsilon}_i(f)$ contains strictly less than $\omega(f)$ elements. These elements are special cases of the concept of state. Let f be a formula of $\mathcal{L}_1(\mathbf{U})$. A structure:

$$(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0);$$

where $\widehat{S}_{-|f|}, \ldots, \widehat{S}_0$ are subsets of $AF(f)$ and $\widetilde{S}_0$ is a subset of $SF(f)$ will be defined to be a f-state. Assuming that the set of all variables in $var(f)$ is arranged in some determinate order $x_1, \ldots, x_N$, we will always suppose that for all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{- \mid f \mid, \ldots, 0\}$ and $l_2 \in \{- \mid f \mid, \ldots, 0\}$ then, according to lemma 2, there is exactly one atomic relation P such that for some integer k, $k \in \{- \mid f \mid, \ldots, 0\}$, $l_1 - k \in \{0, \ldots, \mid f \mid\}$, $l_2 - k \in \{0, \ldots, \mid f \mid\}$ and $P(\bigcirc^{l_1-k} x_{n_1}, \bigcirc^{l_2-k} x_{n_2}) \in \widehat{S}_k$. The interesting result is that the structure $(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0)$ can be linked with the atomic constraint satisfaction problem $(\mathcal{X}, \mathcal{R})$ defined as follows:

- Let $\mathcal{X}$ be $\{X_{1,-|f|}, \ldots, X_{1,0}, \ldots, X_{N,-|f|}, \ldots, X_{N,0}\}$;
- For all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{- \mid f \mid, \ldots, 0\}$ and $l_2 \in \{- \mid f \mid, \ldots, 0\}$ then let $\mathcal{R}(X_{n_1,l_1}, X_{n_2,l_2})$ be the only atomic relation P such that for some integer k, $k \in \{- \mid f \mid, \ldots, 0\}$, $l_1 - k \in \{0, \ldots, \mid f \mid\}$, $l_2 - k \in \{0, \ldots, \mid f \mid\}$ and $P(\bigcirc^{l_1-k} x_{n_1}, \bigcirc^{l_2-k} x_{n_2}) \in \widehat{S}_k$.

f-state $(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0)$ will be defined to be consistent if the atomic constraint satisfaction problem $(\mathcal{X}, \mathcal{R})$ linked with the structure $(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0)$ as above is consistent and:

- If $P(\bigcirc^m x, \bigcirc^n y) \in SF(f)$ then $P(\bigcirc^m x, \bigcirc^n y) \in \widetilde{S}_0$ iff $P(\bigcirc^m x, \bigcirc^n y) \in \widehat{S}_0$;
- If $\neg g \in SF(f)$ then $\neg g \in \widetilde{S}_0$ iff $g \notin \widetilde{S}_0$;
- If $g \vee h \in SF(f)$ then $g \vee h \in \widetilde{S}_0$ iff $g \in \widetilde{S}_0$ or $h \in \widetilde{S}_0$.

f-state $(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0)$ will be defined to be **U**-consistent with respect to f-state $(\widehat{T}_{-|f|}, \ldots, \widehat{T}_0, \widetilde{T}_0)$ if $\widehat{T}_{-|f|} = \widehat{S}_{-|f|+1}$, …, $\widehat{T}_{-1} = \widehat{S}_0$ and:

- If $g\mathbf{U}h \in SF(f)$ then $g\mathbf{U}h \in \widetilde{S}_0$ iff $h \in \widetilde{S}_0$ or $g \in \widetilde{S}_0$ and $g\mathbf{U}h \in \widetilde{T}_0$.

6 Upper Bound

In accordance with the chain of reasoning put forward by Sistla and Clarke [18], one can establish the following remarkable lemmas for all forms of qualitative spatial and temporal reasoning such that consistent atomic constraint satisfaction problems are globally consistent.

Lemma 3. *Let ϵ be a qualitative model, i, j be integers and f be a formula of $\mathcal{L}_1(\mathbf{U})$ such that $i < j$ and $\overline{\epsilon}_i(f) = \overline{\epsilon}_j(f)$. Then there is a qualitative model ϵ' such that for all integers k, if $k < i$ then $\widehat{\epsilon'}_k(f) = \widehat{\epsilon}_k(f)$ and if $k \geq i$ then $\widehat{\epsilon'}_k(f) = \widehat{\epsilon}_{k+j-i}(f)$. Added to that, for all integers k, if $k < i$ then $\widetilde{\epsilon'}_k(f) = \widetilde{\epsilon}_k(f)$ and if $k \geq i$ then $\widetilde{\epsilon'}_k(f) = \widetilde{\epsilon}_{k+j-i}(f)$.*

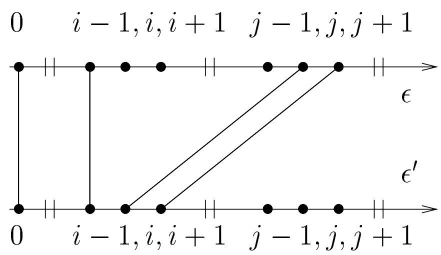

Fig. 3. The relationship between qualitative models ϵ and ϵ' in lemma 3.

Proof. If $\mid f \mid= 0$ then let ϵ' be a function with domain $VAR \times \mathbb{N}$ and range the set VAL defined as follows. For all integers k, if $k < i$ then for all individual variables x in $var(f)$, let $\epsilon'(x, k)$ be $\epsilon(x, k)$. For all integers k, if $k \geq i$ then for all individual variables x in $var(f)$, let $\epsilon'(x, k)$ be $\epsilon(x, k + j - i)$. It is beyond all doubt that ϵ' satisfies the required conditions. If $\mid f \mid\geq 1$ then the previous function ϵ' might not satisfy the required conditions. In that case we let ϵ' be a function with domain $VAR \times \mathbb{N}$ and range the set VAL defined as follows, see figure 3. For all integers k, if $k < i$ then for all individual variables x in $var(f)$, let $\epsilon'(x, k)$ be $\epsilon(x, k)$. For all integers k, if $k \geq i$ then assuming that the set of all individual variables in $var(f)$ is arranged in some determinate order $x_1, \ldots, x_N$, let $\epsilon'(x_1, k), \ldots, \epsilon'(x_N, k)$ be the values defined as follows. Let us consider the atomic constraint satisfaction problem $(\mathcal{X}_k, \mathcal{R}_k)$ defined as follows:

- Let $\mathcal{X}_k$ be $\{X_{1,k-|f|}, \ldots, X_{1,k}, \ldots, X_{N,k-|f|}, \ldots, X_{N,k}\}$;
- For all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{k- \mid f \mid, \ldots, k\}$ and $l_2 \in \{k- \mid f \mid, \ldots, k\}$ then let $\mathcal{R}_k(X_{n_1,l_1}, X_{n_2,l_2})$ be the only atomic relation P such that $P(\epsilon(x_{n_1}, l_1 + j - i), \epsilon(x_{n_2}, l_2 + j - i))$.

Obviously, the constraint satisfaction problem $(\mathcal{X}_k, \mathcal{R}_k)$ is consistent. Hence, it is globally consistent. All this goes to show that given the values $\epsilon'(x_1, k- \mid f \mid)$, $\ldots$, $\epsilon'(x_1, k - 1)$, $\ldots$, $\epsilon'(x_N, k- \mid f \mid)$, $\ldots$, $\epsilon'(x_N, k - 1)$ such that:

- For all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{k- \mid f \mid, \ldots, k - 1\}$ and $l_2 \in \{k- \mid f \mid, \ldots, k - 1\}$ then $\epsilon'(x_{n_1}, l_1)$ and $\epsilon'(x_{n_2}, l_2)$ satisfy the constraint $\mathcal{R}_k(X_{n_1,l_1}, X_{n_2,l_2})$;

there are values $\epsilon'(x_1, k)$, $\ldots$, $\epsilon'(x_N, k)$ such that:

- For all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{k- \mid f \mid, \ldots, k\}$ and $l_2 \in \{k- \mid f \mid, \ldots, k\}$ then $\epsilon'(x_{n_1}, l_1)$ and $\epsilon'(x_{n_2}, l_2)$ satisfy the constraint $\mathcal{R}_k(X_{n_1,l_1}, X_{n_2,l_2})$.

It is beyond all doubt that ϵ' satisfies the required conditions.

Lemma 4. *Let ϵ be a qualitative model, i, j be integers and f be a formula of $\mathcal{L}_1(\mathbf{U})$ such that $i < j$, $\overline{\epsilon}_i(f) = \overline{\epsilon}_j(f)$ and every $\mathbf{U}$-formula in $\widetilde{\epsilon}_i(f)$ is fulfilled between i and j in ϵ. Then there is a qualitative model ϵ' such that for all integers k, if $k < j$ then $\widehat{\epsilon'}_k(f) = \widehat{\epsilon}_k(f)$ and if $k \geq j$ then $\widehat{\epsilon'}_k(f) = \widehat{\epsilon'}_{k+i-j}(f)$. Added to that, for all integers k, if $k < j$ then $\widetilde{\epsilon'}_k(f) = \widetilde{\epsilon}_k(f)$ and if $k \geq j$ then $\widetilde{\epsilon'}_k(f) = \widetilde{\epsilon'}_{k+i-j}(f)$. ϵ' is said to be a periodic qualitative model with starting index i and starting period j.*

Proof. If $\mid f \mid= 0$ then let ϵ' be a function with domain $VAR \times \mathbb{N}$ and range the set VAL defined as follows. For all integers k, if $k < j$ then for all individual variables x in $var(f)$, let $\epsilon'(x, k)$ be $\epsilon(x, k)$. For all integers k, if $k \geq j$ then for all individual variables x in $var(f)$, let $\epsilon'(x, k)$ be $\epsilon'(x, k + i - j)$. It is beyond

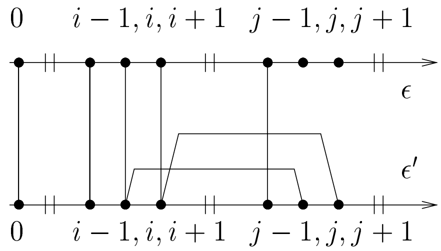

Fig. 4. The relationship between qualitative models ϵ and ϵ' in lemma 4.

all doubt that ϵ' satisfies the required conditions. If $\mid f \mid \geq 1$ then the previous function ϵ' might not satisfy the required conditions. In that case we let ϵ' be a function with domain $VAR \times \mathbb{N}$ and range the set VAL defined as follows, see figure 4. For all integers k, if $k < j$ then for all individual variables x in $var(f)$, let $\epsilon'(x, k)$ be $\epsilon(x, k)$. For all integers k, if $k \geq j$ then assuming that the set of all individual variables in $var(f)$ is arranged in some determinate order x_1, ..., x_N, let $\epsilon'(x_1, k)$, ..., $\epsilon'(x_N, k)$ be the values defined as follows. Let us consider the atomic constraint satisfaction problem $(\mathcal{X}_k, \mathcal{R}_k)$ defined as follows:

- Let $\mathcal{X}_k$ be $\{X_{1,k-|f|}, \ldots, X_{1,k}, \ldots, X_{N,k-|f|}, \ldots, X_{N,k}\}$;
- For all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{k- \mid f \mid, \ldots, k\}$ and $l_2 \in \{k- \mid f \mid, \ldots, k\}$ then let $\mathcal{R}_k(X_{n_1,l_1}, X_{n_2,l_2})$ be the only atomic relation P such that $P(\epsilon'(x_{n_1}, l_1 + i - j), \epsilon'(x_{n_2}, l_2 + i - j))$.

Obviously, the constraint satisfaction problem $(\mathcal{X}_k, \mathcal{R}_k)$ is consistent. Hence, it is globally consistent. All this goes to show that given the values $\epsilon'(x_1, k- \mid f \mid)$, ..., $\epsilon'(x_1, k-1)$, ..., $\epsilon'(x_N, k- \mid f \mid)$, ..., $\epsilon'(x_N, k-1)$ such that:

- For all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{k- \mid f \mid, \ldots, k-1\}$ and $l_2 \in \{k- \mid f \mid, \ldots, k-1\}$ then $\epsilon'(x_{n_1}, l_1)$ and $\epsilon'(x_{n_2}, l_2)$ satisfy the constraint $\mathcal{R}_k(X_{n_1,l_1}, X_{n_2,l_2})$;

there are values $\epsilon'(x_1, k)$, ..., $\epsilon'(x_N, k)$ such that:

- For all integers n_1, n_2 and for all integers l_1, l_2, if $n_1 \in \{1, \ldots, N\}$, $n_2 \in \{1, \ldots, N\}$, $l_1 \in \{k- \mid f \mid, \ldots, k\}$ and $l_2 \in \{k- \mid f \mid, \ldots, k\}$ then $\epsilon'(x_{n_1}, l_1)$ and $\epsilon'(x_{n_2}, l_2)$ satisfy the constraint $\mathcal{R}_k(X_{n_1,l_1}, X_{n_2,l_2})$.

It is beyond all doubt that ϵ' satisfies the required conditions.

The reasoning behind the proof of lemma 3 and the proof of lemma 4 deserves especial consideration, for the simple reason that it makes use of the fact that within the form of qualitative temporal reasoning in terms of intervals we have considered, consistent atomic constraint satisfaction problems are globally consistent. Combining lemma 3 with lemma 4, we obtain the following theorem.

Theorem 3. *Let f be a formula of $\mathcal{L}_1(\mathbf{U})$ such that f is satisfiable. Then there is a qualitative model ϵ and there are integers i, j such that $i < j$, ϵ is a periodic qualitative model with starting index i and starting period j, $i < \omega(f)$, $j - i < Card(SF(f)) \times \omega(f)$ and $\epsilon, 0 \models f$.*

Proof. Let $\epsilon^{(0)}$ be a qualitative model such that $\epsilon^{(0)}, 0 \models f$. The reader may easily verify that there are integers $i^{(0)}$, $j^{(0)}$ such that:

- $i^{(0)} < j^{(0)}$, $\overline{\epsilon^{(0)}}_{i^{(0)}}(f) = \overline{\epsilon^{(0)}}_{j^{(0)}}(f)$ and every $\mathbf{U}$-formula in $\widetilde{\epsilon^{(0)}}_{i^{(0)}}(f)$ is fulfilled between $i^{(0)}$ and $j^{(0)}$ in $\epsilon^{(0)}$.

If $i^{(0)} \geq \omega(f)$ then there are integers k, l such that $k < l$, $l \leq i^{(0)}$ and $\overline{\epsilon^{(0)}}_k(f) = \overline{\epsilon^{(0)}}_l(f)$. By applying lemma 3, we infer from this that there is a qualitative model $\epsilon^{(1)}$ such that $\epsilon^{(1)}, 0 \models f$ and there are integers $i^{(1)}$, $j^{(1)}$ such that:

- $i^{(1)} < j^{(1)}$, $\overline{\epsilon^{(1)}}_{i^{(1)}}(f) = \overline{\epsilon^{(1)}}_{j^{(1)}}(f)$, every $\mathbf{U}$-formula in $\widetilde{\epsilon^{(1)}}_{i^{(1)}}(f)$ is fulfilled between $i^{(1)}$ and $j^{(1)}$ in $\epsilon^{(1)}$ and $i^{(1)} < i^{(0)}$.

Applying this reduction as far as possible, we gather from this that there is a qualitative model $\epsilon^{(2)}$ such that $\epsilon^{(2)}, 0 \models f$ and there are integers $i^{(2)}$, $j^{(2)}$ such that:

- $i^{(2)} < j^{(2)}$, $\overline{\epsilon^{(2)}}_{i^{(2)}}(f) = \overline{\epsilon^{(2)}}_{j^{(2)}}(f)$, every $\mathbf{U}$-formula in $\widetilde{\epsilon^{(2)}}_{i^{(2)}}(f)$ is fulfilled between $i^{(2)}$ and $j^{(2)}$ in $\epsilon^{(2)}$ and $i^{(2)} < \omega(f)$.

If $j^{(2)} - i^{(2)} \geq Card(SF(f)) \times \omega(f)$ then there are integers k, l such that $i^{(2)} \leq k$, $k < l$, $l \leq j^{(2)}$, $\overline{\epsilon^{(2)}}_k(f) = \overline{\epsilon^{(2)}}_l(f)$ and for all $\mathbf{U}$-formulas $g\mathbf{U}h$ in $\widetilde{\epsilon^{(2)}}_{i^{(2)}}(f)$ and for all integers m, if $k < m$ and $m < l$ then $\epsilon^{(2)}, m \not\models h$. By applying lemma 3, we infer from this that there is a qualitative model $\epsilon^{(3)}$ such that $\epsilon^{(3)}, 0 \models f$ and there are integers $i^{(3)}$, $j^{(3)}$ such that:

- $i^{(3)} < j^{(3)}$, $\overline{\epsilon^{(3)}}_{i^{(3)}}(f) = \overline{\epsilon^{(3)}}_{j^{(3)}}(f)$, every $\mathbf{U}$-formula in $\widetilde{\epsilon^{(3)}}_{i^{(3)}}(f)$ is fulfilled between $i^{(3)}$ and $j^{(3)}$ in $\epsilon^{(3)}$, $i^{(3)} < \omega(f)$ and $j^{(3)} - i^{(3)} < j^{(2)} - i^{(2)}$.

Applying this reduction as much as possible, we gather from this that there is a qualitative model $\epsilon^{(4)}$ such that $\epsilon^{(4)}, 0 \models f$ and there are integers $i^{(4)}$, $j^{(4)}$ such that:

- $i^{(4)} < j^{(4)}$, $\overline{\epsilon^{(4)}}_{i^{(4)}}(f) = \overline{\epsilon^{(4)}}_{j^{(4)}}(f)$, every $\mathbf{U}$-formula in $\widetilde{\epsilon^{(4)}}_{i^{(4)}}(f)$ is fulfilled between $i^{(4)}$ and $j^{(4)}$ in $\epsilon^{(4)}$, $i^{(4)} < \omega(f)$ and $j^{(4)} - i^{(4)} < Card(SF(f)) \times \omega(f)$.

By applying lemma 4, we therefore conclude that there is a qualitative model ϵ and there are integers i, j such that $i < j$, ϵ is a periodic qualitative model with starting index i and starting period j, $i < \omega(f)$, $j - i < Card(SF(f)) \times \omega(f)$ and $\epsilon, 0 \models f$.

By theorem 3, we infer the following theorem.

Theorem 4. *Determining of any given formula of $\mathcal{L}_1(\mathbf{U})$ whether it is satisfiable or not is in $PSPACE$.*

Proof. To test a formula f of $\mathcal{L}_1(\mathbf{U})$ for satisfiability, we present the following nondeterministic algorithm: Guess integers i, j such that $i < j$, $i < \omega(f)$ and $j - i < Card(SF(f)) \times \omega(f)$;
Guess a consistent f-state $(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0)$ such that $\widehat{S}_{-|f|} = \emptyset$, ..., $\widehat{S}_{-1} = \emptyset$ and $f \in \widetilde{S}_0$;
$k := 0$;
While $k < i$ do
Guess a consistent f-state $(\widehat{T}_{-|f|}, \ldots, \widehat{T}_0, \widetilde{T}_0)$ such that $(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0)$ is $\mathbf{U}$-consistent with respect to $(\widehat{T}_{-|f|}, \ldots, \widehat{T}_0, \widetilde{T}_0)$;
$(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0) := (\widehat{T}_{-|f|}, \ldots, \widehat{T}_0, \widetilde{T}_0)$;
$k := k + 1$;
$(\widehat{U}_{-|f|}, \ldots, \widehat{U}_0, \widetilde{U}_0) := (\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0)$;
While $k < j$ do
For all $\mathbf{U}$-formulas $g\mathbf{U}h$ in $SF(f)$, if $g\mathbf{U}h \in \widetilde{U}_0$ and $h \in \widetilde{S}_0$ then mark $g\mathbf{U}h$ in $\widetilde{U}_0$;
Guess a consistent f-state $(\widehat{T}_{-|f|}, \ldots, \widehat{T}_0, \widetilde{T}_0)$ such that $(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0)$ is $\mathbf{U}$-consistent with respect to $(\widehat{T}_{-|f|}, \ldots, \widehat{T}_0, \widetilde{T}_0)$;
$(\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0) := (\widehat{T}_{-|f|}, \ldots, \widehat{T}_0, \widetilde{T}_0)$;
$k := k + 1$;
Check whether for all $\mathbf{U}$-formulas $g\mathbf{U}h$ in $SF(f)$, if $g\mathbf{U}h \in \widetilde{U}_0$ then $g\mathbf{U}h$ is marked in $\widetilde{U}_0$;
Check whether $(\widehat{U}_{-|f|}, \ldots, \widehat{U}_0, \widetilde{U}_0) := (\widehat{S}_{-|f|}, \ldots, \widehat{S}_0, \widetilde{S}_0)$. Remark that the previous nondeterministic algorithm works correctly and that it is polynomial-space bounded in $length(f)$. It follows that determining of any given formula of $\mathcal{L}_1(\mathbf{U})$ whether it is satisfiable or not is in $NPSPACE$. Seeing that $NPSPACE = PSPACE$, see Savitch [17], we therefore conclude that determining of any given formula of $\mathcal{L}_1(\mathbf{U})$ whether it is satisfiable or not is in $PSPACE$.

Referring to theorem 2 and theorem 4, we easily obtain a proof of theorem 1.

7 Conclusion

We have considered the language introduced by Wolter and Zakharyaschev [20] and obtained by mixing the model of the regions and the propositional linear temporal logic. In particular, we have proposed alternative languages where the model of the regions is replaced by different forms of qualitative spatial or temporal reasoning. In these languages, qualitative formulas describe the movement and the relative positions of spatial or temporal entities in some spatial or temporal universe. On the basis of the argument displayed by Sistla and Clarke [18], we have demonstrated that for all forms of qualitative spatial and temporal reasoning in which consistent atomic constraint satisfaction problems are globally consistent, determining of any given qualitative formula whether it is satisfiable or not is $PSPACE$-complete. Much remains to be done, given that there are many ways we could extend our results. An important development in the theory of propositional linear temporal logic is the introduction of the **S** operator, the informal meaning of (f**S**g) being that "f holds at all previous time points since a time at which g holds". Therefore, we plan to investigate the question whether our line of reasoning is still valid for a propositional linear temporal logic with both **U** and **S** operators. The reasoning behind the proof of lemma 3 and the proof of lemma 4 makes use of the fact that within the form of qualitative spatial or temporal reasoning we have considered, consistent atomic constraint satisfaction problems are globally consistent. Consequently, another promising direction of research is the issue whether our line of reasoning can be adapted to the forms of qualitative spatial or temporal reasoning which does not fit this requirement. In other respects, an important development in the applications of propositional linear temporal logic is the model-checking algorithm used to determine whether a given finite-state program meets a particular correctness specification. Thus, we intend to illustrate how the model-checking algorithm works within the context of our propositional linear temporal logic based on qualitative spatial or temporal reasoning.

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Axiomatizing the cyclic interval calculus

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Axiomatizing the Cyclic Interval Calculus

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Abstract

A model for qualitative reasoning about intervals on a cyclic time has been recently proposed by Balbiani and Osmani (Balbiani & Osmani 2000). In this formalism, the basic entities are intervals on a circle, and using considerations similar to Allen's calculus, sixteen basic relations are obtained, which form a jointly disjunctive and pairwise distinct (JEPD) set of relations. The purpose of this paper is to give an axiomatic description of the calculus, based on the properties of the *meets* relation, from which all other fifteen relations can be deduced. We show how the corresponding theory is related to cyclic orderings, and use the results to prove that any countable model of this theory is isomorphic to the cyclic interval structure based on the rational numbers. Our approach is similar to Ladkin's axiomatization of Allen's calculus, although the cyclic structures introduce specific difficulties.

Keywords: qualitative temporal reasoning, cyclic interval calculus, cyclic orderings, completeness, $\aleph_0$–categorical theories

Introduction

In the domain of qualitative temporal reasoning, a great deal of attention has been devoted to the study of temporal formalisms based on a dense and unbounded linear model of time. Most prominently, this is the case of Allen's calculus, where the basic entities are intervals of the real time line, and the 13 basic relations (Allen's relations) correspond to the possible configurations of the endpoints of two intervals (Allen 1981). Other calculi such as the cardinal direction calculus (Ligozat 1998a; 1998b), the n-point calculus (Balbiani & Condotta 2002), the rectangle calculus (Balbiani, Condotta, & Fariñas del Cerro 1999), the n-block calculus (Balbiani, Condotta, & Farinas del Cerro 2002) are also based on products of the real line equipped with its usual ordering relation, hence on products of dense and unbounded linear orderings.

However, many situations call for considering orderings which are *cyclic* rather than linear. In particular, the set of directions around a given point of reference has such a cyclic structure. This fact has motivated several formalisms in this direction: Isli and Cohn (Isli & Cohn 2000) and Balbiani *et al.* (Balbiani, Condotta, & Ligozat 2002) consider a calculus about points on a circle, based on qualitative ternary relations between the points. Schlieder's work on the concepts of orientation and panorama (Schlieder 1993; 1995) is also concerned with cyclic situations. Our work is more closely related to Balbiani and Osmani's proposal (Balbiani & Osmani 2000) which we will refer to as the *cyclic interval calculus*. This calculus is similar in spirit to Allen's calculus: in the same way as the latter, which views intervals on the line as ordered pairs of points (the starting and ending point of the interval), the cyclic interval calculus considers intervals on a circle as pairs of distinct points: two points on a circle define the interval obtained when starting at the first, going (say counterclockwise) around the circle until the second point is reached. The consideration of all possible configurations between the endpoints of two intervals defined in that way leads to sixteen basic relations, each one of which is characterized by a particular qualitative configuration. For instance, the relation *meets* corresponds to the case where the last point of the first interval coincides with the first point of the other, and the two intervals have no other point in common. Another interesting relation, which has no analog in the linear case, is the *mmi* relation[1], where the last point of each interval is the first point of the other (as is the case with two serpents, head to tail, each one of them devouring the other).

This paper is concerned with giving suitable axioms for the *meets* relation in the cyclic case. This single relation can be used to define all other 15 relations of the formalism (there is a similar fact about the *meets* relation in Allen's calculus). We give a detailed description of the way in which the axiomatization of cyclic orderings – using a ternary relation described in (Balbiani, Condotta, & Ligozat 2002)– relates to the axiomatization of cyclic intervals based on the binary relation *meets*. Our approach is very similar to the approach followed by Ladkin in his PhD thesis (Ladkin 1987) where he shows how the axiomatization of linear dense and unbounded linear orderings relates to the axiomatization proposed by Allen and Hayes for the interval calculus, in terms of the relation *meets*.

The core of the paper, apart from the choice of an appropriate set of axioms, rests on two constructions:

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[1]The notation is mnemonic for *meets* and *meets inverse*.

- Starting from a cyclic ordering, that is a set of points equipped with a ternary order structure satisfying suitable axioms , the first construction defines a set of cyclic intervals equipped with a binary *meets* relation; and conversely.
- Starting from a set of cyclic intervals equipped with a *meets* relation, the second construction yields a set of points (the intuition is that two intervals which meet define a point, their meeting point) together with a ternary relation which has precisely the properties necessary to define a cyclic ordering.

The next step involves studying how the two constructions interact. In the linear case, a result of Ladkin's can be expressed in the language of category theory by saying that the two constructions define an equivalence of categories. Using Cantor's theorem, this implies that the corresponding theories are $\aleph_0$ categorical. In the cyclic case, we prove an analogous result: here again, the two constructions define an equivalence of categories. On the other hand, as shown in (Balbiani, Condotta, & Ligozat 2002), all countable cyclic orderings are isomorphic. As a consequence, the same fact is true of the cyclic interval structures which satisfy the axioms we give for the relation *meets*. This is the main result of the paper. We further examine the connections of these results to the domain of constraint-based reasoning in the context of the cyclic interval calculus, and we conclude by pointing to possible extensions of this work.

Building cyclic interval structures from cyclic orderings

This section is devoted to a construction of the cyclic interval structures we will consider in this paper, starting from cyclic orderings. In the next section, we will propose a set of axioms for these structures. Intuitively, each model can be visualized in terms of a set of oriented arcs (intervals) on a circle (an interval is identified by a starting point and an ending point on the circle), together with a binary *meets* relation on the set of intervals. Specifically, two cyclic intervals (m, n) and (m', n') are such that (m, n) *meets* (m', n') if $n = m'$ and n' is not between m and n, see Figure 1 (as a consequence, $n = m'$ is the only point that the two intervals have in common).

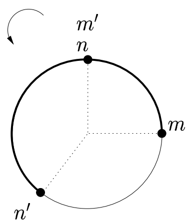

Figure 1: Two cyclic intervals (m, n) and (m', n') satisfying the *meets* relation.

In order to build interval structures, we start from cyclic orderings[2](Balbiani, Condotta, & Ligozat 2002). Intuitively, the cyclic ordering on a circle is similar to the usual ordering on the real line. In formal terms, a cyclic ordering is a pair $(\mathcal{P}, \prec)$ where $\mathcal{P}$ is a nonempty set of points, and $\prec$ is a ternary relation on $\mathcal{P}$ such that the following conditions are met, for all $x, y, z, t \in \mathcal{P}$:

P1. $\neg \prec (x, y, y)$;

P2. $\prec (x, y, z) \wedge \prec (x, z, t) \rightarrow \prec (x, y, t)$;

P3. $x \neq y \wedge x \neq z \rightarrow y = z \vee \prec (x, y, z) \vee \prec (x, z, y)$;

P4. $\prec (x, y, z) \leftrightarrow \prec (y, z, x) \leftrightarrow \prec (z, x, y)$;

P5. $x \neq y \rightarrow (\exists z \prec (x, z, y)) \wedge (\exists z \prec (x, y, z))$;

P6. $\exists x, y \quad x \neq y$.

Definition 1 (The cyclic interval structure associated to a cyclic ordering) *Let $(\mathcal{P}, \prec)$ be a cyclic ordering. The cyclic interval structure* $\mathsf{CycInt}((\mathcal{P}, \prec))$ *associated to $(\mathcal{P}, \prec\ )$ is the pair $(\mathcal{I}, meets)$ where:*

- *$\mathcal{I} = \{(x, y) \in \mathcal{P} \times \mathcal{P} : \exists z \in \mathcal{P}\ with \prec (x, y, z)\}$. The elements of $\mathcal{I}$ are called (cyclic) intervals.*
- *meets is the binary relation defined by $meets = \{((x, y), (x', y')) : y = x'\ and\ \prec (x, y, y')\}$.*

As an example, consider the set $\mathcal{C}$ of all rational numbers contained in the interval $[0, 2\pi[$. Each rational number in that range represents a point in the unit circle centered at the origin in the Euclidean plane: $n \in [0, 2\pi[$ corresponds to the point with polar coordinates $(1, n)$. Let $\prec_{\mathcal{C}}$ the binary relation $[0, 2\pi[$ as follows: $\prec_{\mathcal{C}} (x, y, z)$ if and only if either $x < y < z$ or $y < z < x$ or $z < x < y$, where $x, y, z \in [0, 2\pi[$. We can easily check that the structure $(\mathcal{C}, \prec_{\mathcal{C}})$ we get is a cyclic ordering. Hence $\mathsf{C}ycInt((\mathcal{C}, \prec_{\mathcal{C}}))$ is a cyclic interval structure $(\mathcal{I}, meets)$. Each element $u = (x, y)$ of $\mathcal{I}$ can be viewed as the oriented arc containing all points between the points represented by x and y (we will refer to these two points as to the endpoints of the cyclic interval u and denote by u^- and u^+, respectively, the points associated to x and y). For instance, the cyclic intervals $(0, \pi/2)$, $(\pi/2, 0)$ and $(3\pi/2, \pi/2)$ are shown in Figure 2. Notice that no cyclic interval contains only one point (there are no punctual intervals), and that no interval covers the whole circle. Intuitively, two cyclic intervals are in the relation *meets* if

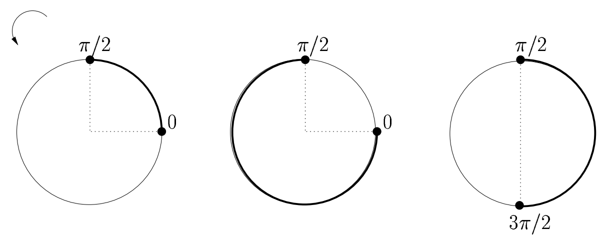

Figure 2: Three cyclic intervals.

[2]Actually, we use what are called "standard cyclic orderings" in (Balbiani, Condotta, & Ligozat 2002). We use the shorter term "cyclic ordering" in this paper.

and only if the ending point of the first coincides with the starting point of the other, and the intervals have no other point in common. For instance, $((3\pi/2, \pi/2), (\pi/2, \pi)) \in meets$, while $((3\pi/2, \pi/2), (\pi/2, 5\pi/3)) \notin meets$.

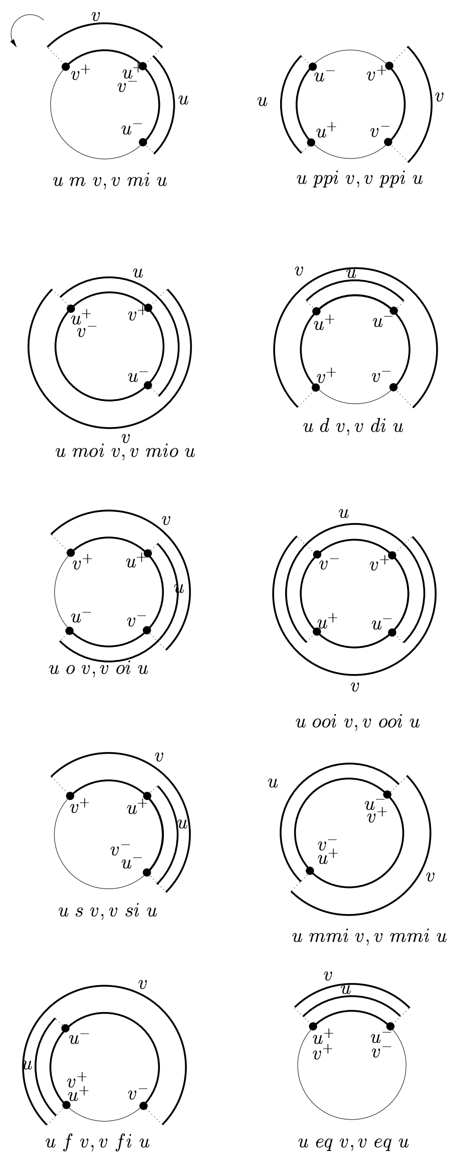

Figure 3: The 16 basic relations of the cyclic interval calculus.

Let $(\mathcal{I}, meets)$ be a cyclic interval structure. We now show how the other fifteen basic relations of the cyclic interval calculus defined by Balbiani and Osmani (Balbiani & Osmani 2000) can be defined using the *meets* relation. The 16 relations are denoted by the set of symbols $\{m, mi, ppi, mmi, d, di, f, fi, o, oi, s, si, ooi, moi, mio, eq\}$ (where m is the *meets* relation). Figure 3 shows examples of these relations. More formally, the relations other than *meets* are defined as follows[3]:

- u **ppi** $v \stackrel{def}{\equiv} \exists w, x \quad u\ \mathbf{m}\ w\ \mathbf{m}\ v\ \mathbf{m}\ x\ \mathbf{m}\ u$,
- u **mmi** $v \stackrel{def}{\equiv} \exists w, x, y, z \quad w\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ z\ \mathbf{m}\ w \wedge z\ \mathbf{m}\ u\ \mathbf{m}\ y \wedge x\ \mathbf{m}\ v\ \mathbf{m}\ w$,
- u **d** $v \stackrel{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ u\ \mathbf{m}\ y\ \mathbf{m}\ w \wedge v\ \mathbf{mmi}\ w$,
- u **f** $v \stackrel{def}{\equiv} \exists w, x \quad w\ \mathbf{m}\ x\ \mathbf{m}\ u\ \mathbf{m}\ w \wedge v\ \mathbf{mmi}\ w$,
- u **o** $v \stackrel{def}{\equiv} \exists w, x, y, z \quad u\ \mathbf{m}\ v\ \mathbf{m}\ x\ \mathbf{m}\ u \wedge v\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ v \wedge y\ \mathbf{m}\ z\ \mathbf{m}\ w$,
- u **s** $v \stackrel{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ v\ \mathbf{m}\ w \wedge x\ \mathbf{m}\ u\ \mathbf{m}\ y\ \mathbf{m}\ w$,
- u **ooi** $v \stackrel{def}{\equiv} \exists w, x \quad w\ \mathbf{f}\ u \wedge w\ \mathbf{s}\ v \wedge x\ \mathbf{s}\ u \wedge x\ \mathbf{f}\ v$,
- u **moi** $v \stackrel{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ w \wedge y\ \mathbf{ppi}\ u \wedge x\ \mathbf{ppi}\ v$,
- u **mio** $v \stackrel{def}{\equiv} \exists w, x, y \quad w\ \mathbf{m}\ x\ \mathbf{m}\ y\ \mathbf{m}\ w \wedge x\ \mathbf{ppi}\ u \wedge y\ \mathbf{ppi}\ v$,
- u **eq** $v \stackrel{def}{\equiv} \exists w, x \quad w\ \mathbf{m}\ u\ \mathbf{m}\ x \wedge w\ \mathbf{m}\ v\ \mathbf{m}\ x$.

The relations mi, di, fi, oi, si are the converse relations of m, d, f, o, s, respectively.

Axioms for cyclic interval structures: The CycInt theory

In this section, we give a set of axioms allowing to characterise the relation *meets* of cyclic intervals. Several axioms are motivated by intuitive properties owned by models of cyclic intervals. Other axioms are axioms of the relation *meets* of the intervals of the line (Ladkin 1987; Allen & Hayes 1985) adapted to the cyclic case.

In the sequel $u, v, w, \ldots$ will denote variables representing cyclic intervals. The symbol $|$ corresponds to the relation *meets*. The expression $v_1|v_2|\ldots|v_n$ with $v_1, v_2, \ldots, v_n$ n variables ($n > 2$) is an abbreviation for the conjunction $\bigwedge_{i=1}^{n-1} v_i|v_{i+1}$. Note that the expression $v_1|v_2|\ldots|v_n|v_1$ is equivalent to $v_2|\ldots|v_n|v_1|v_2$.

Another abbreviation used in the sequel is $\mathsf{X}(u, v, w, x)$. It is defined by the expression $u|v \wedge w|x \wedge (u|x \vee w|v)$. Intuitively, the satisfaction of $\mathsf{X}(u, v, w, x)$ expresses the fact that the cyclic interval u meets (is in relation *meets* with) the cyclic interval v, the cyclic interval w meets (is in relation *meets* with) the cyclic interval x and the two meeting points are the same points. In Figure 4 are represented the three possible cases for which $\mathsf{X}(u, v, w, x)$ is satisfied by cyclic intervals onto an oriented circle :

(a) $u|v$, $w|x$, $u|x$, $w|v$ are satisfied,

(b) $u|v$, $w|x$, $w|v$ are satisfied and $u|x$ is not satisfied,

(c) $u|v$, $w|x$, $u|x$ are satisfied and $w|v$ is not satisfied.

Now, it is possible for us to give the $\mathcal{CycInt}$ axioms defined to axiomatize the relation *meets* of the cyclic interval models. After each axiom is given an intuitive idea of what

[3]Here we use the notation $v_1\ m\ v_2\ m\ \ldots\ m\ v_n$ where $v_1, v_2, \ldots, v_n$ are n variables ($n > 2$) as a shorthand for the conjunction $\bigwedge_{i=1}^{n-1} v_i\ m\ v_{i+1}$.

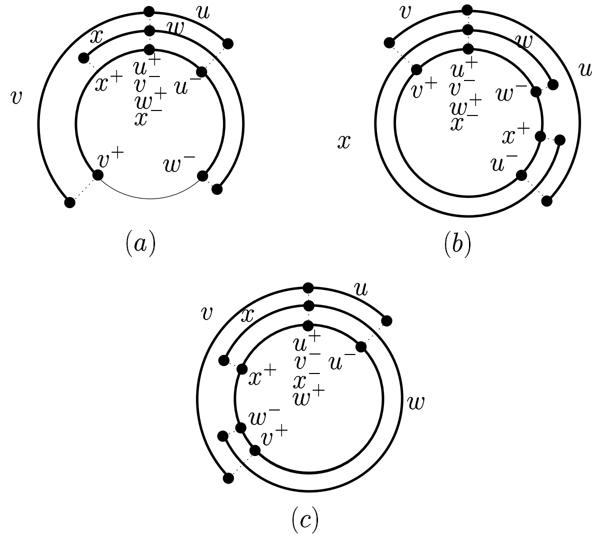

Figure 4: Satisfaction of $\mathsf{X}(u, v, w, x)$.

it expresses.

Definition 2 (The $\mathcal{C}yc\mathcal{I}nt$ axioms)

A1. $\forall u, v, w, x, y, z \quad X(u, v, w, x) \wedge X(y, z, w, x) \rightarrow X(u, v, y, z)$

Given three pairs of meeting cyclic intervals, if the meeting point defined by the first pair is the same as the one defined by the second pair and the meeting point defined by the second pair is the same as the one defined by the third pair then, the first pair and the second pair of meeting cyclic intervals define the same meeting point.

A2. $\forall u, v, w, x, y, z \quad X(u, v, w, x) \wedge X(y, u, x, z) \rightarrow \neg u|x \wedge \neg x|u$

Two cyclic intervals with the same endpoints do not satisfy the relation meets.

A3. $\forall u, v, w, x, y, z \quad u|v \wedge w|x \wedge y|z \wedge \neg u|x \wedge \neg w|v \wedge \neg u|z \wedge \neg y|v \wedge \neg w|z \wedge \neg y|x \rightarrow \exists r, s, t \quad r|s|t|r \wedge X(u, v, r, s) \wedge (X(w, x, s, t) \wedge X(y, z, t, r)) \vee (X(w, x, t, r) \wedge X(y, z, s, t))$

Three distinct meeting points can be defined by three cyclic intervals satisfying the relation meets so that these three meeting cyclic intervals cover the circle in its entirety.

A4. $\forall u, v, w, x, \quad u|v \wedge w|x \wedge \neg u|x \wedge \neg w|v \rightarrow (\exists y, z, t, \quad y|z|t|y \quad \wedge \quad X(y, z, w, x) \quad \wedge \quad X(t, y, u, v)) \wedge (\exists y, z, t, \quad y|z|t|y \wedge X(y, z, u, v) \wedge X(t, y, w, x))$

Two meeting points are the endpoints of two cyclic intervals. Each one can be defined by two other cyclic intervals.

A5. $\forall u, v \quad (\exists w, x \quad u|w|x|v|u) \rightarrow (\exists y \quad u|y|v|u)$

Two meeting cyclic intervals define another cyclic interval corresponding to the union of these cyclic intervals.

A6. $\exists u \quad u = u$ *and* $\forall u \exists v, w \quad u|v|w|u$

There exists a cyclic interval and for every cyclic intervals there exist two other cyclic intervals such that they satisfy the relation meets in a cyclic manner (they satisfy the relation meets so that they cover the circle in its entirety).

A7. $\forall u, v \quad (\exists w, x \quad w|u|x \wedge w|v|x) \leftrightarrow u = v$

There does not exist two distinct cyclic intervals with the same endpoints.

A8. $\forall u, v, w \qquad u|v|w \rightarrow \neg u|w$

Two cyclic intervals separated by a third one cannot satisfy the relation meets.

From these axioms we can deduce several theorems which will be used in the sequel.

Proposition 1 *Every structure $(\mathcal{I}, |)$ satisfying the $\mathcal{C}yc\mathcal{I}nt$ axioms satisfies the following formulas:*

B1. $\forall u, v \quad u|v \rightarrow \neg v|u$

B2. $\forall u, v, w, x, y, z \quad X(u, v, w, x) \wedge X(y, u, x, z) \rightarrow w|v \wedge y|z$

B3. $\forall u, v \quad (\exists w \quad u|w|v|u) \rightarrow (\exists x, y \quad u|x|y|v|u)$

Proof

- $(B1)$ Let u, v be two cyclic intervals satisfying $u|v$. Suppose that $v|u$ is satisfied. It follows that $\mathsf{X}(u, v, u, v)$ and $\mathsf{X}(v, u, v, u)$ are satisfied. From Axiom **A2** follows that $u|v$ and $v|u$ cannot be satisfied. There is a contradiction.
- $(B2)$ Let u,v,w,x,y,z be cyclic intervals satisfying $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, u, x, z)$. From Axiom **A2** we can deduce that $u|x$ and $x|u$ are not satisfied. As $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, u, x, z)$ are satisfied, we can assert that $y|z$ and $w|v$ are satisfied.
- $(B3)$ Let u, v, w be cyclic intervals satisfying $u|w|v|u$. We have $u|w$, $w|v$ and $v|u$ which are satisfied. Moreover, since $v|u$ is satisfied, from **B1** we can deduce that $u|v$ and $w|w$ cannot be satisfied. From Axiom **A4** follows that there exists cyclic intervals x, y, z satisfying $x|y|z|x$, $\mathsf{X}(x, y, u, w)$ and $\mathsf{X}(z, x, w, v)$. From Axiom **A2** we can assert that $x|w$ and $w|x$ are not satisfied. From it and the satisfaction of $\mathsf{X}(x, y, u, w) \wedge \mathsf{X}(z, x, w, v)$, we can assert that $u|y$ and $z|v$ are satisfied. We can conclude that u, v, y, z satisfy $u|y|z|v|u$.

□

From cyclic interval structures back to cyclic orderings

In this section, we show how to define a cyclic ordering $\prec$ onto a set of points from a set of cyclic intervals and a relation *meets* onto these cyclic intervals satisfying the $\mathcal{C}yc\mathcal{I}nt$ axioms. The line of reasoning used is similar to the one used by Ladkin (Ladkin 1987) in the linear case. Indeed, intuitively, a set of pairs of meeting cyclic intervals satisfying the relation *meets* at a same place will represent a cyclic point. Hence, a cyclic point will correspond to a meeting place. Three cyclic points l, m, n defined in this way will be in relation $\prec$ if, and only if, there exist three cyclic intervals satisfying the relation *meets* in a cyclic manner (so that they cover the circle in its entirety) so that their meeting points are successively l, m and n. Now, let us give more formally the definition of this cyclic ordering.

Let $(\mathcal{I}, |)$ be a pair defined by a set $\mathcal{I}$ and a binary relation $|$ onto $\mathcal{I}$ satisfying the $\mathcal{C}yc\mathcal{I}nt$ axioms. Let $\mathcal{J}$ be the subset of $\mathcal{I} \times \mathcal{I}$ defined by $\mathcal{J} = \{(u, v) \in \mathcal{I} \times \mathcal{I} : u|v\}$.

Definition 3 *Let $\doteq$ be the binary relation onto $\mathcal{J}$ defined by $(u, v) \doteq (w, x)$ iff $u|x$ or $w|v$.*

Note that since $u|v$ and $w|x$ are satisfied, we have $(u, v) \doteq (w, x)$ iff $\mathsf{X}(u, v, w, x)$ for all $u, v, w, x \in \mathcal{I}$.

Proposition 2 *$\doteq$ is a relation of equivalence.*

Proof From the definition of the relation $\doteq$ we can easily establish the properties of reflexivity and symmetry. Axiom **A1** allows us to assert that the relation $\doteq$ is a transitive relation. □

Given an element $(u, v) \in \mathcal{J}$, $\overline{uv}$ will denote the equivalence class corresponding to (u, v) with respect to the relation $\doteq$. Let $\mathcal{P}$ be the set of all equivalence classes of $\doteq$. We define the ternary relation $\prec$ onto $\mathcal{P}$ in the following way.

Definition 4 *Let $\overline{uv}$, $\overline{wx}$, $\overline{yz} \in \mathcal{P}$. $\prec(\overline{uv}, \overline{wx}, \overline{yz})$ iff $\exists r, s, t \in \mathcal{I}$ with $r|s|t|r$, $\overline{rs} = \overline{uv}$, $\overline{st} = \overline{wx}$ and $\overline{tr} = \overline{yz}$.*

See Figure 5 for an illustration of this definition. The

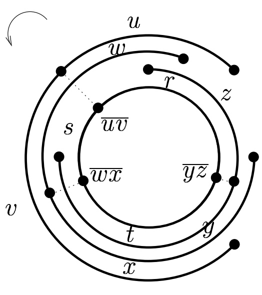

Figure 5: Satisfaction of $\prec(\overline{uv}, \overline{wx}, \overline{yz})$.

structure $(\mathcal{P}, \prec)$ obtained from $(\mathcal{I}, |)$ will be denoted by $\mathsf{CycPoint}((\mathcal{I}, |))$ in the sequel.

Theorem 1 *The structure $(\mathcal{P}, \prec)$ is a cyclic ordering.*

Proof We give the proof for Axioms $P1$ and $P2$ only. The proof for the other axioms is in the annex.

• $\forall \overline{uv}, \overline{wx} \in \mathcal{P}, \quad \neg \prec(\overline{uv}, \overline{wx}, \overline{wx}) \quad (P1)$

Let $\overline{uv}, \overline{wx} \in \mathcal{P}$. Suppose that $\prec(\overline{uv}, \overline{wx}, \overline{wx})$ is satisfied. From the definition of $\prec$, there exist $y, z, t \in \mathcal{I}$ satisfying $y|z|t|y$ and such that $(y, z) \doteq (u, v)$, $(z, t) \doteq (w, x)$, $(t, y) \doteq (w, x)$. $\doteq$ owns the properties of transitivity and symmetry, in consequence, we can assert that $(z, t) \doteq (t, y)$. From it and from the definition of $\doteq$, we have $z|y$ or $t|t$ which are satisfied. As $|$ is an irreflexive relation, we can assert that $z|y$ is satisfied. Moreover, $y|z$ is also satisfied. There is a contradiction since the relation $|$ is an asymmetric relation.

• $\forall \overline{uv}, \overline{wx}, \overline{yz}, \overline{st} \in \mathcal{P}, \prec(\overline{uv}, \overline{wx}, \overline{yz}) \wedge \prec(\overline{uv}, \overline{yz}, \overline{st}) \rightarrow \prec(\overline{uv}, \overline{wx}, \overline{st}) \quad (P2)$

Let $\overline{uv}, \overline{wx}, \overline{yz}, \overline{st} \in \mathcal{P}$ which satisfy $\prec(\overline{uv}, \overline{wx}, \overline{yz})$ and $\prec(\overline{uv}, \overline{yz}, \overline{st})$. From the definition of $\prec$ we can deduce that there exist $m, n, o \in \mathcal{I}$ satisfying $m|n|o|m$, $\overline{mn} = \overline{uv}$, $\overline{no} = \overline{wx}$, $\overline{om} = \overline{yz}$. On the other hand, we can assert that there exist $p, q, r \in \mathcal{I}$ satisfying $p|q|r|p$, $\overline{pq} = \overline{uv}$, $\overline{qr} = \overline{yz}$ and $\overline{rp} = \overline{st}$. From the property of transitivity of the relation $\doteq$ and the equalities $\overline{mn} = \overline{uv}$, $\overline{pq} = \overline{uv}$, $\overline{om} = \overline{yz}$, $\overline{qr} = \overline{yz}$, we obtain the equalities $\overline{mn} = \overline{pq}$ and $\overline{om} = \overline{qr}$. Hence, from the definition of $\doteq$, we can assert that $\mathsf{X}(m, n, p, q)$ and $\mathsf{X}(o, m, q, r)$ are satisfied. From Theorem **B2**, it follows that $p|n$ and $o|r$ are also satisfied. From all this, we can deduce that $n|o|r|p|n$ is satisfied. From Axiom **A5**, we can assert that there exists l satisfying $n|l|p|n$. By rotation, we deduce that $p|n|l|p$ is satisfied. $n|l$ and $n|o$ are satisfied, in consequence, we have $\overline{nl} = \overline{no}$. From this equality, the transitivity of the relation $\doteq$ and the equality $\overline{no} = \overline{wx}$, we can assert that $\overline{nl} = \overline{wx}$. As $l|p$ and $r|p$ are satisfied, we have the equality $\overline{lp} = \overline{rp}$. From this equality, the transitivity of the relation $\doteq$ and the equality $\overline{rp} = \overline{st}$, we can deduce that $\overline{lp} = \overline{st}$. Consequently, $p|n|l|p$, $\overline{pn} = \overline{uv}$, $\overline{nz} = \overline{wx}$ and $\overline{zp} = \overline{st}$ are satisfied. Hence, from the definition of $\prec$, we can conclude that $\prec(\overline{uv}, \overline{wx}, \overline{st})$ is satisfied. □

Cyclic orderings yield models of CycInt

In this section, we prove that every structure of cyclic intervals defined from a cyclic ordering is a model of $\mathcal{C}yc\mathcal{I}nt$.

Theorem 2 *Let $(\mathcal{P}, \prec)$ be a cyclic ordering. $(\mathcal{I}, |) = \mathsf{CycInt}((\mathcal{P}, \prec))$ is a model of the $\mathcal{C}yc\mathcal{I}nt$ axioms.*

Proof In the sequel, given an element $u = (m, n) \in \mathcal{I}$, u^- (resp. u^+) will correspond to m (resp. to n). Let us prove that the axioms of $\mathcal{C}yc\mathcal{I}nt$ are satisfied by $(\mathcal{I}, |)$.

- **(A1)** Let $u, v, w, x, y, z \in \mathcal{I}$ satisfying $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, z, w, x)$. From the definition of X we can assert that $u|v$ and $y|z$ are satisfied. Hence the equalities $u^+ = v^-$, $w^+ = x^-$ and $y^+ = z^-$. Moreover, from the definition of

X, it follows that $u|x$ or $w|v$ and $y|x$ or $w|z$ are satisfied. Let us consider all the possible situations exhaustively:

- $u|x$ and $y|x$ are satisfied. It follows that $u^+ = x^-$ and $y^+ = x^-$ are satisfied. Hence, we have $u^+ = v^- = w^+ = x^- = y^+ = z^-$.
- $u|x$ and $w|z$ are satisfied. It follows that $u^+ = x^-$ and $w^+ = z^-$ are satisfied. Consequently, $u^+ = v^- = w^+ = x^- = y^+ = z^-$ is satisfied.
- $w|v$ and $y|x$ are satisfied. It follows that $w^+ = v^-$ and $y^+ = x^-$ are satisfied. Therefore, $u^+ = v^- = w^+ = x^- = y^+ = z^-$ is satisfied.
- $w|v$ and $w|z$ are satisfied. It follows that $w^+ = v^-$ and $w^+ = z^-$ are satisfied. Hence, $u^+ = v^- = w^+ = x^- = y^+ = z^-$ is satisfied.

Let us denote by l the identical points $u^+, v^-, w^+, x^-, y^+, z^-$. Suppose that $\mathsf{X}(u, v, y, z)$ is falsified. By using the fact that $u|v$ and $y|z$ are satisfied, we deduce that $u|z$ and $y|v$ are not satisfied. Since $u^+ = z^-$ and $y^+ = v^-$, $\prec (u^-, l, z^+)$ and $\prec (y^-, l, v^+)$ are not satisfied. From $\mathbf{P5}$, we get the satisfaction of $\prec (u^-, z^+, l)$ and the one of $\prec (y^-, v^+, l)$. As $u|v$ and $y|z$ are satisfied, $\prec (u^-, l, v^+)$ and $\prec (y^-, l, z^+)$ are also satisfied. Hence, by using $\mathbf{P4}$, we can assert that $\prec (l, y^-, v^+)$ and $\prec (l, v^+, u^-)$ are satisfied. From $\mathbf{P2}$, it follows that $\prec (l, y^-, u^-)$ is also satisfied. From the satisfaction of $\prec (u^-, z^+, l)$ and the one of $\mathbf{P4}$, it follows that $\prec (l, u^-, z^+)$ is satisfied. By using $\mathbf{P2}$, it results that $\prec (l, y^-, z^+)$ is satisfied. Recall that $\prec (y^-, l, z^+)$ is satisfied. From $\mathbf{P4}$ and $\mathbf{P2}$, it results that $\prec (y^-, z^+, z^+)$ is satisfied. From $\mathbf{P1}$, a contradiction follows. Consequently, we can conclude that $\mathsf{X}(u, v, y, z)$ is satisfied.

- **(A2)** Let $u, v, w, x, y, z \in \mathcal{I}$ satisfy $\mathsf{X}(u, v, w, x)$ and $\mathsf{X}(y, u, x, z)$. The following equalities are satisfied: $u+ = x-$ and $x^+ = u^-$. By using $\mathbf{P4}$ and $\mathbf{P1}$, we can assert that $\prec (u^-, u^+, x^+)$ and $\prec (x^-, x^+, u^+)$ cannot be satisfied. Hence, $u|x$ and $x|u$ are not satisfied.
- **(A3)** Let us prove the satisfaction of Axiom $\mathbf{A3}$. Let $u, v, w, x, y, z \in \mathcal{I}$ satisfying $u|v$, $w|x$, $y|z$, $\neg u|x$, $\neg w|v$, $\neg u|z$, $\neg y|v$, $\neg w|z$, $\neg y|x$. From the satisfaction of $u|v$ (resp. $w|x$ and $y|z$), it follows that $u^+ = v^-$ (resp. $w^+ = x^-$ and $y^+ = z^-$). Let l (resp. m and n) the point defined by $l = u^+ = v^-$ (resp. $m = w^+ = x^-$ and $n = y^+ = z^-$). Suppose that $l = m$. the equality $u^+ = v^- = w^+ = x^-$ is satisfied. Since $w|x$ is true, we can deduce that $\prec (w^-, x^-, x^+)$ is also satisfied. Consequently, w^- and x^+ are distinct points. Let us consider the three points u^-, w^-, x^+. From $\mathbf{P3}$, we can assert that only four cases are possible: $u^- = w^-$ is satisfied, $u^- = x^+$ is satisfied, $\prec (w^-, x^+, u^-)$ is satisfied, or $\prec (w^-, u^-, x^+)$ is satisfied. By using, $\mathbf{P2}$, $\mathbf{P3}$ and $\mathbf{P4}$, we obtain for every case a contradiction:
  - $u^- = w^-$ is satisfied. As $w|x$ is satisified, $\prec (u^-, x^-, x^+)$ is also satisfied. Recall that $u^+ = x^-$. It follows that $u|x$ is satisfied. There is a contradiction.
  - $u^- = x^+$ is satisfied. As $u|v$ and $w|x$ are satisfied, we can assert that $\prec (u^+, v^-, v^+)$ and $\prec (w^-, x^-, x^+)$ are satisfied. Hence, $\prec (x^+, v^-, v^+)$ and $\prec (w^-, v^-, x^+)$ are also satisfied. By using $\mathbf{P4}$, we can deduce that $\prec (v^-, v^+, x^+)$ and $\prec (v^-, x^+, w^-)$ are satisfied. From $\mathbf{P2}$ it follows that $\prec (v^-, v^+, w^-)$ is also satisfied. From $\mathbf{P4}$ follows the satisfaction of $\prec (w^-, v^-, v^+)$. Moreover, we have the equality $w^+ = v^-$. Consequently, $w|v$ is satisfied. There is a contradiction.
  - $\prec (w^-, x^+, u^-)$ is satisfied. From $\mathbf{P4}$, we obtain the satisfaction of $\prec (x^+, u^-, w^-)$. As $w|x$ is satisfied, we deduce that $\prec (w^-, x^-, x^+)$ is satisfied. Hence, $\prec (x^+, w^-, x^-)$ is also satisfied ($\mathbf{P4}$). From $\mathbf{P2}$, we can assert that $\prec (x^+, u^-, x^-)$ is satisfied. From $\mathbf{P4}$, $\prec (u^-, x^-, x^+)$ is satisfied. As $x^- = u^+$ is satisfied, we can assert that $u|x$ is satisfied. There is a contradiction.
  - $\prec (w^-, u^-, x^+)$ is satisfied. Hence, u^- and x^+ are distinct points. Moreover, we know that u^+ and x^+ are distinct points from the fact that x^- and u^+ are equal. From $\mathbf{P3}$, $\prec (u^-, x^+, x^-)$ or $\prec (u^-, x^-, x^+)$ is satisfied. Suppose that $\prec (u^-, x^-, x^+)$ is satisfied. Since we have the equality $u^+ = x^-$, $u|x$ is satisfied. There is a contradiction. It results that $\prec (u^-, x^+, x^-)$ must be satisfied. From the satisfaction of $\prec (w^-, u^-, x^+)$ and $\mathbf{P4}$, we deduce that $\prec (u^-, x^+, w^-)$ is satisfied. From the satisfaction of $w|x$ and from $\mathbf{P4}$, we can assert that $\prec (x^-, x^+, w^-)$ is satisfied. $\prec (u^-, x^+, x^-)$ is satisfied, hence, from $\mathbf{P4}$ we can deduce that $\prec (u^-, x^+, x^-)$ is satisfied. From $\mathbf{P4}$, we obtain the satisfaction of $\prec (x^-, u^-, x^+)$. From $\mathbf{P2}$, it results that $\prec (x^-, u^-, w^-)$ is satisfied. Hence, $\prec (u^+, u^-, w^-)$ is satisfied. From the satisfaction of $u|v$ and from $\mathbf{P4}$ it follows that $\prec (u^+, v^+, u^-)$ is satisfied. From $\mathbf{P2}$ we can assert that $\prec (u^+, v^+, w^-)$ is satisfied. In consequence, $\prec (w^+, v^+, w^-)$ is satisfied. Hence, from $\mathbf{P4}$, $\prec (w^-, w^+, v^+)$ is satisfied. It results that $w|v$ is satisfied. There is a contradiction.

Consequently, we can assert that $l \neq m$. In a similar way, we can prove that $l \neq n$ and $m \neq n$. Now, we know that l, m, n are distinct points. From $\mathbf{P3}$, we can just examine two cases:

- $\prec (l, m, n)$ is satisfied. Let $r = (n, l)$, $s = (l, m)$ and $t = (m, n)$. We have $r|s|t|r$ which is satisfied. Suppose that $u|s$ is falsified. It follows that $\prec (u^-, l, m)$ is also falsified. As l is different from u^- and m, we have $u^- = m$ or $\prec (u^-, m, l)$ which is satisfied.
  - ∗ Suppose that $u^- = m$ is satisfied. Since $u|v$ is satisfied, it follows that $\prec (u^-, u^+, v^+)$ is satisfied. Consequently, $\prec (m, l, v^+)$ is true. From $\mathbf{P4}$, it follows that $\prec (l, v^+, m)$ is satisfied. From all this, the satisfaction of $\prec (l, m, n)$ and $\mathbf{P2}$, we can assert that $\prec (l, v^+, n)$ is satisfied. From $\mathbf{P4}$, we deduce that $\prec (n, l, v^+)$ is satisfied. As $l = v^-$, $r|v$ is satisfied.
  - ∗ Suppose that $\prec (u^-, m, l)$ is satisfied. From $\mathbf{P4}$, it follows that $\prec (l, u^-, m)$ is satisfied. From all this, the satisfaction of $\prec (l, m, n)$ and $\mathbf{P2}$, we can assert that $\prec (l, u^-, n)$ is satisfied. As $u|v$ is satisfied, we can deduce that $\prec (u^-, u^+, v^+)$ is satisfied. Consequently, $\prec (u^-, l, v^+)$ is also satisfied. From $\mathbf{P4}$, it results that $\prec (l, v^+, u^-)$ is satisfied. From all this

and the satisfaction of $\prec (l, u^-, n)$, we can deduce that $\prec (l, v^+, n)$ is satisfied. By using **P4**, we obtain the satisfaction of $\prec (n, l, v^+)$. As $l = v^-$, we deduce that $r|v$ is satisfied.

It results that $u|s$ or $r|v$ is satisfied. Hence, $\mathsf{X}(u, v, r, s)$ is satisfied. With a similar line of reasoning, we can prove that $\mathsf{X}(w, x, s, t)$ and $\mathsf{X}(y, z, t, r)$ are satisfied.

– $\prec (l, n, m)$ is satisfied. Let $r = (m, l)$, $s = (l, n)$ and $t = (n, m)$. We have $r|s|t|r$ which is satisfied. In a similar way, we can prove that $\mathsf{X}(u, v, r, s)$, $\mathsf{X}(y, z, s, t)$ and $\mathsf{X}(w, x, t, r)$ are satisfied.

- For Axioms **A4-A5-A6-A7-A8**, the proofs can be found in the annex.

□

Categoricity of $\mathcal{C}yc\mathcal{I}nt$

In this section, we establish the fact that the countable models satisfying the $\mathcal{C}yc\mathcal{I}nt$ axioms are isomorphic. In order to prove this property, let us show that for every cyclic interval there exist two unique "endpoints".

Proposition 3 *Let $\mathcal{M} = (\mathcal{I}, |)$ a model of $\mathcal{C}yc\mathcal{I}nt$. Let $(\mathcal{P}, \prec)$ be the structure* $\mathsf{CycPoint}(\mathcal{M})$. *For every $u \in \mathcal{I}$ there exist $L_u, U_u \in \mathcal{P}$ such that :*

1. *$\exists v \in \mathcal{I}$ such that $(v, u) \in L_u$,*

2. *$\exists w \in \mathcal{I}$ such that $(u, w) \in U_u$,*

3. *L_u (resp. U_u) is the unique element of $\mathcal{P}$ satisfying (1.) (resp. (2.)),*

4. *$L_u \neq U_u$.*

Proof From Axiom **A6**, we can assert that there exist $v, w \in \mathcal{I}$ such that $u|w|v|u$ is satisfied. Consequently, $u|w$ and $v|u$ are satisfied. By defining L_u by $L_u = \overline{vu}$ and U_u by $U_u = \overline{uw}$, the properties (1) and (2) are satisfied. Now, let us prove that the property (3) is satisfied. Suppose that there exists L'_u such that there exists $x \in \mathcal{I}$ with $(x, u) \in L'_u$. We have $(v, u) \equiv (x, u)$. It follows that $L_u = L'_u$. Now, suppose that there exists U'_u such that there exists $y \in \mathcal{I}$ with $(u, y) \in U'_u$. We have $(u, w) \equiv (u, y)$. It follows that $U_u = U'_u$. Hence, we can assert that property (3) is true. Now, suppose that $L_u = U_u$. It follows that $(v, u) \equiv (u, w)$. As a result, $v|w$ or $u|u$ is satisfied. We know that $|$ is an irreflexive relation. Moreover, from Axiom **A8** we can assert that $v|w$ cannot be satisfied. It results that there is a contradiction. Hence, L_u and U_u are distinct elements. □

From an initial model of $\mathcal{C}yc\mathcal{I}nt$, we have seen that we can define a cyclic ordering. Moreover, from this cyclic ordering we can generate a cyclic interval model. We are going to show that this generated cyclic interval model is isomorphic to the initial cyclic interval model.

Proposition 4 *Let $\mathcal{M} = (\mathcal{I}, |)$ a model of the $\mathcal{C}yc\mathcal{I}nt$ axioms. $\mathcal{M}$ is isomorphic to $(\mathcal{I}', |') = \mathsf{CycInt}(\mathsf{CycPoint}(\mathcal{M}))$.*

Proof Let f be the mapping from $\mathcal{I}$ onto $\mathcal{I}'$ defined by $f(u) = (L_u, U_u)$, *i.e.* $f(u) = (\overline{vu}, \overline{uw})$ for any $v, w \in \mathcal{I}$ satisfying $v|u$ and $u|w$. Let us show that f is a one-to-one mapping. Let $(\overline{uv}, \overline{wx}) \in \mathcal{I}'$. We have $u|v$ and $w|x$ which are satisfied and $u|x$ and $w|v$ which are falsified (in the contrary case we would have $\overline{uv} = \overline{wx}$). From **A4**, it follows that there exist y, z, t satisfying $y|z|t|y$, $\mathsf{X}(y, z, w, x)$ and $\mathsf{X}(t, y, u, v)$. Note that $L_y = \overline{ty} = \overline{uv}$ and $U_y = \overline{yz} = \overline{wx}$. Consequently, there exists $y \in \mathcal{I}$ such that $f(y) = (\overline{uv}, \overline{wx})$. Now, suppose that there exist $u, v \in \mathcal{I}$ such that $f(u) = f(v)$. Suppose that $f(u) = (\overline{wu}, \overline{ux})$ and $f(v) = (\overline{yv}, \overline{vz})$. We have $\overline{wu} = \overline{yv}$ and $\overline{ux} = \overline{vz}$. It follows that $(w, u) \doteq (y, v)$ and $(u, x) \doteq (v, z)$. From all this, we have $w|u$, $y|v$, $u|x$ and $v|z$ which are satisfied. Four possible situations must be considered:

- $w|v$ and $u|z$ are satisfied. It follows that $w|v|z$ and $w|u|z$ are satisfied.
- $w|v$ and $v|x$ are satisfied. It follows that $w|v|x$ and $w|u|x$ are satisfied.
- $y|u$ and $u|z$ are satisfied. It follows that $y|v|z$ and $y|u|z$ are satisfied.
- $y|u$ and $v|x$ are satisfied. It follows that $y|v|z$ and $y|u|z$ are satisfied.

For each case, by using **A7**, we can deduce the equality $u = v$. Consequently, f is a one-to-one mapping.

Now, let us show that $u|v$ if, and only if, $f(u)|'f(v)$. We will denote $f(u)$ by $(\overline{wu}, \overline{ux})$ and $f(v)$ by $(\overline{yv}, \overline{vz})$. Suppose that $u|v$ is satisfied. It follows that $(u, x) \doteq (y, v)$, hence, $\overline{ux} = \overline{yv}$. For this reason, $f(u)|'f(v)$ is satisfied. Now, suppose that $f(u)|'f(v)$ is satisfied. It follows that $\prec (\overline{wu}, \overline{ux}, \overline{vz})$ and $\overline{ux} = \overline{yv}$ are satisfied. Hence, there exist $r, s, t \in \mathcal{I}$ such that $r|s|t|r$, $\overline{rs} = \overline{wu}$, $\overline{st} = \overline{ux}$ and $\overline{tr} = \overline{vz}$ are satisfied. From the equalities $\overline{rs} = \overline{wu}$ and $\overline{st} = \overline{ux}$, we can assert that $u|x$, $s|t$, $r|s$ and $w|u$ are satisfied. Moreover, one of the following cases is satisfied:

- $r|u$ and $u|t$ are satisfied. It follows that $r|u|t$ and $r|s|t$ are satisfied.
- $r|u$ and $s|x$ are satisfied. It follows that $r|s|x$ and $r|u|x$ are satisfied.
- $w|s$ and $u|t$ are satisfied. It follows that $w|u|t$ and $w|s|t$ are satisfied.
- $w|s$ and $s|x$ are satisfied. It follows that $w|s|x$ and $w|u|x$ are satisfied.

For each case, from **A7**, we can deduce the equality $u = s$. From the equalities $\overline{st} = \overline{yv}$ and $\overline{tr} = \overline{vz}$, we can deduce that $s|t$, $y|v$, $t|r$ and $v|z$ are satisfied. Moreover, one of the following cases is satisfied:

- $s|v$ and $t|z$ are satisfied. It follows that $s|t|z$ and $s|v|z$ are satisfied.
- $s|v$ and $v|r$ are satisfied. It follows that $s|v|r$ and $s|t|r$ are satisfied.
- $y|t$ and $t|z$ are satisfied. It follows that $y|t|z$ and $y|v|z$ are satisfied.
- $y|t$ and $v|r$ are satisfied. It follows that $y|t|r$ and $y|v|r$ are satisfied.

For each case, from Axiom **A7**, we can deduce that $v = t$. Hence, we have the equalities $u = s$ and $v = t$. We can conclude that $u|v$ is satisfied. □

Now, let us show that two cyclic interval models generated by two countable cyclic orderings are isomorphic.

Proposition 5 *Let* $(\mathcal{P}, \prec)$ *and* $(\mathcal{P}', \prec')$ *be two cyclic orderings with* $\mathcal{P}$ *and* $\mathcal{P}'$ *two countable sets of points.* $\mathsf{CycInt}((\mathcal{P}, \prec))$ *and* $\mathsf{CycInt}((\mathcal{P}', \prec'))$ *are isomorphic.*

Proof Let $(\mathcal{I}, |)$ and $(\mathcal{I}', |')$ be defined by $\mathsf{CycInt}((\mathcal{P}, \prec))$ and $\mathsf{CycInt}((\mathcal{P}', \prec'))$. We know that $(\mathcal{P}, \prec)$ and $(\mathcal{P}', \prec')$ are isomorphic (Balbiani, Condotta, & Ligozat 2002). Let g be an isomorphism from $(\mathcal{P}, \prec)$ to $(\mathcal{P}', \prec')$. Let h be the mapping from $\mathcal{I}$ onto $\mathcal{I}'$ defined by $h((l, m)) = (g(l), g(m))$. First, let us show that $(g(l), g(m)) \in \mathcal{I}'$. As $(l, m) \in \mathcal{I}$, there exists $n \in \mathcal{P}$ satisfying $\prec (l, m, n)$. It follows that $\prec' (g(l), g(m), g(n))$ is satisfied. It results that $(g(l), g(m)) \in \mathcal{I}'$. Now, let us show that for every $(l, m) \in \mathcal{I}'$, there exists $(n, o) \in \mathcal{I}$ such that $h((n, o)) = (l, m)$. We can define n and o by $n = g^{-1}(l)$ and $o = g^{-1}(m)$. Indeed, $h(g^{-1}(l), g^{-1}(m)) = (g(g^{-1}(l)), g(g^{-1}(m))) = (l, m)$. Now, let $(l, m), (n, o) \in \mathcal{I}$ such that $h((l, m)) = h((n, o))$. It follows that $g(l) = g(n)$ and $g(m) = g(o)$. Therefore, we have $l = n$ and $m = o$. Hence, we obtain the equality $(l, m) = (n, o)$. Finally, let us show that for all $(l, m), (n, o) \in \mathcal{I}$, $(l, m)|(n, o)$ is satisfied iff $h((l, m))|'h((n, o))$ is satisfied. $(l, m)|(n, o)$ is satisfied iff $\prec (l, m, o)$ and $m = n$ are satisfied. Hence, $(l, m)|(n, o)$ is satisfied iff $\prec' (g(l), g(m), g(o))$ and $g(m) = g(n)$ are satisfied. For these reasons, we can assert that $(l, m)|(n, o)$ is satisfied iff $h((l, m))|'h((n, o))$ is satisfied. We can conclude that h is an isomorphism. □

In the sequel, $(\mathcal{Q}, \prec)$ will correspond to the cyclic ordering on the set of rational numbers $\mathcal{Q}$, defined by $\prec (x, y, z)$ iff $x < y < z$ or $y < z < x$ or $z < x < y$, with $x, y, z \in \mathcal{Q}$ and $<$ the usual linear order on $\mathcal{Q}$. It is time to establish the main result of this section.

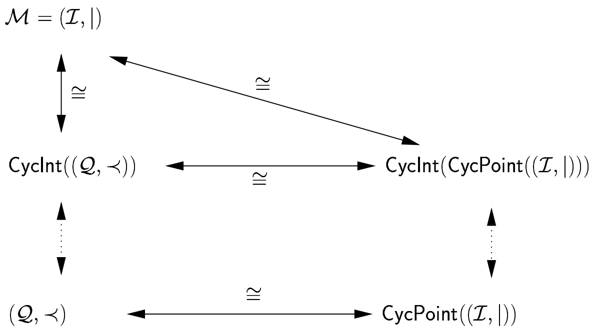

Figure 6: Every countable model of $\mathcal{CycInt}$ $(\mathcal{I}, |)$ is isomorphic to $\mathsf{CycInt}((\mathcal{Q}, \prec))$.

Theorem 3 *The theory axiomatized by* $\mathcal{CycInt}$ *is* $\aleph_0$*–categorical. Moreover, its countable models are isomorphic to* $\mathsf{CycInt}((\mathcal{Q}, \prec))$.

Proof Let $\mathcal{M}$ a model of $\mathcal{CycInt}$. $\mathcal{M}$ is isomorphic to $\mathsf{CycInt}(\mathsf{CycPoint}(\mathcal{M}))$. $\mathsf{CycInt}(\mathsf{CycPoint}(\mathcal{M}))$ is isomorphic to $\mathsf{CycInt}((\mathcal{Q}, \prec))$. By composing the isomorphisms, we have $\mathsf{CycInt}((\mathcal{Q}, \prec))$ which is isomorphic to $\mathcal{M}$. □

As a direct consequence of this theorem we have that the set of the theorems of $\mathcal{CycInt}$ is syntactically complete and decidable.

Application to constraint networks

Balbiani and Osmani (Balbiani & Osmani 2000) use constraint networks to represent the qualitative information about cyclic intervals. A network is defined as a pair (V, C), where V is a set of variables representing cyclic intervals and C is a map which, to each pair of variables (V_i, Vj) associates a subset C_{ij} of the set of all sixteen basic relations. The main problem in this context is the consistency problem, which consists in determining whether the network has a so-called solution: a solution is a map m from the set of variables V_i to the set of cyclic intervals in $\mathcal{C}$ such that all constraints are satisfied. The constraint C_{ij} is satisfied if and only if, denoting by m_i and m_j the images of V_i and V_j respectively, the cyclic interval m_i is in one of the relations in the set C_{ij} with respect to m_j (the set C_{ij} is consequently given a disjunctive interpretation in terms of constraints).

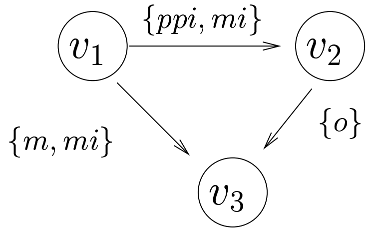

Figure 7: A constraint network on cyclic intervals.

A first interesting point is the fact that the axiomatization we have obtained allows us to check the consistency of a constraint network on cyclic intervals by using a theorem prover. Indeed, the procedure goes as follows: First, translate the network (V, C) into an *equivalent* logical formula Φ. Then, test the validity of the formula (or its validity in a specific model) by using the $\mathcal{CycInt}$ axiomatization.

As an example, consider the constraint network in Figure 7. The corresponding formula is $\Phi = (\exists v_1, v_2, v_3)\ ((v_1\ ppi\ v_2 \vee v_1\ mi\ v_2) \wedge (v_1\ m\ v_3 \vee v_1\ mi\ v_3) \wedge (v_2\ o\ v_3))$.

In order to show that this network is consistent, we would have to prove that this formula is valid with respect to $\mathcal{CycInt}$, or satisfiable for a model such as $\mathcal{C}$. In order to show inconsistency, we have to consider the negation of Φ.

Usually a local constraint propagation method, called the path-consistency method, is used to solve this kind of constraint network. The method [4] consists in removing from

[4] In the case of cyclic interval networks, the path-consistency

each constraint C_{ij} all relations which are not compatible with the constraints in C_{ik} and C_{kj}, for all 3-tuples i, j, k. This is accomplished by using the composition table of the cyclic interval calculus which, for each pair (a, b) of basic relations, gives the composition of a with b, that is the set of all basic relations c such that there exists a configuration of three cyclic intervals u, v, w with $u\ a\ v$, $v\ b\ w$ and $u\ c\ w$. For instance, the composition of m with d consists in the relation ppi.The composition table of the cyclic interval calculus can be automatically computed by using our axiomatization. Indeed, in order to decide whether c belongs to the composition of a with b, it suffices to prove that the formula $(\exists u, v, w)\ \ (u\ a\ v \wedge v\ b\ w \wedge u\ c\ w)$ is valid. In order to prove that, conversely, c does not belong to this composition, one has to consider the negated formula $\neg(\exists u, v, w)\ \ (u\, a\, v \wedge v\, b\, w \wedge u\, c\, w)$.

Conclusions and further work

We have shown in in paper how the theory of cyclic orderings, on the one hand, and the theory of cyclic intervals, on the other hand, can be related. We proposed a set of axioms for cyclic intervals and showed that each countable model is isomorphic to the model based on cyclic intervals on the rational circle. Determining whether the first order theory of the *meets* relation between cyclic orderings admits the elimination of quantifiers is to our knowledge an open problem we are currently examining. Another question is whether the axioms of the $\mathcal{C}ycInt$ theory are independent. Still another interesting direction of research is the study of finite models of cyclic intervals. To this end, we will have to consider discrete cyclic orderings (which consequently do not satisfy axiom **P5**). This could lead to efficient methods for solving the consistency problem for cyclic interval networks: Since these involve only a finite number of variables, they should prove accessible to the use of finite models.

Annex

Proof (End of proof of Theorem 1)

- $\forall \overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}, \overline{uv} \neq \overline{wx} \wedge \overline{wx} \neq \overline{yz} \wedge \overline{uv} \neq \overline{yz} \rightarrow\ \prec(\overline{uv}, \overline{wx}, \overline{yz}) \vee \prec(\overline{uv}, \overline{yz}, \overline{wx})\ \ (P3)$

  Let $\overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}$ satisfying $\overline{uv} \neq \overline{wx}$, $\overline{wx} \neq \overline{yz}$ and $\overline{uv} \neq \overline{yz}$. From the definitions of $\mathcal{P}$ and $\doteq$ we can assert that $u|v$, $w|x$, $y|z$, $\neg u|x$, $\neg w|v$, $\neg u|z$, $\neg y|v$, $\neg w|z$, $\neg y|x$ are satisfied. From Axiom **A3** we can deduce that there exist r, s, t satisfying $r|s|t|r$ and such that $\mathsf{X}(u, v, r, s)$, $\mathsf{X}(w, x, s, t)$, $\mathsf{X}(y, z, t, r)$ or $\mathsf{X}(u, v, r, s)$, $\mathsf{X}(w, x, t, r)$, $\mathsf{X}(y, z, s, t)$ are satisfied. From all this, we can conclude that $\prec(\overline{uv}, \overline{wx}, \overline{yz}) \vee \prec(\overline{uv}, \overline{yz}, \overline{wx})$ is satisfied.

- $\forall \overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}, \prec(\overline{uv}, \overline{wx}, \overline{yz}) \leftrightarrow \prec(\overline{wx}, \overline{yz}, \overline{uv}) \leftrightarrow \prec(\overline{yz}, \overline{uv}, \overline{wx})\ \ (P4)$

  Let $\overline{uv}, \overline{wx}, \overline{yz} \in \mathcal{P}$ satisfying $\prec(\overline{uv}, \overline{wx}, \overline{yz})$. From the definition of $\prec$, we have $u|v$, $w|x$ and $y|z$ which are satisfied and there exist r, s, t satisfying $r|s|t|r$, $\overline{rs} = \overline{uv}$, $\overline{st} = \overline{wx}$ and $\overline{tr} = \overline{yz}$. By rotation, we can assert that $s|t|r|s$ is also satisfied. From this, we can deduce that $\prec(\overline{wx}, \overline{yz}, \overline{uv})$ is satisfied. In a similar way, we can prove that $\prec(\overline{wx}, \overline{yz}, \overline{uv}) \rightarrow \prec(\overline{yz}, \overline{uv}, \overline{wx})$ and $\prec(\overline{yz}, \overline{uv}, \overline{wx}) \rightarrow \prec(\overline{uv}, \overline{wx}, \overline{yz})$ are satisfied.

- $\forall \overline{uv}, \overline{wx} \in \mathcal{P}, \overline{uv} \neq \overline{wx} \rightarrow ((\exists \overline{yz} \in \mathcal{P}, \prec(\overline{uv}, \overline{wx}, \overline{yz})) \wedge (\exists \overline{rs} \in \mathcal{P}, \prec(\overline{uv}, \overline{rs}, \overline{wx})))\ \ (P5)$

  Let $\overline{uv}, \overline{wx} \in \mathcal{P}$ such that $\overline{uv} \neq \overline{wx}$. From the definition of $\mathcal{P}$ and the one of the relation $\doteq$ we can assert that $u|v$, $w|x$, $\neg u|x$ and $\neg w|v$ are satisfied. From Axiom **A4** we deduce that there exist y, z, t such that $y|z|t|y\ \wedge\ \mathsf{X}(y, z, w, x)\ \wedge\ \mathsf{X}(t, y, u, v))$ is satisfied and that there exist q, r, s such that $q|r|s|q\ \wedge\ \mathsf{X}(q, r, u, v)\ \wedge\ \mathsf{X}(s, q, w, x))$ is satisfied. Consequently, there exists y, z, t such that $\prec(\overline{yz}, \overline{zt}, \overline{ty})$, $\overline{yz} = \overline{wx}$, $\overline{ty} = \overline{uv}$ are satisfied and there exist q, r, s such that $\prec(\overline{qr}, \overline{rs}, \overline{sq})$, $\overline{qr} = \overline{uv}$, $\overline{sq} = \overline{wx}$ are satisfied. Hence, there exists $\overline{zt} \in \mathcal{P}$ such that $\prec(\overline{wx}, \overline{zt}, \overline{uv})$ is satisfied , and there exists $\overline{rs} \in \mathcal{P}$ such that $\prec(\overline{uv}, \overline{rs}, \overline{wx})$ is satisfied. From **C3** we can conclude that there exists $\overline{zt} \in \mathcal{P}$ satisfying $\prec(\overline{uv}, \overline{wx}, \overline{zt})$, and that there exists $\overline{rs} \in \mathcal{P}$ satisfying $\prec(\overline{uv}, \overline{rs}, \overline{wx})$.

- $\exists \overline{uv}, \overline{wx} \in \mathcal{P},\ \ \overline{uv} \neq \overline{wx}.\ \ (P6)$

  From Axiom **A6** we can assert that there exist u, v, w satisfying $u|v|w|u$. Hence, there exist $\overline{uv}, \overline{vw}, \overline{wu} \in \mathcal{P}$ such that $\prec(\overline{uv}, \overline{vw}, \overline{wu})$ is satisfied. From **P1** we deduce that $\overline{uv}$ and $\overline{vw}$ are distinct classes.

□

Proof (End of proof of Theorem 2)

- **(A4)** Let $u, v, w, x \in \mathcal{I}$ satisfying $u|v$, $w|x$, $\neg u|x$, and $\neg w|v$. $\prec(u^-, u^+, v^+)$, $\prec(w^-, w^+, x^+)$ with $u^+ = v^-$ and $w^+ = x^-$ are satisified. Let l and m defined by $l = u^+ = v^-$ and $m = w^+ = x^-$. Suppose that $l = m$. As $\prec(u^-, u^+, v^+)$ and $\prec(w^-, w^+, x^+)$ are satisfied, we have $\prec(u^-, l, v^+)$ and $\prec(w^-, l, x^+)$ which are also satisfied. Hence, we have $u^- \neq l$ and $x^+ \neq l$. From **P3**, we can just consider three cases: $u^- = x^+$ is satisfied, $\prec(u^-, l, x^+)$ is satisfied, or $\prec(u^-, x^+, l)$ is satisfied. From **P2** and **P4**, we can deduce a contradiction for every case. We can assert that $l \neq m$. From **P5**, we can deduce there exist $n, o \in \mathcal{P}$ satisfying $\prec(l, m, n)$ and $\prec(l, o, n)$. Let us define three cyclic intervals y, z, t by $y = (l, m)$, $z = (m, n)$ and $t = (n, l)$. From the satisfaction of $\prec(l, m, n)$ and **P4**, we can deduce that $y|z|t|y$ is satisfied. Let us suppose that $y|x$ is not satisfied. As $y^+ = x^-$, it follows that $\prec(y^-, y^+, x^+)$ is not satisfied. We have $y^- \neq y^+$ and $y^+ \neq x^+$. From **P3**, it follows that $y^- = x^+$ or $\prec(y^-, x^+, y^+)$ is satisfied. Let us examine these two possible cases.

method is not complete even for atomic networks: path-consistency does not insure consistency.

– $y^- = x^+$ is satisfied. It follows that $x^+ = l = u^+ = v^-$. From the satisfaction of $w|x$, we have $\prec (w^-, w^+, x^+)$ which is satisfied, with $w^+ = x^-$. Since $\prec (l, m, n)$ is satisfied, $\prec (x^+, w^+, n)$ is also satisfied. From $\mathbf{P4}$, we can deduce that $\prec (w^+, x^+, w^-)$ and $\prec (w^+, n, x^+)$ are satisfied. From $\mathbf{P2}$ follows that $\prec (w^+, n, w^-)$ is satisfied. Hence, from $\mathbf{P4}$, we obtain the satisfaction of $\prec (w^-, w^+, n)$. As $w^+ = m$, $w|z$ is satisfied.

– $\prec (y^-, x^+, y^+)$ is satisfied. Hence, $\prec (l, x^+, w^+)$ is satisfied. As $\prec (l, m, n)$ is satisfied, $\prec (l, w^+, n)$ is also satisfied. From $\mathbf{P4}$, it follows that $\prec (w^+, n, l)$ and $\prec (w^+, l, x^+)$ are satisfied. From $\mathbf{P2}$, we can deduce that $\prec (w^+, n, x^+)$ is satisfied. As $w|x$ is satisfied, $\prec (w^-, w^+, x^+)$ is satisfied, with $w^+ = x^-$. From $\mathbf{P4}$, we have $\prec (w^+, x^+, w^-)$ which is satisfied. From $\mathbf{P2}$, we deduce that $\prec (w^+, n, w^-)$ is satisfied. From $\mathbf{P4}$, it follows that $\prec (w^-, w^+, n)$ is satisfied. We have $w^+ = m$. It results that $w|z$ is satisfied.

Hence, $\mathsf{X}(y, z, w, x)$ is satisfied. In a similar way, we can prove that $\mathsf{X}(t, y, u, v)$ is satisfied. By defining y, z, t by $y = (m, l)$, $z = (l, o)$ and $t = (o, m)$, we can also prove that $\mathsf{X}(y, z, u, v)$ and $\mathsf{X}(t, y, w, x)$ are satisfied.

- **(A5)** Let $u, v, w, x \in \mathcal{I}$ satisfying $u|w|x|v|u$. We have the following equalities: $u^+ = w^-$, $w^+ = x^-$, $x^+ = v^-$ and $v^+ = u^-$. Let us define l_1 (resp. l_2, l_3 and l_4) by $l_1 = u^+ = w^-$ (resp. $l_2 = w^+ = x^-$, $l_3 = x^+ = v^-$ and $l_4 = v^+ = u^-$). Consider the pair $y = (l_1, l_3)$. As $w|x$ is satisfied, we can deduce the satisfaction of $\prec (l_1, l_2, l_3)$. Hence, we can assert that $l_1 \neq l_3$. From $\mathbf{P5}$, it follows that there exists l satisfying $\prec (l_1, l_3, l)$. It results that $y = (l_1, l_3)$ belongs to $\mathcal{I}$.
  Suppose that $u|y$ is not satisfied. Since $u^+ = l_1$, $\prec (u^-, l_1, l_3)$ is not satisfied. u^- and l_1 are distinct points and, l_1 and l_3 are also distinct points. From the satisfaction of $v|u$, we can deduce that $\prec (l_3, u^-, u^+)$ is satisfied. It follows that $l_3 \neq u^-$. Consequently, Axiom $\mathbf{P3}$ and the non satisfaction of $u|y$ allow us to assert that $\prec (u^-, l_3, l_1)$ is satisfied. As $v|u$ is satisfied, $\prec (l_3, u^-, l_1)$ is also satisfied. From $\mathbf{P4}$ and from $\mathbf{P2}$, it follows that $\prec (l_3, u^-, u^-)$ is satisfied. From Axiom $\mathbf{P1}$, it results a contradiction. In consequence, $u|y$ is satisfied. With a similar line of reasoning, by supposing that $y|v$ is not satisfied, we obtain a contradiction. Hence, $u|y|v|u$ is satisfied.
- **(A6)** From $\mathbf{P6}$, we can deduce that there exist $l, m \in \mathcal{P}$ such that $l \neq m$. From $\mathbf{P5}$, it follows that there exists n satisfying $\prec (l, m, n)$. Let $u = (l, m)$, we have $u \in \mathcal{I}$ and $u = u$. Now, let us prove the second part of the axiom. Let $u = (l, m) \in \mathcal{I}$. By definition of $\mathcal{I}$, there exists $n \in \mathcal{P}$ such that $\prec (l, m, n)$. Let $v = (m, n)$ and $w = (n, l)$. From $\mathbf{P4}$, $\prec (m, n, l)$ and $\prec (n, l, m)$ are satisfied. From all this, we deduce that $u|v$, $v|w$ and $w|u$ are satisfied.
- **(A7)** Let $u, v, w, x \in \mathcal{I}$ satisfying $w|u|x$ and $w|v|x$. The following equalities are satisfied: $w^+ = u^-$, $u^+ = x^-$, $w^+ = v^-$, $v^+ = x^-$. It follows that $(u^-, u^+) = (v^-, v^+)$. Consequently, we can assert that $u = v$. Let $u, v \in \mathcal{I}$ such that $u = v$. We know that $u^- \neq u^+$. From $\mathbf{P5}$, it follows that there exists $l \in \mathcal{P}$ satisfying $\prec (u^-, u^+, l)$. Let $w = (l, u^-)$ and $x = (u^+, l)$. From $\mathbf{P4}$, we deduce that $\prec (l, u^-, u^+)$ is satisfied. From all this, we can assert that $w, x \in \mathcal{I}$ and that $w|u$ and $u|x$ are satisfied. Since $(u^-, u^+) = (v^-, v^+)$, we can assert that $w|v|x$ is satisfied.
- **(A8)** Let $u, v, w \in \mathcal{I}$ satisfying $u|v|w$. It follows that $u^+ = v^-$ and $v^+ = w^-$. Moreover, as $\prec (u^-, v^-, v^+)$ is satisfied, we have $v^- \neq v^+$. In consequence, $u^+ \neq w^-$. Hence, we can assert that $u|w$ is not satisfied.

□

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A general qualitative framework for temporal and spatial reasoning

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A General Qualitative Framework for Temporal and Spatial Reasoning[1]

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Abstract. To offer a generic framework which groups together several interval algebra generalizations, we simply define a generalized interval as a tuple of intervals. An atomic relation between two generalized intervals is a matrix of atomic relations of Interval Algebra. After introducing the generalized relations we focus on the consistency problem of generalized constraint networks and we present sets of generalized relations for which this problem is tractable, in particular the set of the strongly-preconvex relations.

Keywords: temporal and spatial qualitative reasoning, interval algebra, generalized intervals, constraint networks, complexity, preconvexity

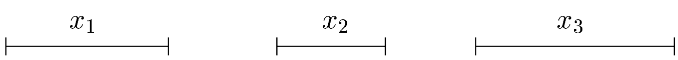
1. Introduction

The field of temporal representation and reasoning has been a central research topic in Artificial Intelligence since many years. It has been studied in many areas such as natural language understanding, specification and verification of programs and systems, temporal databases, scheduling, *etc*.

The framework proposed by Allen [1, 2], called Interval Algebra (IA), is one of the best-known models for qualitative reasoning about temporal data. Allen takes intervals as primitive temporal entities and considers 13 atomic relations between these intervals (Figure 1). To represent temporal information about several intervals he uses constraint networks whose constraints are defined by subsets of the 13 atomic relations. He proposes a variant of the well-known constraint-propagation algorithm, the path-consistency algorithm [17, 18], to reason with these temporal networks.

For some particular applications, Allen's framework seems to be much restrictive. In consequence IA was generalized in numerous ways and more specifically at the level of the basic entities considered. Notably, many formalisms [5, 13, 16, 19] consider tuples of intervals satisfying particular atomic relations of IA as basic entities instead of simple convex intervals.

To represent interrupted events like system processes or system tasks, Ladkin [13] and Khatib *et al.* [11, 12, 19] take tuples of non-overlapping intervals $x_1, \ldots, x_p$ as temporal entities. In such tuples, an interval x_i and the following interval x_{i+1} satisfy the atomic relation "precedes". For illustration, in the following picture is represented a Ladkin's 3-interval:



Ligozat [16] defines a generalized p-interval as a tuple of $p - 1$ intervals $x_1, \ldots, x_{p-1}$ such that x_i satisfies with x_{i+1} the atomic relation "meets". With these temporal entities

Relation	Symbol	Reverse	Meaning
precedes	b	bi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

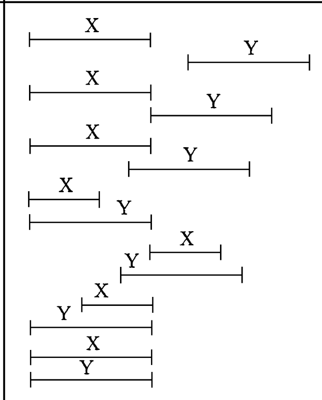
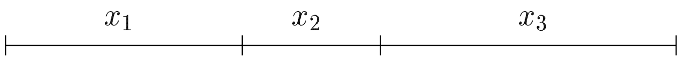
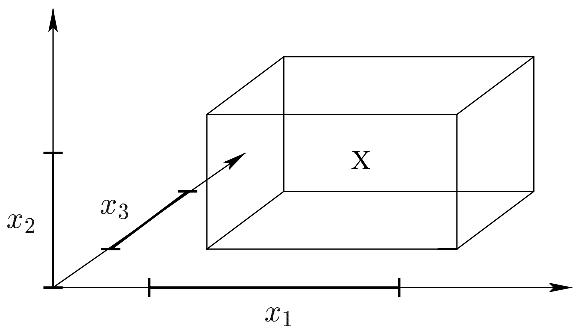


Figure 1. The set $\mathcal{A}_{int}$ of the basic relations between intervals.

we can represent activities characterized by a sequence of important events. Here is an example of a Ligozat's 4-interval:



The n-blocks [5, 10] are the blocks of the Euclidean space of dimension n whose sides are parallel to the axes of some orthogonal base. A n-block is characterized by its n projections onto the n axes. An example of a 3-block follows:



Although the projections of a n-block are on distinct axes, we can represent them on a same line. Hence, a n-block can be represented by a n-tuple of intervals $x_1, \dots, x_n$. Unlike the previous two cases, every pair (x_i, x_j) can satisfy any IA atomic relation. In these three extensions of IA, the qualitative relations can be defined by constraints between the components of tuples expressible by IA relations.

We define a framework subsuming all these formalisms by introducing generalized intervals. This representation enables us to reason with qualitative constraints on generalized intervals still more "general". Actually, we define a generalized interval as a p-tuple of intervals $x_1, \dots, x_p$ of the real line. We do not impose constraints between intervals forming the tuple. This definition is very general and includes the definitions of the generalized intervals previously cited. An atomic relation between two generalized intervals will be characterized by a matrix of IA atomic relations. A relation is a subset of atomic relations. Like Allen, to represent information between generalized intervals we use constraint networks.

The relation defining the constraint between a variable and itself allows to constrain the structure of the generalized interval represented by the variable. Thus, we can force a variable to represent a generalized interval in the sense of Ligozat or Ladkin for example. We can imagine more complicated structures of generalized interval. Let us suppose a system in which exist processes. Each process is realized in three steps S1, S2 and S3 respecting the following ordering constraints:

– Step S2 starts after the beginning of Step S1,
– if Step S2 terminates before the end of S1 then S3 starts after the end of S1, else Step S3 is during Step S1.

These constraints cannot be expressed by a constraint network of the interval algebra. But, as we will see in Subsection 3.3, we can easily define a generalized network whose variables represent such particular processes, *i.e.* three convex intervals representing the three steps and respecting the ordering constraints given above. This simple example illustrates the strong power of expression of our formalism.

Given a generalized constraint network, the main problem is to know whether it is consistent, *i.e.*, whether the information represented by the network is coherent. In general, this problem is a NP-complete problem. By considering the convexity and the preconvexity – both notions introduced respectively by Nökel [22] and by Ligozat [14] for IA – we characterize tractable subsets of generalized relations: the convex generalized relations and the strongly-preconvex generalized relations. We prove that for these sets the closure method and the weak closure method – methods respectively similar to the path-consistency method and the weak path-consistency method – are complete. This path-consistency method was introduced in the frame of the Rectangle Algebra [4] which corresponds to the n-block algebra for $n = 2$. In the following section we define the generalized intervals and the relations we consider between these entities. Section 3 is devoted to the generalized constraint networks. In Section 4 we define the convex generalized relations and the weakly-preconvex generalized relations which are respectively characterized by means of convexity and preconvexity. In Section 5 we prove that convex generalized relations is a tractable set whereas, contrary to the interval algebra case, weak-preconvexity leads to an intractable set. In Section 6 we define the strongly-preconvex generalized relations and prove its tractability. Section 7 concludes with some suggestions for further work.

2. The Generalized Interval Algebra

2.1. The Generalized Intervals and Relations

We define a generalized interval X to dimension p (with $p > 0$), called a p-interval, as a tuple of intervals $x_1, \ldots, x_p$ of the real line. We do not fix p, consequently, we can consider several generalized intervals with different dimensions at the same time. Moreover, we do not assume that the intervals in a generalized interval are in particular. Our definition is very general and as we will see later allows us to represent objects like the Ladkin's generalized intervals [13, 19], the Ligozat's generalized intervals [16] and the rectangles of the rectangle algebra or more generally the n-blocks of the n-block algebra [3–5]. These objects are tuples of intervals satisfying specific constraints expressible with Allen's relations. Given a set W, $\mathcal{M}(W)_{p\times q}$ will denote the set of $p \times q$ matrices over the elements of W. $\mathcal{A}_{int}$ will correspond to the 13 basic relations of IA. The set of the atomic relations between a p-interval and a q-interval, $\mathcal{A}_{p,q}$, is defined by the set of the matrices $p \times q$ of the IA atomic relations:

$$\mathcal{A}_{p,q} = \{A : A \in \mathcal{M}(\mathcal{A}_{int})_{p\times q}\}.$$

In the sequel we will denote the atomic relations by the capital letters A, B, C, *etc*. Let X be a p-interval, Y a q-interval and $A \in \mathcal{A}_{p,q}$, X and Y satisfy A, denoted by $X\ A\ Y$, iff $\forall_i \in 1, \dots, p$ and $\forall_j \in 1, \dots, q$, $x_i\ A_{ij}\ y_j$. A p-interval and a q-interval satisfy one, and only one, atomic relation from $\mathcal{A}_{p,q}$.

Example 1. As an illustration, in the following figure are represented a 2-interval $X = (x_1, x_2)$ and a 3-interval $Y = (y_1, y_2, y_3)$:

We have $X \begin{pmatrix} b & m & oi \\ oi & bi & bi \end{pmatrix} Y$, $X \begin{pmatrix} eq & b \\ bi & eq \end{pmatrix} X$, $Y \begin{pmatrix} eq & bi & bi \\ b & eq & bi \\ b & b & eq \end{pmatrix} Y$.

Let us remark that there exist generalized atomic relations which can be never satisfied. For instance no pair of 2-intervals $X = (x_1, x_2)$ and $Y = (y_1, y_2)$ can satisfy the atomic relation $\begin{pmatrix} b & b_i \\ b_i & b \end{pmatrix}$. Indeed, let us assume that the 2-intervals X and Y satisfy this atomic relation, we have $x_1\ b\ y_1$ and $x_1\ bi\ y_2$, it follows that the interval y_2 is before y_1. Moreover, $x_2\ bi\ y_1$ and $x_2\ b\ y_2$, consequently y_1 is before y_2. There is a contradiction, X and Y cannot satisfy such a generalized atomic relation.

The set of relations considered between a p-interval and a q-interval in the generalized interval algebra is the power set of $\mathcal{A}_{p,q}$, i.e. $2^{\mathcal{A}_{p,q}}$. We will denote the relation by the capital letters R, S, T, *etc*. Given a relation $R \in 2^{\mathcal{A}_{p,q}}$, a p-interval X and a q-interval Y, X and Y satisfy R iff there exists $A \in R$ such that $X\ A\ Y$. A relation R can be seen as the disjunction of its atomic relations.

Example 2. The 2-interval X and the 3-interval Y previously represented in example 1 satisfy the following relation of $2^{\mathcal{A}_{2,3}}$:

$$\left\{ \begin{pmatrix} bi & m & m \\ oi & si & bi \end{pmatrix}, \begin{pmatrix} b & m & oi \\ oi & bi & bi \end{pmatrix}, \begin{pmatrix} eq & m & eq \\ oi & s & s \end{pmatrix} \right\}.$$

Let a matrix $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ be given, we will denote $\prod M$ the relation of $2^{\mathcal{A}_{p,q}}$ defined by:

$$\prod M = \{A \in \mathcal{A}_{p,q} : A_{ij} \in M_{ij}, 1 \leq i \leq p \text{ and } 1 \leq j \leq q\}.$$

Example 3. $\prod \begin{pmatrix} \{s,m\} & \{bi\} \\ \{b\} & \{s,o\} \end{pmatrix}$ is the following relation of $2^{\mathcal{A}_{2,2}}$:

$$\left\{ \begin{pmatrix} s & bi \\ b & s \end{pmatrix}, \begin{pmatrix} s & bi \\ b & o \end{pmatrix}, \begin{pmatrix} m & bi \\ b & s \end{pmatrix}, \begin{pmatrix} m & bi \\ b & o \end{pmatrix} \right\}.$$

Consider a relation $R \in 2^{\mathcal{A}_{p,q}}$. We define $R_{\downarrow ij}$, with $1 \leq i \leq p$ and $1 \leq j \leq q$, by the interval relation: $R_{\downarrow ij} = \{A \in \mathcal{A}_{int} : \exists B \in R, B_{ij} = A\}$, and $R_{\downarrow}$ by the matrix of $\mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ defined by $(R_{\downarrow})_{ij} = R_{\downarrow ij}$. $R_{\downarrow ij}$ will be called the projection of R *w.r.t.* i and j.

Definition 1 Let R be a generalized relation. R is a saturated relation iff $R = \prod R_{\downarrow}$.

Example 4. Let R be the relation of $2^{\mathcal{A}_{2,2}}$ defined by: $R = \left\{ \begin{pmatrix} b & b \\ bi & s \end{pmatrix}, \begin{pmatrix} m & b \\ bi & s \end{pmatrix}, \begin{pmatrix} o & b \\ bi & s \end{pmatrix}, \right.$ $\left. \begin{pmatrix} b & b \\ bi & eq \end{pmatrix}, \begin{pmatrix} m & b \\ bi & eq \end{pmatrix}, \begin{pmatrix} o & b \\ bi & eq \end{pmatrix} \right\}$. R is saturated and equals $\prod \begin{pmatrix} \{b,m,o\} & \{b\} \\ \{bi\} & \{s,eq\} \end{pmatrix}$. Now, let us consider $S \in 2^{\mathcal{A}_{1,2}}$ defined by: $S = \{(b\ o), (m\ b)\}$. S is not saturated since it is not equal to $\prod (\{b, m\}\ \{o, b\})$.

We can easily prove the following proposition:

Proposition 1 *Let R, S* $\in 2^{\mathcal{A}_{p,q}}$.

(a) $R \subseteq \prod R_{\downarrow}$,
(b) if R $\subseteq S$ *then* $R_{\downarrow ij} \subseteq S_{\downarrow ij}$, *with* $1 \leq i \leq p$ *and* $1 \leq j \leq q$.

Proof:

(a) Let $A \in R$. $A_{ij} \in R_{\downarrow ij}$ we deduce that $A \in \prod R_{\downarrow}$ because $R_{\downarrow ij} = (R_{\downarrow})_{ij}$.
(b) Let $A \in R_{\downarrow ij}$. There exists $B \in R$ such that $B_{ij} = A$. Since $R \subseteq S$ we have $B \in S$. We deduce that $A \in S_{\downarrow ij}$. ■

2.2. The Fundamental Operations

The fundamental operations, intersection ($\cap$), union ($\cup$), weak composition ($\odot$) and inverse ($^{-1}$) are defined on the set of the generalized relations. The binary operations intersection and union are two operations from $2^{\mathcal{A}_{p,q}} \times 2^{\mathcal{A}_{p,q}}$ onto $2^{\mathcal{A}_{p,q}}$ and are the usual homonymous set operations. The unary operation inverse is an operation from $2^{\mathcal{A}_{p,q}}$ onto $2^{\mathcal{A}_{q,p}}$ defined from this one of IA (see Figure 1) :

Definition 2

– Let $A \in \mathcal{A}_{p,q}$, $A^{-1} = B$ with $B \in \mathcal{A}_{q,p}$ and $B_{ji} = A_{ij}^{-1}$, $1 \leq i \leq p$ and $1 \leq j \leq q$.
– Let $R \in 2^{\mathcal{A}_{p,q}}$, $R^{-1} = \{A^{-1} : A \in R\}$.

We can note that for each atomic relation A and for each relation R, $(A^{-1})^{-1} = A$ and $(R^{-1})^{-1} = R$. The weak composition is an operation from $2^{\mathcal{A}_{p,q}} \times 2^{\mathcal{A}_{q,r}}$ and $2^{\mathcal{A}_{p,r}}$. We define it from the interval algebra composition (denoted by the symbol $\circ$):

Definition 3 Let $R \in 2^{\mathcal{A}_{p,q}}$ and $S \in 2^{\mathcal{A}_{q,r}}$, $R \odot S = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p \times r}$ and $\forall_i \in 1, \ldots, p$ and $j \in 1, \ldots, r$, $M_{ij} = \bigcap_{1 \leq k \leq q} \{R_{\downarrow ik} \circ S_{\downarrow kj}\}$.

Example 5. Let R and S be two relations of $2^{\mathcal{A}_{2,2}}$ defined by:

$$R = \left\{ \begin{pmatrix} b & o \\ m & eq \end{pmatrix}, \begin{pmatrix} b & s \\ m & mi \end{pmatrix} \right\}, \ S = \left\{ \begin{pmatrix} m & d \\ b & b \end{pmatrix} \right\}.$$

$$R \odot S \text{ is equal to } \prod \begin{pmatrix} \{b\} & \{b\} \\ \{b\} & \{o\} \end{pmatrix} = \left\{ \begin{pmatrix} b & b \\ b & o \end{pmatrix} \right\}.$$

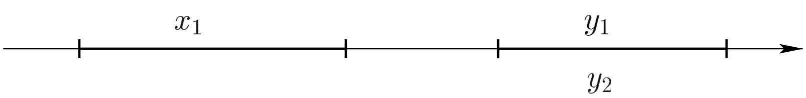
These fundamental operations satisfy the following properties:

Property 1 Let two generalized interval X, Y and two generalized relations R and S be. We have:

– $X \; R \cap S \; Y$ iff $X \; R \; Y$ and $X \; S \; Y$,
– $X \; R \cup S \; Y$ iff $X \; R \; Y$ or $X \; S \; Y$,
– $X \; R^{-1} \; Y$ iff $Y \; R \; X$,
– If $\exists \; Z$ such as $X \; R \; Z$ and $Z \; S \; Y$ then $X \; R \odot S \; Y$.

Contrary to the composition in IA and more generally contrary to the usual composition, the converse of this last property is not true for the operation of weak composition which we define. This is why we use the term "weak" to qualify this composition. Consider the following example:

Example 6. Let $R \in 2^{\mathcal{A}_{1,1}}$ and $S \in 2^{\mathcal{A}_{1,2}}$ defined by: $R = \{(b)\}$ and S $= \{(b \; eq)\}$. We have $R \odot S = \{(b \; b)\}$. Now by considering the two generalized intervals $x = (x_1)$ and $y = (y_1, y_2)$ in the following figure:



it is easy to see that there does not exist a generalized interval $z = (z_1)$ satisfying with y the relation $\{(b \; eq)\}$ since y_1 and y_2 are two equal intervals.

We will see in the sequel that this lost of equivalence does not matter, just the converse implication is essential. For the saturated generalized relations we can prove the following proposition:

Proposition 2 *Let R, S* $\in 2^{\mathcal{A}_{p,q}}$ *be two saturated relations.*

(a) $R^{-1} = \prod M$, *with* $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{q\times p}$ *and* $\forall_i \in 1,\ldots,p$ *and* $j \in 1,\ldots,q$, $M_{ji} = (R_{\downarrow ij})^{-1}$,
(b) $R \cap S = \prod M$, *with* $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$. *and* $\forall_i \in 1,\ldots,p$ *and* $j \in 1,\ldots,q$, $M_{ij} = R_{\downarrow ij} \cap S_{\downarrow ij}$.

Proof:

(a) Let $A \in R^{-1}$. As $A^{-1} \in R$, $\forall_i \in 1,\ldots,p$ and $\forall_j \in 1,\ldots,q$, $(A_{ji})^{-1} \in R_{\downarrow ij}$. Hence $A_{ji} \in (R_{\downarrow ij})^{-1}$; it follows that $A \in \prod M$. Let $A \in \prod M$. $A_{ji} \in (R_{\downarrow ij})^{-1}$, therefore $(A_{ji})^{-1} \in R_{\downarrow ij}$ and $(A^{-1})_{ij} \in R_{\downarrow ij}$. It follows that $A^{-1} \in \prod R_{\downarrow}$. Since $R = \prod R_{\downarrow}$, $A^{-1} \in R$ and $A \in R^{-1}$.
(b) Let $A \in R \cap S$, $A \in R$ and $A \in S$, consequently $A_{ij} \in R_{\downarrow ij} \cap S_{\downarrow ij}$. This implies that $A \in \prod M$. Let $A \in \prod M$. $A_{ij} \in R_{\downarrow ij} \cap S_{\downarrow ij}$, therefore $A_{ij} \in R_{\downarrow ij}$ and $A_{ij} \in S_{\downarrow ij}$. Hence, $A \in \prod R_{\downarrow}$ and $A \in \prod S_{\downarrow}$. As R and S are saturated we deduce that $A \in R$ and $A \in S$. ■

In [15] Ligozat defines the operation of dimension onto relations of IA. He represents an interval $x = (x^-, x^+)$ in the real Euclidean plane by a point of coordinates (x^-, x^+). Given a point (x_0^-, x_0^+) representing a reference interval x_0, an atomic relation A of $\mathcal{A}_{int}$ is represented by the region: $\{(y^-, y^+) \in \mathbb{R}^2 : (y^-, y^+)\ A\ (x_0^-, x_0^+)\}$. The resulting regions are: a point (for *eq*), some semi-lines (for *m*, *mi*, *f*, *fi*, *s*, *si*) and regions of dimension 2 (for *b*, *bi*, *d*, *di*, *o*, *oi*), see Figure 2. Given a relation R of $2^{\mathcal{A}_{int}}$, the region corresponding to R is the union of regions representing the atomic relations. The dimension of A, denoted by $dim(A)$, is the dimension of the region representing it (see Figure 3). Given a relation $R \in 2^{\mathcal{A}_{int}}$, $dim(R) = max\{dim(A) : A \in R\}$. We define the dimension of a generalized relation in the following way:

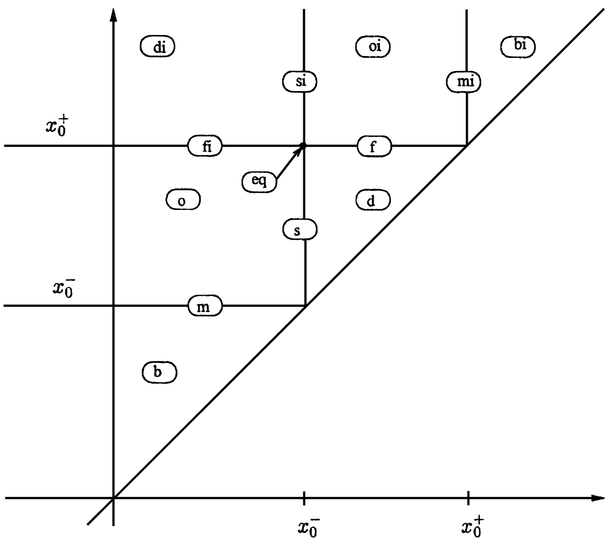


Figure 2. Geometric representation of $\mathcal{A}_{int}$.

Atomic Relation	b bi o oi d di	m mi s si f fi	eq
Dimension	2	1	0

Figure 3. The dimensions of the basic relations of $\mathcal{A}_{int}$.

Definition 4 Let $A \in \mathcal{A}_{p,q}$ and $R \in 2^{\mathcal{A}_{p,q}}$. Then:

- $dim(A) = \sum_{1 \leq i \leq p, 1 \leq j \leq q} dim(A_{ij})$,
- $dim(R) = max\{dim(A) : A \in R\}$.

Example 7. For illustration, we have $dim\left(\begin{pmatrix} m & o & b \\ eq & d & b \end{pmatrix}\right) = 9$ whereas $dim\left(\begin{pmatrix} s & si & b \\ eq & si & o \end{pmatrix}\right)$ $= 7$, and hence $dim\left(\left\{\begin{pmatrix} m & o & b \\ eq & d & b \end{pmatrix}, \begin{pmatrix} s & si & b \\ eq & si & o \end{pmatrix}\right\}\right) = 9$.

The dimension of a saturated generalized relation follows from the dimensions of its projections:

Proposition 3 *Let* $R \in 2^{\mathcal{A}_{p,q}}$ be a saturated relation. Then:

$$dim(R) = \sum_{1 \leq i \leq p, 1 \leq j \leq q} dim(R_{\downarrow ij})$$

Proof: Let $A \in R$ be such that $dim(A) = dim(R)$. Let us prove that for each $i \in 1, \ldots, p$ and $j \in 1, \ldots, q$ $dim(A_{ij}) = dim(R_{\downarrow ij})$. Let us suppose that there exists $B'_{kl} \in R_{\downarrow kl}$ with $dim(B'_{kl}) > dim(A_{kl})$, $k \in 1, \ldots, p$ and $l \in 1, \ldots, q$. Let us consider the atomic relation $B \in \mathcal{A}_{p,q}$ defined by $B_{ij} = A_{ij}$ if $i \neq k$ and $j \neq l$, $B_{ij} = B'_{kl}$ else. We can assert that $dim(B) > dim(A)$. Moreover, B belongs to R because R is saturated. From this we can deduce that $dim(R) > dim(A)$, which is a contradiction. It follows that for each $i \in 1, \ldots, p$ and $j \in 1, \ldots, q$, $dim(A_{ij}) \geq dim(R_{\downarrow ij})$. Moreover, as $A_{ij} \in R_{\downarrow ij}$ we can assert that $dim(A_{ij}) \leq dim(R_{\downarrow ij})$. Consequently, for each $i \in 1, \ldots, p$ and $j \in 1, \ldots, q$, $dim(A_{ij}) = dim(R_{\downarrow ij})$. Hence $dim(R) = dim(A) = \sum_{1 \leq i \leq p, 1 \leq j \leq q} dim(A_{ij}) = \sum_{1 \leq i \leq p, 1 \leq j \leq q} R_{\downarrow ij}$. ■

3. The Generalized Networks

3.1. Definition

Information between several generalized intervals is represented by a special binary constraint satisfaction problem: a network of generalized intervals. A network of generalized intervals (henceforth we will say generalized network) $\mathcal{N}$ is a structure (V, C) where :

- V is a set of variables $V_1, \ldots, V_l$ (with $l = |V|$) ranging over generalized intervals,
- and C is a mapping from $V \times V$ onto the set of generalized relations which corresponds to the binary constraints between the generalized intervals.

In the sequel, we will sometimes denote the relation $C(V_i, V_j)$ by C_{ij}. Morevoer, we assume that the C is such that:

- $\forall i, j \in 1, \ldots, |V|$, $C_{ij} \in 2^{\mathcal{A}_{p,q}}$, with V_i and V_j being respectively a p-interval and a q-interval.
- $\forall i, j \in 1, \ldots, |V|$, $C_{ij} = C_{ji}^{-1}$.
- $\forall i \in 1, \ldots, |V|$, $\forall A \in C_{ii}$, we have $\forall k \in 1, \ldots, p$, $A_{kk} = eq$ (with $C_{ii} \in 2^{\mathcal{A}_{p,p}}$).

A generalized network whose all variables represent 1-intervals is an Allen's interval network. A generalized network $\mathcal{N}$ will be said to be saturated iff all its constraints are saturated.
We assume that each variable of a generalized network represents a p-interval with p bounded by an integer number M. In other terms, given a generalized network $\mathcal{N} = (\mathcal{V}, \mathcal{C})$, each constraint C_{ii} (with $i \in 1, \ldots, |V|$, belongs to $2^{\mathcal{A}_{p,p}}$ with p an integer less than the fixed constant M. In consequence the cardinality of a constraint is less than the constant 13^{M^2}. In the sequel, this assumption will be important for the complexity results.

3.2. The Consistency Problem

For reasonning with a constraint network, the first step to accomplish is to check whether this constraint network is consistant. This problem is called the consistency problem. According to the nature of the constraint network the complexity of this problem differs. For example, the consistency problem in the point algebra is polynomial, whereas this one of the interval algebra is NP-complete. Since a interval network is a generalized network, the consistency problem of the generalized networks is NP-hard. As we will see in the following section, by using only relations from some subsets of the generalized algebra this problem becomes polynomial. Before going on, we give some formal definitions:

Definition 5 Let $\mathcal{N} = (V, C)$ be a network.

- A consistent instantiation m of $\mathcal{N}$ is a mapping which associates to each variable $V_i \in V$ representing a p-interval, a p-interval noted $m(V_i)$ such that $m(V_i)\ C_{ij}\ m(V_j)$, $\forall i, j \in 1, \ldots, |V|$. The atomic relation satisfied between $m(V_i)$ and $m(V_j)$ will be denoted $m(V_i, V_j)$.
- A consistent instantiation m is maximal iff $dim(m(V_i, V_j)) = dim(C_{ij})$ for every $i, j \in 1, \ldots, |V|$.
- $\mathcal{N}$ is consistent iff it admits a consistent instantiation.
- $\mathcal{N}$ is minimal iff for every $i, j \in 1, \ldots, |V|$ and for every $A \in C_{ij}$ there exists a consistant instantiation m such that $m(V_i, V_j) = A$.
- $\mathcal{N}$ is closed for $\odot$ iff for every $i, j, k \in 1, \ldots, |V|$, $C_{ij} \subseteq C_{ik} \odot C_{kj}$ and $C_{ij} \neq \{\}$.
- $\mathcal{N}$ is equivalent to another network $\mathcal{N}' = (V, C')$ iff they have the same consistent instantiations.

Let us remark that an interval network closed for $\odot$ is path-consistent since the operation $\odot$ restricted to the interval relations corresponds to the usual operation of composition $\circ$. We call the closure method, the method which consists of transforming a given generalized network $\mathcal{N} = (V, C)$ into an equivalent generalized network closed for $\odot$ or having empty constraints by iterating the triangulation operation:

$$C_{ij} \leftarrow C_{ij} \cap (C_{ik} \odot C_{kj})$$

until a fixed point is reached. The closure method applied to an interval network $\mathcal{N}$ corresponds to the well-known path-consistency method. The closure method can be implemented by an algorithm of complexity $O(|V|^3)$ in time [17]. For the consistency problem this method is sound but not complete: if the empty relation is a constraint of the resulting network then the initial network is inconsistent, otherwise we cannot assert the consistency of the initial network because we are not sure that all the unsatisfiable atomic relations have been removed. In the past, it has been proved that this method is complete for particular subclasses of IA [15, 20, 24]. by an algorithm of complexity $O(|V|^3)$ in time [17].

3.3. Representation of Particular Generalized Intervals

With the help of the relation C_{ii} we can constrain the structure of the p-interval represented by V_i to represent particular generalized intervals.

3.3.1. Representation of Ligozat's Generalized Interval

A Ligozat's p-interval can be defined as a tuple of $p-1$ intervals $x_1, \ldots, x_{p-1}$ such that x_i satisfies with x_{i+1} the Allen's atomic relation "meets". If we want that a variable V_i represents a Ligozat's p-interval, we just need to take for C_{ii} the following singleton relation of $2^{\mathcal{A}_{p,p}}$:

$$C_{ii} = \left\{ \begin{pmatrix} eq & m & b & \cdots & b \\ mi & eq & \ddots & \ddots & \vdots \\ bi & \ddots & \ddots & \ddots & b \\ \vdots & \ddots & \ddots & eq & m \\ bi & \cdots & bi & mi & eq \end{pmatrix} \right\}.$$

3.3.2. Representation of Ladkin's Generalized Interval

A Ladkin's generalized interval can be defined as a sequence of intervals $x_1, \ldots, x_p$ such that x_i is before x_{i+1} for all $i \in 1, \ldots, p-1$. Consequently, to take into account a Ladkin's generalized interval, the generalized relation C_{ii} must be the following relation:

$$C_{ii} = \left\{ \begin{pmatrix} eq & b & \cdots & b \\ bi & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & b \\ bi & \cdots & bi & eq \end{pmatrix} \right\}.$$

3.3.3. *Representation of Rectangles*

The rectangle algebra [3] is the extension of the interval algebra to dimension 2. The considered objects are the isothetic rectangles (henceforward we will just say rectangles) of the Euclidean plane. An atomic relation between two of these rectangles is characterized by a pair of IA atomic relation (A, B) where A is the atomic relation satisfied by the orthogonal projection of the rectangles onto the first axis, and B is the atomic relation satisfied by the orthogonal projection of the rectangles onto the second axis. The two orthogonal projections of a rectangle are onto two distinct axes, but we can put them onto a same axis. Consequently, we can represent a rectangle by a 2-interval, C_{ii} will be the relation composed by all the atomic relations of $\mathcal{A}_{2,2}$ having only the atomic relation *eq* onto their descending diagonal:

$$C_{ii} = \bigcup_{A \in \mathcal{A}_{int}} \left\{ \begin{pmatrix} eq & A \\ A^{-1} & eq \end{pmatrix} \right\}.$$

Moreover, by using one axis we must just consider relative positions between intervals corresponding to rectangle projections *w.r.t* a same axis. Consequently, the constraints between two 2-intervals representing rectangles must just include relations of the form:

$$\prod \begin{pmatrix} \{A\} & \mathcal{A}_{int} \\ \mathcal{A}_{int} & \{B\} \end{pmatrix}, \text{ with } A, B \in \mathcal{A}_{int}.$$

Example 8. For example, the constraint $\{(b, m), (o, eq)\}$ of the rectangle algebra will be represented by the generalized relation

$$\prod \begin{pmatrix} \{b\} & \mathcal{A}_{int} \\ \mathcal{A}_{int} & \{m\} \end{pmatrix} \cup \prod \begin{pmatrix} \{o\} & \mathcal{A}_{int} \\ \mathcal{A}_{int} & \{eq\} \end{pmatrix}.$$

In a similar way, we can represent the blocks of the p-blocks algebra.

3.3.4. *Representation of Particular Processes*

Again, let us consider the processes realized in three steps S1, S2 and S3 respecting the following ordering constraints:

- Step S2 starts after the beginning of Step S1,
- if Step S2 terminates before the end of S1 then S3 starts after the end of S1, else Step S3 is during Step S1.

We can represent each process by a 3-interval $X = (x_1, x_2, x_3)$ where x_1 (resp. x_2 and x_3) represents the first step (resp. the second step and the third step). Because of the first ordering constraint x_2 and x_1 must satisfy the relation $\{b_i, m_i, oi, d, f\}$ of the interval algebra. Moreover, from the second ordering constraint x_3 and x_1 satisfy $\{bi\}$ if x_2 and x_1 satisfy $\{d, f\}$, else in the situation where x_2 and x_1 are in relation $\{bi, mi, oi\}$ we must have x_3 and x_1 satisfying $\{d\}$. To take into account such constraints, for the generalized relation C_{ii} we can take the following relation:

$$C_{ii} = \bigcup_{A \in \mathcal{A}_{int}} \left\{ \begin{pmatrix} eq & b & di \\ bi & eq & A \\ d & A^{-1} & eq \end{pmatrix}, \begin{pmatrix} eq & m & di \\ mi & eq & A \\ d & A^{-1} & eq \end{pmatrix}, \begin{pmatrix} eq & o & di \\ oi & eq & A \\ d & A^{-1} & eq \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} eq & di & b \\ d & eq & A \\ bi & A^{-1} & eq \end{pmatrix}, \begin{pmatrix} eq & fi & b \\ f & eq & A \\ bi & A^{-1} & eq \end{pmatrix} \right\}.$$

4. Particular Subsets of Generalized Subsets

In this section we will define two particular subsets of the set of the generalized relations : the convex generalized relations and the weakly-preconvex generalized relations. The definitions of both these subsets follow closely these ones given by Ligozat in [15] to define the convex relations and the preconvex relations of IA. These two sets are two tractable subclass of IA [15, 21–23, 25] for the consistency problem, it is why we extend these two concepts to the generalized interval algebra. In the next section we will prove the tractability of the convex generalized relations and the intractability of the weakly-preconvex generalized relations. Seeing this last fact, we will refine our concept of preconvexity to define the strongly-preconvex generalized relations, which is tractable.

4.1. The Convex Relations

Now, we are going to define the notion of convexity for the generalized relations. For this purpose we will extend some notions introduced by Ligozat [14, 15] to redefine the convex relations [22]. Ligozat arranges the atomic relations of $\mathcal{A}_{int}$ in a partial order which defines a lattice: the interval lattice (see Figure 4). From this order we organize the atomic relations of $\mathcal{A}_{p,q}$ in a partial order $\sqsubseteq$:

$$\forall A, B \in \mathcal{A}_{p,q}, \quad A \sqsubseteq B \text{ iff } \forall_i \in 1, \ldots, p \text{ and } \forall j \in 1, \ldots, q, \quad A_{ij} \leq B_{ij}$$

($\mathcal{A}_{p,q}$, $\sqsubseteq$) defines a lattice too, called the generalized (p,q)-lattice. Given A, $B \in \mathcal{A}_{p,q}$ with $A \sqsubseteq B$, $[A, B]$ will denote an interval of the (p,q)-lattice, *i.e.* the set (the generalized relation) $\{C : A \sqsubseteq C$ and $C \sqsubseteq B\}$. Since ($\mathcal{A}_{p,q}$, $\sqsubseteq$) is the product order of ($\mathcal{A}_{int}$, $\leq$), each

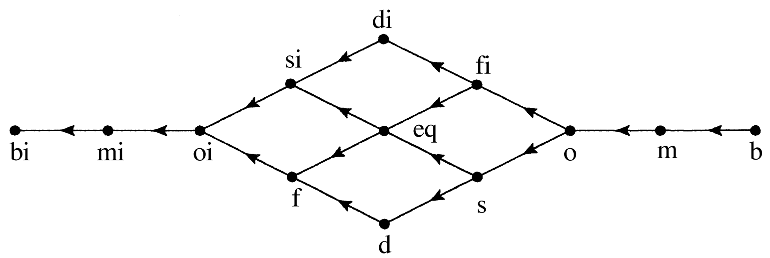


Figure 4. The interval lattice ($\mathcal{A}_{int}$, $\leq$).

interval of the (p,q)-lattice corresponds to a Cartesian product of $p \times q$ intervals of the interval lattice:

Proposition 4 *Let* $[A, B]$ *be an interval of the (p,q)-lattice. We have:*

$$[A,B] = \prod M, \text{ with } M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q} \text{ and } \forall_i \in 1,\ldots,p \text{ and } j \in 1,\ldots,q,$$
$$M_{ij} = [A_{ij}, B_{ij}].$$

Now we extend the definition of convex closure in the following way:

Definition 6 Let $R \in 2^{\mathcal{A}_{p,q}}$. The convex closure of R, denoted by $I(R)$, is the relation of $2^{\mathcal{A}_{p,q}}$ corresponding to the smallest interval of the generalized (p,q)-lattice containing R.

Example 9. Let the relation $R \in 2^{\mathcal{A}_{2,2}}$ be defined by R $= \left\{ \begin{pmatrix} b & o \\ eq & o \end{pmatrix}, \begin{pmatrix} o & m \\ eq & o \end{pmatrix} \right\}$, we have: $I(R) = \left[\begin{pmatrix} b & m \\ eq & o \end{pmatrix}, \begin{pmatrix} o & o \\ eq & o \end{pmatrix} \right] = \prod \begin{pmatrix} \{b,m,o\} & \{m,o\} \\ \{eq\} & \{o\} \end{pmatrix} = \left\{ \begin{pmatrix} b & m \\ eq & o \end{pmatrix}, \right.$
$\left. \begin{pmatrix} b & o \\ eq & o \end{pmatrix}, \begin{pmatrix} m & m \\ eq & o \end{pmatrix}, \begin{pmatrix} m & o \\ eq & o \end{pmatrix}, \begin{pmatrix} o & m \\ eq & o \end{pmatrix}, \begin{pmatrix} o & o \\ eq & o \end{pmatrix} \right\}.$

$I(R)$ always exists since the intersection of two intervals of the (p,q)-lattice is an interval too. We have the following properties:

Proposition 5 *Let* $R, S \in 2^{\mathcal{A}_{p,q}}$.

(a) $R \subseteq I(R)$ *and* $I(I(R)) = I(R)$,
(b) *if* $R \subseteq S$ *then* $I(R) \subseteq I(S)$,
(c) $I(R) = \prod M$, *with* $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ *and* $\forall_i \in 1, \ldots, p$ *and* $j \in 1, \ldots, q$, $M_{ij} = I(R_{\downarrow ij})$.

Proof:

– (a) and (b) follows directly from the definition of the convex closure.
– (c) Let us denote the interval $I(R_{\downarrow ij})$ of the interval lattice by $[A'_{ij}, B'_{ij}]$ with $A'_{ij}, B'_{ij} \in \mathcal{A}_{int}$. Let us denote by A and B the atomic relations of $\mathcal{A}_{p,q}$ defined by $A_{ij} = A'_{ij}$ and $B_{ij} = B'_{ij}$. From Prop. 4 $\prod M = [A, B]$. As $R_{\downarrow ij} \subseteq I(R_{\downarrow ij})$ we deduce that $R \subseteq \prod M$. Consequently, $\prod M$ is an interval of the generalized (p,q)-lattice containing R. Now, we must show that $\prod M$ is the smallest interval containing R. Let a relation $S = [C, D] \in 2^{\mathcal{A}_{p,q}}$ be such that $R \subseteq S$. From Prop. 1 (b), $R_{\downarrow ij} \subseteq S_{\downarrow ij}$. It follows that $I(R_{\downarrow ij}) \subseteq I(S_{\downarrow ij})$, but $I(S_{\downarrow ij}) = S_{\downarrow ij}$ because $S_{\downarrow ij}$ is an interval of the interval lattice (Prop. 4). Thus, $I(R_{\downarrow ij}) \subseteq S_{\downarrow ij}$, from it we deduce that $\prod M \subseteq \prod S$. As S is a saturated relation (Prop. 4) we conclude that $\prod M \subseteq S$. ■

Concerning the fundamental operations and the convex closure we can prove the following proposition:

Proposition 6 *Let* $R, S \in 2^{\mathcal{A}_{p,q}}$ *and* $T \in 2^{\mathcal{A}_{q,r}}$.

(a) $I(R^{-1}) = I(R)^{-1}$,
(b) $I(R \odot T) \subseteq I(R) \odot I(T)$,

(c) $I(I(R) \cap I(S)) = I(R) \cap I(S)$,
(d) $I(I(R) \odot I(T)) = I(R) \odot I(T)$.

Proof:

(a) From propositions 5 (c) and 2. we deduce that $I(R)^{-1} = \prod M$ with $M_{ij} = I(R_{\downarrow ji})^{-1}$. We know that in IA the property is true, consequently $I(R_{\downarrow ji})^{-1} = I((R_{\downarrow ji})^{-1})$. From $(R_{\downarrow ji})^{-1} = (R^{-1})_{\downarrow ij}$ it follows that $M_{ij} = I((R^{-1})_{\downarrow ij})$. From 5 ($c$) we can deduce that $\prod M = I(R^{-1})$, consequently $I(R^{-1}) = I(R)^{-1}$.
(b) From Proposition 5 (c) and Definition 3 $I(R \odot T) = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times r}$ and $\forall i \in 1, \ldots, p$ and $j \in 1, \ldots, r$, $M_{ij} = I(\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\})$. Still from Proposition 5 (c) and Definition 3, $I(R) \circ I(T) = \prod N$, with $N \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times r}$ and $\forall i \in 1, \ldots, p$ and $j \in 1, \ldots, r$, $N_{ij} = \bigcap_{1\leq k\leq q}\{I(R_{\downarrow ik}) \circ I(T_{\downarrow kj})\}$. From Proposition 5 ($a$), $R_{\downarrow ik} \circ T_{\downarrow kj} \subseteq I(R_{\downarrow ik} \circ T_{\downarrow kj})$. It follows that $\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\} \subseteq \bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})$. From Proposition 5 ($b$), ($b$), $I(\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\}) \subseteq I(\bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj}))$. As the intersection of two intervals of the interval lattice is an interval too, we have $I(\bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})) = \bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})$. It follows that $I(\bigcap_{1\leq k\leq q}\{R_{\downarrow ik} \circ T_{\downarrow kj}\}) \subseteq \bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj})$. In IA the property is true, consequently we have $\bigcap_{1\leq k\leq q} I(R_{\downarrow ik} \circ T_{\downarrow kj}) \subseteq \bigcap_{1\leq k\leq q}\{I(R_{\downarrow ik}) \circ I(T_{\downarrow kj})\}$. Hence $M_{ij} \subseteq N_{ij}$. We can conclude that $\prod M \subseteq \prod N$.
(c) Let us show that $I(R) \cap I(S)$ corresponds to an inteval of the generalized (p,q)-lattice. From Proposition 5 (c) and Proposition 2 (b), $I(R) \cap I(S) = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times q}$ and $\forall i \in 1, \ldots, p$ and $j \in 1, \ldots, q$, $M_{ij} = I(R_{\downarrow ij}) \cap I(S_{\downarrow ij})$. We know that the intersection of two IA relations corresponding to two intervals of the interval lattice is too an interval of this lattice. It follows that M_{ij} is an interval of the interval lattice and thus $\prod M$ is an interval of the (p,q)-lattice (Proposition 4).
(d) Let us show that $I(R) \odot I(T)$ is an interval of the (p,r)-lattice. From Proposition 5 (c) and Definition 3, $I(R) \odot I(T) = \prod M$, with $M \in \mathcal{M}(2^{\mathcal{A}_{int}})_{p\times r}$ and $\forall i \in 1, \ldots, p$ and $j \in 1, \ldots, r$, $M_{ij} = \bigcap_{1\leq k\leq q}\{I(R_{\downarrow ik}) \circ I(T_{\downarrow kj})\}$. Moreover we know that the intersection and the composition of two relations corresponding to two intervals of the interval lattice is also an interval of the interval lattice. From this, we deduce that $M_{ij} = [A'_{ij}, B'_{ij}]$ with $A'_{ij}, B'_{ij} \in \mathcal{A}_{int}$. Let $A, B \in \mathcal{A}_{p,r}$ be defined by $A_{ij} = A'_{ij}$ and $B_{ij} = B'_{ij}$, from Prop. 4 $\prod M = [A, B]$. Consequently $I(R) \odot I(T)$ is an interval of the (p,r)-lattice. ∎

The convex relations of IA [15, 22] correspond to the intervals of the interval lattice. In a natural way, we define the convex generalized relations in the following way :

Definition 7 Let $R \in 2^{\mathcal{A}_{p,q}}$, R is convex iff R corresponds to an interval of the generalized (p,q)-lattice.

From Proposition 4 we can assert the following properties:

Proposition 7 *Let R be a relation of* $2^{\mathcal{A}_{p,q}}$.

– *If R is convex then R is a saturated relation and for all i ∈ 1, . . . , p, j ∈ 1, . . . , q,* $R_{\downarrow ij}$ *is a convex relation of* $2^{\mathcal{A}_{int}}$.
– *R is a convex relation iff* $I(R) = R$.

From this and Proposition 6 we can prove the following theorem:

Theorem 1 *The set of the convex relations of* $2^{\mathcal{A}_{p,q}}$ *is closed with respect to the fundamental operations* $\cap$, $\odot$ *and* $^{-1}$.

Proof: Let $R, S \in 2^{\mathcal{A}_{p,q}}$ and $T \in 2^{\mathcal{A}_{q,r}}$ be three convex relations.

- As $R = I(R)$ we have $R^{-1} = I(R)^{-1}$. From proposition 6 (a) we deduce that $R^{-1} = I(R^{-1})$. Consequently R^{-1} is convex.
- $R = I(R)$ and $S = I(S)$, therefore $R \cap S = I(R) \cap I(S)$. From proposition 6 (b), $R \cap S = I(I(R) \cap I(S))$, consequently $R \cap S$ is convex.
- $R = I(R)$ and $T = I(T)$, it follows that $R \odot T = I(R) \odot I(T)$. From proposition 6 (c), $R \odot T = I(I(R) \odot I(T))$, consequently $R \odot T$ is convex. ■

4.2. The Weakly-Preconvex Relations

Seeing that the preconvex interval relations correspond to the maximal tractable set containing the singleton relations [20], it is natural to extend these relations to the generalized algebra. There are several ways to define the preconvex interval relations [15], for the generalized relations we have chosen the definition using the concepts of dimension and convex closure:

Definition 8 Let $R \in 2^{\mathcal{A}_{p,q}}$. R is weakly-preconvex iff $dim(I(R) \setminus R) < dim(R)$.

The reader has noticed that we use the term of *weak-preconvexity* to design preconvexity. Intuitively, a relation is weakly-preconvex iff to compute its convex closure, we only add its atomic relations of dimension strictly lower than its own dimension. Let us notice that the weakly-preconvex relations of $2^{\mathcal{A}_{1,1}}$ are the preconvex relations of IA.

Example 10. For example, let $R = \left\{ \begin{pmatrix} m & m \\ b & b \end{pmatrix}, \begin{pmatrix} o & o \\ b & b \end{pmatrix} \right\}$ *be*. R is weakly-preconvex because $I(R) = \left\{ \begin{pmatrix} m & m \\ b & b \end{pmatrix}, \begin{pmatrix} m & o \\ b & b \end{pmatrix}, \begin{pmatrix} o & m \\ b & b \end{pmatrix}, \begin{pmatrix} o & o \\ b & b \end{pmatrix} \right\}$, thus $dim\,(R) = 8$ and $dim\,(I(R) \setminus R) = 7$.

The reader can notice that we extended the concept of preconvexity to the generalized interval algebra in a different way than the one proposed by Balbiani *et al.* in [6]. With their extension the resulting "preconvex" generalized relations – which we will call the saturated-preconvex relations – correspond to the saturated generalized relations whose projections are preconvex. Our notion of weakly-preconvexity subsumes that one:

Proposition 8 **Let** $R \in 2^{\mathcal{A}_{p,q}}$**.**

- *If R is convex then R is saturated-preconvex.*
- *If R is saturated-preconvex then R is weakly-preconvex.*

Proof:

- Let R be a convex relation. From Proposition 7 we deduce that R is a saturated generalized relation whose projections are convex relations of the interval algebra. We know that each convex relation of IA is also a preconvex relation. We can conclude that R is a saturated-preconvex generalized relation.
- Let R be a saturated-preconvex relation and let $A \in I(R)$. If $A \notin R$ and $dim(A) \geq dim(R)$ then from Proposition 5 (c) and Proposition 3 it follows that a projection of R is not preconvex. We know that $\forall i \in 1, \ldots, p$ and $\forall j \in 1, \ldots, q$ $R_{\downarrow ij}$ is a preconvex relation of IA since R is saturated-preconvex. It follows that $dim(I(R_{\downarrow ij}) \setminus R_{\downarrow ij}) \leq dim(R_{\downarrow ij})$, and hence $dim(I(R_{\downarrow ij})) \leq dim(R_{\downarrow ij})$ for each $i \in 1, \ldots, p$ and $j \in 1, \ldots, q$. As $I(R_{\downarrow ij}) = (I(R))_{\downarrow ij}$ (proposition 5 (c)), we deduce that $dim((I(R))_{\downarrow ij}) \leq dim(R_{\downarrow ij})$. Therefore $dim(A_{ij}) \leq dim(R_{\downarrow ij})$ for all i, j. As R is saturated and $dim(A) \geq dim(R)$, from proposition 3, $\sum_{1 \leq i \leq p, 1 \leq j \leq q} dim(A_{ij}) \geq \sum_{1 \leq i \leq p, 1 \leq j \leq q} dim(R_{\downarrow ij})$. Hence, $\sum_{1 \leq i \leq p, 1 \leq j \leq q} dim(A_{ij}) = \sum_{1 \leq i \leq p, 1 \leq j \leq q} dim(R_{\downarrow ij})$ and $dim(A_{ij}) = dim(R_{\downarrow ij})$ for each i, j. Since $A_{ij} \in I(R_{\downarrow ij})$ and $R_{\downarrow ij}$ is preconvex we have $A_{ij} \in R_{\downarrow ij}$. As R is saturated we can conclude that $A \in R$; we obtain a contradiction. Consequently, each atomic relation $A \in I(R) \setminus R$ has a dimension strictly lower than this one of R. Hence, R is weakly-preconvex. ■

Let us notice that a weakly-preconvex relation is not necessarily a saturated-preconvex relation. From the previous example we know that the relation $R = \left\{ \begin{pmatrix} m & m \\ b & b \end{pmatrix}, \begin{pmatrix} o & o \\ b & b \end{pmatrix} \right\}$ is weakly-preconvex. This relation is not saturated since $R \neq \prod \begin{pmatrix} \{m, o\} & \{m, o\} \\ \{b\} & \{b\} \end{pmatrix}$. Consequently R is weakly-preconvex and not saturated-preconvex.

5. First Complexity Results

5.1. Tractability of Convexity is Preserved

In this section we are going to prove that the subset of the convex generalized relations is a tractable set for the consistency problem. Before this, we define the projection of a generalized network in the following way:

Definition 9 Let $\mathcal{N} = (V, C)$ be a generalized network. $\mathcal{N}_{\downarrow}$ is the interval network (V', C') such that:

- For each variable $Vi \in V$ representing a p-interval, p variables $V_i^1, \ldots, V_i^p$ belong to V'. V_i^j is the variable which represents the j^{th} subinterval of Vi;
- Let $V_i^k, V_j^l \in V'$. The constraint $C'(V_i^k, V_j^l)$ is the relation $(C_{ij})_{\downarrow kl}$ of the interval algebra.

We can prove the following proposition:

Proposition 9 *Let $\mathcal{N} = (V, C)$ be a generalized network. If $\mathcal{N}$ is closed for $\odot$ then $\mathcal{N}_{\downarrow}$ is closed for $\odot$* (and hence path-consistent).

Proof: Let $i, j, k \in 1, \ldots, |V|$. Let us now suppose that $C_{ik} \in 2^{\mathcal{A}_{p,q}}$, $C_{kj} \in 2^{\mathcal{A}_{q,r}}$ and let $m \in 1, \ldots, p$ and $n \in 1, \ldots, r$. As $C_{ij} \subseteq C_{ik} \odot C_{kj}$, from Proposition 1 (b) we can deduce that: $(C_{ij})_{\downarrow mn} \subseteq (C_{ik} \odot C_{kj})_{\downarrow mn}$. From def. 3, it follows that $(C_{ij})_{\downarrow mn} \subseteq \bigcap_{1 \leq l \leq q}\{(C_{ik})_{\downarrow ml} \circ (C_{kj})_{\downarrow ln}\}$. Consequently we have $(C_{ij})_{\downarrow mn} \subseteq (C_{ik})_{\downarrow ml} \circ (C_{kj})_{\downarrow ln}$. Hence $\mathcal{N}_{\downarrow}$ is path-consistent. ■

Concerning the saturated generalized networks we have:

Proposition 10 *Let $\mathcal{N}$ be a saturated generalized network. For each (maximal) consistent instantiation m of $\mathcal{N}$ we can build a (maximal) consistent instantiation of $\mathcal{N}_{\downarrow}$ and* vice versa.

Proof: Let $\mathcal{N} = (V, C)$ and $\mathcal{N}_{\downarrow} = (V', C')$.

- Let m be a consistent instantiation of $\mathcal{N}$. Let us denote $m(V_i)^k$ the k^{th} interval of $m(V_i)$ associated to the k^{th} subinterval of the p-interval represented by $V_i \in V$ (with $1 \leq k \leq p$). Let m' be the instantiation of $\mathcal{N}_{\downarrow}$ which associates to each variable $V_i^k \in V'$ the interval $m(V_i)^k$. Let $V_i^k, V_j^l \in V'$. Since $m(V_i, V_j) \in C_{ij}$ we deduce that $m'(V_i^k, V_j^l) = (m(V_i, V_j))_{kl} \in (C_{ij})_{\downarrow kl}$, m' is a consistent instantiation of $\mathcal{N}_{\downarrow}$. Moreover, if $dim(m(V_i, V_j)) = dim(C_{ij})$, as C_{ij} is saturated we have $dim((m(V_i, V_j))_{kl}) = dim((C_{ij})_{\downarrow kl})$. Consequently, if m is maximal then m' is maximal too.
- Let m' be a consistent instantiation of $\mathcal{N}_{\downarrow}$. Let m be the instantiation of $\mathcal{N}$ defined by: let $V_i \in V$ represent a p-interval, $m(V_i)^k = m'(V_i^k)$ with $k \in 1, \ldots, p$ and $V_i^k \in V'$. Let $V_i, V_j \in V$ represent respectively a p-interval and a q-interval. Let $A \in 2^{\mathcal{A}_{p,q}}$ be defined by $A_{kl} = m'(V_i^k, V_j^l)$. We have $m(V_i, V_j) = A$. Since $m'(V_i^k, V_j^l) \in (C_{ij})_{\downarrow kl}$ and C_{ij} is saturated, $A \in C_{ij}$. Thus m is a consistent instantiation of $\mathcal{N}$. Now let us suppose that $dim(m'(V_i^k, V_j^l)) = dim((C_{ij})_{\downarrow kl})$. As C_{ij} is saturated we deduce that $dim(A) = dim(C_{ij})$. ■

In [25], van Beek and Cohen prove that any path-consistent convex interval network is minimal. Moreover, Ligozat [15] proves that any path-consistent preconvex interval network has a maximal consistent intantiation. From all this and Proposition 10 we can assert the following properties:

Theorem 2 *Let $\mathcal{N}$ be a generalized network closed for $\odot$.*

- *If $\mathcal{N}$ is convex then $\mathcal{N}$ is a minimal network.*
- *If $\mathcal{N}$ is convex (or more generally saturated-preconvex) then $\mathcal{N}$ admits a maximal consistent instantiation.*

Proof: Since $\mathcal{N}$ is a convex generalized network closed for $\odot$, we deduce from Proposition 4 that $\mathcal{N}$ is saturated and that $\mathcal{N}_{\downarrow}$ is convex. As $\mathcal{N}$ is closed for $\odot$ then $\mathcal{N}_{\downarrow}$ is path-consistent (Proposition 9). Hence $\mathcal{N}_{\downarrow}$ owns a maximal consistent instantiation and is a minimal interval network. From Proposition 10, we conclude that $\mathcal{N}$ admits also a maximal consistent instantiation and is minimal too. ■

In consequence, a simple nondeterministic algorithm to solve the consistency problem of a generalized network $\mathcal{N} = (V, C)$ is:

- guess a network $\mathcal{N}' = (V, C')$ such that $\forall_i, j \in 1, \ldots, |V|$, $C'_{ij} = \{A\}$ with $A \in C_{ij}$.

– Check the consistency of $\mathcal{N}'$ with the closure method. If $\mathcal{N}'$ is consistent then $\mathcal{N}$ is also consistent.

Hence the consistency problem of the generalized networks is a member of NP.

5.2. Intractability of the Weakly-Preconvex Generalized Relations

In the general case, contrary to the convexity, the preconvexity does not preserve tractability. To prove this fact we are going to characterize a polynomial reduction from the consistency problem of the interval networks to the consistency problem of the weakly-preconvex generalized networks.

Theorem 3 *The consistency problem of the weakly-preconvex generalized networks is a NP-complete problem.*

Proof: Let $\mathcal{N} = (V, C)$ be an interval network. From this network we define a generalized network $\mathcal{N}' = (V', C')$ in the following way:

– For each variable $V_i \in V$ we put a variable V_i' in the set V'.
– Each variable $V_i' \in V'$ is intended for representing a 3-interval whose structure is constrained by the following relation: $C_{ii}' = \left\{ \begin{pmatrix} eq & b & b \\ bi & eq & eq \\ bi & eq & eq \end{pmatrix} \right\}$. Intuitively, each variable V_i' represents a generalized interval formed by an interval followed by two equal intervals.
– For all $i, j \in 1, \ldots, n$, with $i \neq j$ and $n = |V|$, we define the constraints C_{ij}' by the following relation:

$$\prod \begin{pmatrix} C_{ij} & \{b\} & \{b\} \\ \{bi\} & \{eq\} & \{eq\} \\ \{bi\} & \{eq\} & \{eq\} \end{pmatrix} \cup \prod \begin{pmatrix} \mathcal{A}_{int} & \{b\} & \{b\} \\ \{bi\} & \{eq\} & \{eq\} \\ \{bi\} & \{eq\} & \{o, oi, d, di\} \end{pmatrix}.$$

Clearly, this construction is polynomial in the size of the set V. For each $i \in 1, \ldots, n$, it is easy to see that C_{ii}' is a weakly-preconvex relation since C_{ii} is a relation composed by one atomic relation. Now, let us consider the generalized relation C_{ij}', for all $i, j \in 1, \ldots, n$ with $i \neq j$. From Proposition 5 (c) the generalized $I(C_{ij}')$ can be easily calculated:

$$I(C_{ij}') = \prod \begin{pmatrix} I(C_{ij} \cup \mathcal{A}_{int}) & I(\{b\}) & I(\{b\}) \\ I(\{bi\}) & I(\{eq\}) & I(\{eq\}) \\ I(\{bi\}) & I(\{eq\}) & I(\{eq, o, oi, d, di\}) \end{pmatrix}$$
$$= \prod \begin{pmatrix} \mathcal{A}_{int} & \{b\} & \{b\} \\ \{bi\} & \{eq\} & \{eq\} \\ \{bi\} & \{eq\} & \{eq, s, si, f, fi, o, oi, d, di\} \end{pmatrix}$$

Let us show that $dim(I(C_{ij}') \setminus C_{ij}') < dim(C_{ij}')$. It is easy to see that $dim(C_{ij}')$ equals to 12. Let $A \in I(C_{ij}') \setminus C_{ij}'$. We have A_{31} which belongs to $\{eq, s, si, f, fi\}$, consequently $dim(A) \leq 11$. It results that C_{ij}' is a weakly-preconvex relation.

We can conclude that $\mathcal{N}'$ is a weakly-preconvex generalized network.
Now, we are going to show the "equivalence" between $\mathcal{N}$ and $\mathcal{N}'$: $\mathcal{N}$ is a consistent network iff $\mathcal{N}'$ is a consistent network.

- Let m be a consistent instantiation of $\mathcal{N}$. Let `aft` be an interval on the real line such that `aft` is after all m_i, with $i \in 1, \ldots, n$. We build an instantiation m' of $\mathcal{N}'$ in the following way: for all $i \in 1, \ldots, n$, $m'_i = (m_i, \texttt{aft}, \texttt{aft})$. We can check that m' is a consistent instantiation of $\mathcal{N}'$.
- Let m' be a consistent instantiation of $\mathcal{N}'$. It is important to noticed that the relation satisfied by m'_i and m'_j for any $i, j \in 1, \ldots, n$, with $i \neq j$, belongs to $\prod \begin{pmatrix} C_{ij} & \{b\} & \{b\} \\ \{bi\} & \{eq\} & \{eq\} \\ \{bi\} & \{eq\} & \{eq\} \end{pmatrix}$ since the relation $\prod \begin{pmatrix} \mathcal{A}_{int} & \{b\} & \{b\} \\ \{bi\} & \{eq\} & \{eq\} \\ \{bi\} & \{eq\} & \{o, oi, d, di\} \end{pmatrix}$ contains uniquely atomic relations which can never be satisfied. Consequently, to define a consistent instantiation of the generalized $\mathcal{N}$ we can take for m_i the interval $(m'_i)_1$, for each $i \in 1, \ldots, n$.

We have characterized a polynomial reduction from the consistency problem in IA to this one of the weakly-preconvex generalized networks. In consequence this problem is NP-hard. Moreover, it is also in NP since the consistency problem of the generalized networks is in NP. ■

6. A Tractable Case: The Strongly-Preconvex Generalized Relations

In the previous section we established the tractability of the set of the convex generalized relations and the intractability of the set of the weakly-preconvex relations. In this section we are going prove the tractability of a superset of the set of the convex generalized relation and a subset of the weakly-preconvex generalized relations: the set of strongly-preconvex generalized relations. We will give the definition of this set after we prove some generic tractability results.

6.1. Networks Weakly Closed for $\odot$

The definition of a network weakly closed for $\odot$ is closely connected to the definition of a weak path-consistent network introduced in [4]. Its definition uses the concept of convex closure defined for qualitative relation:

Definition 10 Let $\mathcal{N}$ be a generalized network. $\mathcal{N}$ is weakly closed for $\odot$ iff for every $i, j, k, \in 1, \ldots, |V|$, $C_{ij} \subseteq I\,(C_{ik} \odot C_{kj})$ and $C_{ij} \neq \{\}$.

Given a network $\mathcal{N}$, in order to get an equivalent a network which is weakly closed for $\odot$ (or empty), we can use the weak closure method which consists of iterating the operation:

$$C_{ij} \leftarrow C_{ij} \cap I(C_{ik} \odot C_{kj}).$$

This operation removes less atomic relations than that of the closure method since $R \subseteq I(R)$. It follows that the weak closure method is also sound and not complete. Like the closure method, it can be implemented in $O(|V|^3)$ too.

We extend the convex closure to the generalized networks:

Definition 11 Let $\mathcal{N}$ be a generalized network. The convex closure of $\mathcal{N}$, denoted by $I(\mathcal{N})$, is the generalized network (V', C') defined by $V' = V$ and $C'_{ij} = I(C_{ij})$.

We can easily note that the convex closure of a network is always a convex network. Moreover, we have the following property:

Proposition 11 *Let $\mathcal{N}$ be a generalized network. If $\mathcal{N}$ is weakly closed for $\odot$ then $I(\mathcal{N})$ is closed for $\odot$.*

Proof: If $\mathcal{N}$ is weakly closed for $\odot$ then $C_{ij} \subseteq I(C_{ik} \odot C_{kj})$. From Proposition 5 (b) and (a) we have $I(C_{ij}) \subseteq I(C_{ik} \odot C_{kj})$. Hence $I(C_{ij}) \subseteq I(C_{ik}) \odot I(C_{kj})$ (Proposition 6 (b)). ■

From this result we can prove the following proposition:

Proposition 12 *Each weakly-preconvex generalized network closed for $\odot$ $\mathcal{N}$ admits a maximal consistent instantiation.*

Proof: From Proposition 11, $I(\mathcal{N})$ is closed for $\odot$, and consequently admits a maximal consistent instantiation m (Theorem 2). We have $dim(m_{ij}) = dim(I(C_{ij}))$ and $m_{ij} \in I(C_{ij})$. As $C_{ij} \subseteq I(C_{ij})$ we can assert that $dim(C_{ij}) \leq dim(I(C_{ij}))$. Moreover $dim(I(C_{ij}) \setminus C_{ij}) < dim(C_{ij})$ since C_{ij} is weakly−preconvex (see Definition 8). In consequence $dim(I(C_{ij}) \setminus C_{ij}) < dim(I(C_{ij}))$. Let us suppose that $m_{ij} \in I(C_{ij}) \setminus C_{ij}$. Hence $dim(m_{ij}) < dim(I(C_{ij}))$. There is a contradiction. We conclude that $m_{ij} \in C_{ij}$. Moreover, we also get the fact that $dim(m_{ij}) = dim(C_{ij})$ since $dim(C_{ij}) \leq dim(I(C_{ij}))$ and $dim(m_{ij}) = dim(I(C_{ij}))$. ■

With this result we cannot deduce that the weak closure method is complete for weakly-preconvex generalized relations since by applying this method on a weakly-preconvex generalized network we are not sure of the weak preconvexity of the resulting network.

From all this we can prove the following theorem:

Theorem 4 *Let $\mathcal{E}$ be a set of weakly-preconvex generalized relations such that for each relation $R \in 2^{\mathcal{A}_{p,q}}$ belonging to $\mathcal{E}$ and for each convex relation $S \in 2^{\mathcal{A}_{p,q}}$ we have $R \cap S \in \mathcal{E}$. The weak closure method is complete for the consistency problem of the generalized networks whose constraints belong to $\mathcal{E}$.*

Proof: Let $\mathcal{N}$ be a generalized network having its constraints in $\mathcal{E}$. By applying the weak closure method to $\mathcal{N}$ we obtain a network $\mathcal{N}'$. If $\mathcal{N}'$ contains the empty relation then $\mathcal{N}$ is inconsistent, else $\mathcal{N}'$ is weakly closed for $\odot$ and its constraints belong to $\mathcal{E}$ because $\mathcal{E}$ is stable for the intersection with the convex relations. From Proposition 12 we deduce that $\mathcal{N}'$ and $\mathcal{N}$ are consistent. ■

In this theorem we can replace the weak closure method by the closure method. Indeed, we can prove that by applying the closure method to a network whose constraints belong to such a set $\mathcal{E}$, we obtain a subnetwork of a weakly-preconvex network closed for $\odot$ (or containing the empty constraint) which is equivalent to the initial network. Moreover, from this theorem we can show that there exists sets of generalized relations which contains the saturated-preconvex relations and for which the consistency problem is polynomial. As very basic example of such set let us consider the set $\mathcal{E}$ defined by the set of saturated-preconvex relations augmented by the two relations $\{(b, b), (m, o), (o, o), (b, o), (o, b)\}$ and $\{(b, b), (m, o), (b, o)\}$. We note that these relations are weakly-preconvex and not saturated. Since each convex relation of Interval Algebra is preconvex we can assert that each convex generalized relation is a saturated-preconvex relation. Moreover as the intersection of two preconvex interval relations is also preconvex, from Proposition 2 (b) we deduce that the set of saturated-preconvex generalized relations of $2^{\mathcal{A}_{p,q}}$ is stable for the intersection operation. We recall that the set of saturated-preconvex generalized relations is a subset of the weakly-preconvex generalized relations. To assert that $\mathcal{E}$ has the wanted properties we have just to note that the intersection of every convex relation of $2^{\mathcal{A}_{1,2}}$ with a relation of $\{(b, b), (m, o), (o, o), (o, b), (b, o)\}$ or $\{(b, b), (m, o), (b, o)\}$ is a relation of $\mathcal{E}$. Hence $\mathcal{E}$ has the claimed properties and from previous theorem we can affirm that the weak closure method is complete for every generalized network whose constraints belong to $\mathcal{E}$. Of course it is a very sample example, but we can build other sets $\mathcal{E}$ more complex with the properties required by Theorem 4. Actually, we are going to define the larger of these sets.

6.2. The Strongly-Preconvex Relations

The definition of a strongly-preconvex generalized relation is directly inspired by Theorem 4:

Definition 12 Let $R \in 2^{\mathcal{A}_{p,q}}$. R is strongly-preconvex iff for each convex relation $S \in 2^{\mathcal{A}_{p,q}}$, $R \cap S$ is a weakly-preconvex relation.

We will denote by S the set of the strongly-preconvex relations. Now, let us prove that S satisfies the requirements of Theorem 4.

Proposition 13 *Let R be a strongly-preconvex relation of* $2^{\mathcal{A}_{p,q}}$.

(*a*) *R is a weakly-preconvex relation,*
(*b*) *$R \cap S \in S$ for each convex relation S of* $2^{\mathcal{A}_{p,q}}$.

Proof:

(a) The total relation $2^{\mathcal{A}_{p,q}}$ is a convex relation. Hence, $R \cap 2^{\mathcal{A}_{p,q}} = R$ is a weakly-preconvex relation.
(b) Let S be a convex relation of $2^{\mathcal{A}_{p,q}}$. We must prove that $R \cap S \in S$. Let T be a convex relation of $2^{\mathcal{A}_{p,q}}$, $(R \cap S) \cap T = R \cap (S \cap T)$. From Theorem 1 we can deduce that $S \cap T$

is also a convex relation of $2^{\mathcal{A}_{p,q}}$. As R is strongly-preconvex it follows that $R \cap (S \cap T)$ is a weakly-preconvex relation. Hence $R \cap S$ is strongly-preconvex. ■

Hence, by applying Theorem 4, the consistency problem of strongly-preconvex networks is polynomial. The set S is the largest set to which we can apply this theorem.

7. Conclusion

We defined a very generic framework which subsumes several previous formalisms extending IA : the Generalized Interval Algebra. The considered entities are tuples of convex intervals and the atomic relations are defined by matrices of Allen's atomic relations.

By extending some concepts like dimension and convex closure we characterized a tractable set, namely the set of the strongly-preconvex relations. Several questions remain open about this set:

- Is the set of the strongly-preconvex generalized relations maximal tractable?
- Are there other larger tractable sets (containing the atomic relations)?

Another complexity result is that,contrary to the case of the Interval Algebra, the set of the weakly-preconvex relations is not tractable for the Generalized Interval Algebra. To obtain this result we exhibited a polynomial reduction from the consistency problem of the interval networks to the consistency problem of the weakly-preconvex generalized networks. The closure method and the weak closure method can be used to solve the consistency problem of the strongly-preconvex generalized networks. Currently, we are studying the respective advantages and drawbacks of these methods.

Another work to be considered is the integration of metric constraints in our formalism. These metric constraints could be defined by means of STPs [9]. With this kind of constraints,we can force either the total duration of a generalized interval or, the duration of an interval belonging to a generalized interval, to be higher than a lower value and smaller than a upper value. We are currently studying the consistency problem of the generalized networks augmented by these quantitative constraints. For this purpose we use our previous work stated for the Interval Algebra and Rectangle Algebra [7].

Note

1. This paper is an extended version of [8].

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On the consistency problem for the $\mathcal{INDU}$ calculus ☆

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Abstract

In this paper, we further investigate the consistency problem for the qualitative temporal calculus $\mathcal{INDU}$ introduced by Pujari et al. [A.K. Pujari, G.V. Kumari, A. Sattar, INDU: An interval and duration network, in: Australian Joint Conference on Artificial Intelligence, 1999, pp. 291–303]. We prove the intractability of the consistency problem for the subset of pre-convex relations, and the tractability of strongly pre-convex relations. Furthermore, we also define another interesting set of relations for which the consistency problem can be decided by the $\diamond$-closure method, a method similar to the usual path-consistency method. Finally, we prove that the $\diamond$-closure method is also complete for the set of atomic relations of $\mathcal{INDU}$ implying that the intervals have the same duration.
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Keywords: Qualitative temporal reasoning; Interval algebra; Qualitative constraint networks; Tractability

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1. Introduction

Temporal reasoning is a central task for numerous applications in many areas such as natural language understanding, specification and verification of programs and systems, scheduling, etc. In the field of qualitative reasoning about temporal data, the framework proposed by Allen [1], the Interval Algebra ($\mathcal{IA}$), is one of the best-known models.

Allen considers as basic temporal entities intervals of the time line and bases the $\mathcal{IA}$ calculus on 13 qualitative binary relations which correspond to all possible configurations between the four end-points of two intervals. In the $\mathcal{IA}$ calculus, temporal information can be represented using constraint networks (interval networks) whose variables correspond to intervals and whose constraints are expressed by disjunctions of the basic relations (interval relations). The consistency problem for interval networks is NP-complete. A large amount of research in the recent past has been devoted to the study and characterization of tractable subclasses of the interval algebra (see for example [4,11]). Now all tractable subclasses of $\mathcal{IA}$ are known.

More recently, a new qualitative formalism, called $\mathcal{INDU}$ has been proposed by Pujari et al. [7,8,12]. $\mathcal{INDU}$ also considers intervals as temporal entities, but it adds information about the relative durations of the intervals considered to the information expressed by Allen's relations. The resulting calculus has 25 basic relations corresponding to refinements of Allen's basic relations. Each one of the 25 basic relations of $\mathcal{INDU}$ can therefore be represented as a pair consisting of one of Allen's basic relations and of a basic relation of the Point Algebra ($<$, $>$ or $=$), which expresses the relation between the durations.

From a structural point of view, $\mathcal{INDU}$ and $\mathcal{IA}$ look very similar. This first impression, however, is quite deceptive. The real fact is that there exist numerous differences between the two formalisms. In particular, contrary to the relations of $\mathcal{IA}$, the relations of $\mathcal{INDU}$ are not closed for the composition operation. We also show that the consistency problem for $\mathcal{INDU}$ networks whose constraints are atomic relations cannot be decided by means of the well known path-consistency method.

In this paper, we are mainly interested in the consistency problem for $\mathcal{INDU}$ networks. Our aim is to characterize several important tractable sets for this problem. To this end we define the set of convex relations of $\mathcal{INDU}$ (in a way which is different from that used by Pujari et al.), the set of pre-convex relations, and a subset of the latter, the set of strongly pre-convex relations [3]. On the negative side, we prove that the consistency problem for $\mathcal{INDU}$ networks whose constraints are pre-convex relations is NP-complete. On the positive side, we show that strongly pre-convex relations can be expressed as conjunctions of Horn clauses [6] and, as a consequence, that the corresponding consistency problem is tractable. We also show that the usual method based on path-consistency cannot be used for strongly pre-convex relations. On the other hand, we define an interesting subclass of $\mathcal{INDU}$ relations for which the consistency problem can be decided by means of that method. Finally, we prove that the $\diamond$-closure method is also complete for the set of those atomic relations of $\mathcal{INDU}$ that imply that the intervals have the same duration.

2. The $\mathcal{INDU}$ calculus: an extension of $\mathcal{IA}$

2.1. The $\mathcal{INDU}$ relations

The framework introduced by Pujari et al. [12], called $\mathcal{INDU}$, is an extension of the well-known Allen's calculus [1], the Interval Algebra ($\mathcal{IA}$).

$\mathcal{IA}$ considers intervals of the line, and is based onto 13 binary relations. Each relation corresponds to a particular relative position between two intervals. An interval can be before (b), meet (m), overlap (o), contain (di), start (s), finish (f), be equal to (eq), another interval (the remaining six relations correspond to the converses of the first six relations). We denote by IA this set of 13 relations: $\mathsf{IA} = \{b, bi, m, mi, o, oi, s, si, f, fi, d, di, eq\}$. Vilain et al. [14] have defined a similar but less expressive formalism, the Point Algebra formalism ($\mathcal{PA}$). $\mathcal{PA}$ is based onto three basic relations $\mathsf{PA} = \{<, >, =\}$, corresponding to the three possible relative positions between two points of the line.

$\mathcal{INDU}$ is based on 25 binary relations between intervals of the line. Each one is defined as a pair of relations. The first relation is a basic relation of IA and denotes the relative position satisfied by the two intervals, the second one is a basic relation of PA and corresponds to the relation satisfied by the durations of the two intervals. Each basic relation of $\mathcal{INDU}$ will be denoted by an expression i^p where $i \in \mathsf{IA}$ and $p \in \mathsf{PA}$. Two intervals x and y satisfy i^p, denoted by $x\ i^p\ y$, iff $x\ i\ y$ and $(x^+ - x^-)\ p\ (y^+ - y^+)$, where x^-, y^- (resp. x^+, y^+) are the lower (resp. upper) endpoints of x and y. For example, the intervals[1] $(1, 2)$ and $(2, 4)$ satisfy the basic relation $m^<$. We denote by INDU the set of the 25 basic $\mathcal{INDU}$ relations, $\mathsf{INDU} = \{b^<, b^>, b^=, bi^<, bi^>, bi^=, m^<, m^>, m^=, mi^<, mi^>, mi^=, o^<, o^>, o^=, oi^<, oi^>, oi^=, s^<, si^>, d^<, di^>, f^<, fi^>, eq^=\}$. Any two intervals satisfy one and only one relation of INDU. In the sequel, we will also consider other basic relations which we call *virtual*. These particular basic relations are the unsatisfiable relations i^p. This set corresponds to $\{eq^<, eq^>, d^=, d^>, di^=, di^<, f^=, f^>, fi^=, fi^<, s^=, s^>, si^=, si^<\}$.

From the 25 basic relations of INDU, 2^{25} relations can be defined. Each one corresponds to an element of the power set of INDU, i.e., 2^{INDU}. Let $r \in 2^{\mathsf{INDU}}$, and x and y be two intervals, $x\ r\ y$ iff $x\ a\ y$ for some $a \in r$. For instance, $(1, 2)$ and $(3, 4)$ satisfy the relation $\{b^<, b^=, m^=, d^<\}$. The particular relation $\{\}$ is called the empty relation (we will also denote this relation by $\emptyset$). Each relation $r \in 2^{\mathsf{INDU}}$ can be seen as the disjunction of the basic relations which it contains. A relation containing only one basic relation is also called an atomic relation. Similarly, 2^{IA} defines 2^{13} interval relations and 2^{PA} defines 2^3 relations.

In some cases, the constraint "two intervals x and y must satisfy a relation $r \in \mathsf{INDU}$" can be expressed by a conjunction of Horn clauses. A Horn clause [6] is a disjunction of literals (inequations or inequalities) of the following forms: $c_1u_1 + \cdots + c_nu_n \leqslant c_{n+1}u_{n+1} + \cdots + c_{n+k}u_{n+k}$ or $c_1u_1 + \cdots + c_nu_n \neq c_{n+1}u_{n+1} + \cdots + c_{n+k}u_{n+k}$, with at most one literal using $\leqslant$ (each c_i is a rational number, each u_i is a rational variable). The basic relations belonging to INDU can be expressed by a conjunction of unitary Horn clauses (a unitary clause is a clause containing only one literal). For example, the satisfaction of

[1] We consider closed intervals of the rational number line. Such an interval will be denoted by (a, b) with a and b two rational numbers satisfying $a < b$.

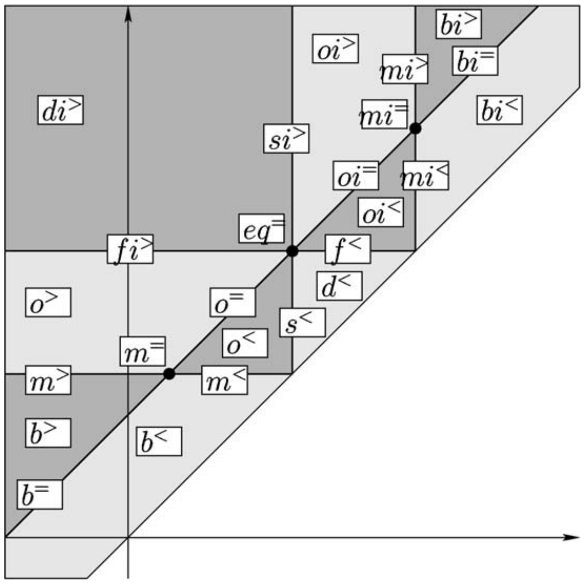

Fig. 1. Geometrical representation of the basic relations of $\mathcal{INDU}$.

$m^<$ by two intervals $x = (x^-, x^+)$ and $y = (y^-, y^+)$ can be expressed by the following conjunction of unitary Horn clauses: $(x^- \leqslant x^+) \wedge (x^- \neq x^+) \wedge (y^- \leqslant y^+) \wedge (y^- \neq y^+) \wedge (x^+ \leqslant y^-) \wedge (y^- \leqslant x^+) \wedge (x^+ - x^- \leqslant y^+ - y^-) \wedge (x^+ - x^- \neq y^+ - y^-)$. A ORD Horn clause is a Horn clause which contains only literals of the forms $u \neq v$ or $u \leqslant v$, with u, v two endpoints. All relations of 2^{PA} can be expressed by a conjunction of ORD Horn clauses, those of 2^{IA} having this property are called the ORD Horn relations [11].

In the same way as for the basic relations of IA, the basic relations of $\mathcal{INDU}$ can be represented by regions of the plane equipped with an orthogonal basis. Each interval $x = (x^-, x^+)$ is represented by a point with coordinates (x^-, x^+). Given an interval of reference $y = (y^-, y^+)$, the region representing the basic relation $a \in \mathsf{INDU}$, denoted by $\mathsf{Reg}(a, y)$, corresponds to the set of points $\{x = (x^-, x^+)\colon\ x\ a\ y\}$. We obtain 25 convex regions partitioning the half plane $\mathsf{H} = \{(x, y)\colon\ y > x\}$ (see Fig. 1). The geometrical representation of $r \in 2^{\mathsf{INDU}}$ is the region defined by $\mathsf{Reg}(r, y) = \bigcup_{a \in r} \{\mathsf{Reg}(a, y)\}$. When the choice of the point of reference y is immaterial, we denote this region by $\mathsf{Reg}(r)$.

2.2. *The fundamental operations*

Similarly to the case of relations of $\mathcal{IA}$ and $\mathcal{PA}$, we define operations for the relations of $\mathcal{INDU}$. Considering a relation of 2^{INDU} as a usual binary relation defined on intervals, the operations union ($\cup$), intersection ($\cap$), converse ($^{-1}$) and composition ($\circ$) can be defined in the usual way: $x(r \cap s)y$ iff $x\ r\ y$ and $x\ s\ y$; $x(r \cup s)y$ iff $x\ r\ y$ or $x\ s\ y$; $x(r \circ s)y$ iff $\exists z, x\ r\ z$ and $z\ s\ y$; $x\ r^{-1}\ y$ iff $y\ r\ x$. It is easy to show that $r \cap s = \{a \in \mathsf{INDU}\colon\ a \in r \text{ and } a \in s\}$, $r \cup s = \{a \in \mathsf{INDU}\colon\ a \in r \text{ or } a \in s\}$. The converse of an atomic relation is an atomic relation, like for IA and PA, and $\{i^p\}^{-1} = \{(i^{-1})^{p^{-1}}\}$, with $i^p \in \mathsf{INDU}$. Hence, $r^{-1} = \bigcup_{a \in r} \{a^{-1}\}$. Hence, 2^{INDU} is closed for $\cap$, $\cup$ and $^{-1}$. The $\mathcal{INDU}$ composition operation has an unusual behavior for a qualitative formalism. Indeed, unlike $\mathcal{IA}$ and $\mathcal{PA}$, 2^{INDU} is not closed under composition: consider the relation $r = \{m^=\}$, the pair of intervals formed by $(1, 2)$ and $(3, 4)$ belongs to $r \circ r$ as $(1, 2)$ and $(2, 3)$ satisfy $m^=$, $(2, 3)$ and $(3, 4)$ satisfy $m^=$. Since $(1, 2)$ and $(3, 4)$ satisfy the relation $b^=$, $b^= \in r \circ r$. Now, given a pair of intervals (x, y) satisfying the basic relation $b^=$, an interval z such that $x\ m^=\ z$ and $z\ m^=\ y$ may not exist. For example, this is the case for $x = (0, 1)$ and $y = (4, 5)$. So, the composition operation is inadequate for qualitative reasoning in $\mathcal{INDU}$ since *basic building blocks* must be the basic relations. It is necessary to define a weaker operation, sometimes called *weak composition*, for which 2^{INDU} is closed; we will denote this operation by $\diamond$. The operation $\diamond$ is defined as follows: Firstly, for atomic relations $a, b \in \mathsf{INDU}$, $a \diamond b = \{c \in \mathsf{INDU}\colon \exists x, y, z$ with $x\ a\ z, z\ b\ y$ and $x\ c\ y\}$; then, for arbitrary relations $r, s \in 2^{\mathsf{INDU}}$, $r \diamond s = \bigcup_{a\in r, b\in s}\{a \diamond b\}$. Equivalently, $r \diamond s$ is the smallest relation of 2^{INDU} containing $r \circ s$. Note that $\circ$ and $\diamond$ are the same operations for $\mathcal{IA}$ and $\mathcal{PA}$. The operation $\diamond$ is not associative, for instance, $(\{bi^>\} \diamond \{mi^>\}) \diamond \{m^>\} = \{oi^>, mi^>, bi^>\}$ and $\{bi^>\} \diamond (\{mi^>\} \diamond \{m^>\}) = \{bi^>\}$. As a result $\diamond$ cannot be used to define a relation algebra [13] on INDU. We also define a binary operation corresponding to the Cartesian product of an interval relation and a point relation by: $r \times s = \{i^p\colon\ i \in r, p \in s\}$, with $r \in 2^{\mathsf{IA}}$ and $s \in 2^{\mathsf{PA}}$. This relation can contain virtual basic relations of $\mathcal{INDU}$. Note that for $i^p, j^q \in \mathsf{INDU}$, $i^p \diamond j^q = ((i \circ j) \times (p \circ q)) \cap \mathsf{INDU}$. The interval and point projections of an $\mathcal{INDU}$ relation r, denoted respectively, by r_I and r_P are defined by $r_I = \{i\colon\ i^p \in r\}$, $r_P = \{p\colon\ i^p \in r\}$. In the sequel we will say that a subset of relations of 2^{INDU} is a subclass iff it is closed for the operations $^{-1}$, $\diamond$, and $\cap$.

3. Qualitative constraint networks

3.1. Definitions

Definition 1. An $\mathcal{INDU}$ constraint network is a pair $\mathcal{N} = (V, C)$, where:

- V is a finite set $\{V_1, \ldots, V_n\}$ (with $n = |V|$) of variables representing intervals of the line;
- C is a mapping associating with each pair $V_i, V_j \in V$ a constraint, denoted by C_{ij}, defined by a relation of 2^{INDU}.

We assume that $C_{ij}^{-1} = C_{ji}$ and $C_{ii} = \{eq^=\}$.

We define constraint networks on $\mathcal{IA}$ (interval networks) and on $\mathcal{PA}$ (point networks) in a similar way. By definition, an atomic network is a network whose constraints are defined by atomic relations.

Definition 2. Let $\mathcal{N} = (V, C)$ be a constraint network on $\mathcal{INDU}$ with $n = |V|$. An instantiation m of $\mathcal{N}$ is a mapping which associates an interval m_i with each variable $V_i \in V$. The basic relation of INDU satisfied by m_i and m_j will be denoted by m_{ij}. The instantiation m will be called a consistent instantiation or a solution of $\mathcal{N}$ iff for every pair of variables $V_i, V_j \in V$, $m_{ij} \in C_{ij}$. In the case where $\mathcal{N}$ has a solution, $\mathcal{N}$ will be said consistent. $\mathcal{N}$ is $k-$consistent (with $k \in \{1, \ldots, n\}$) iff any partial consistent instantiation on $k - 1$ variables can be extended to a new variable while remaining consistent. $\mathcal{N}$ will be said

$\diamond$-closed iff for each $i, j, k \in \{1, \ldots, n\}$, $C_{ij} \subseteq C_{ik} \diamond C_{kj}$. A subnetwork $\mathcal{N}' = (V', C')$ is a network such that $V' = V$ and $C'_{ij} \subseteq C_{ij}$ for each pair of variables V_i and V_j. A network $\mathcal{N}'' = (V, C'')$ is equivalent to $\mathcal{N}$ iff $\mathcal{N}$ and $\mathcal{N}''$ have the same solutions.

Given a constraint network, the main problem is to decide whether it admits a consistent instantiation. This problem is called the consistency problem.

Given a set $\mathcal{E} \subseteq 2^{\mathsf{INDU}}$ (closed for converse and containing the relation $\{eq^=\}$), the consistency problem for $\mathcal{INDU}$ networks whose constraints belong to $\mathcal{E}$ will be denoted by $\mathsf{Cons}(\mathcal{E})$. The consistency problem for interval networks being NP-complete, obviously $\mathsf{Cons}(2^{\mathsf{INDU}})$ is NP-hard. Moreover, we will show in Section 4.2 that the consistency problem for atomic $\mathcal{INDU}$ networks is a polynomial problem. Hence, we can test the consistency of any $\mathcal{INDU}$ network in exponential time by testing the consistency of all its atomic subnetworks. As a consequence, we get the fact that $\mathsf{Cons}(2^{\mathsf{INDU}})$ is NP-complete.

We call $\diamond$-closure method, the method which consists in obtaining from a network $\mathcal{N} = (V, C)$ an equivalent and $\diamond$-closed subnetwork $\mathcal{N}'$ by iterating the operation $C_{ij} := C_{ij} \cap (C_{ik} \diamond C_{kj})$, for $i, j, k \in \{1, \ldots, |V|\}$, until a fixpoint is obtained. This method can be implemented in $O(n^3)$ time (with $n = |V|$) by an algorithm similar to those used to obtain equivalent path-consistent constraint subnetworks from binary constraint networks [10].

3.2. Consistency and the $\diamond$-closure method

Given that the set 2^{INDU} is not closed for the composition operation, several fundamental properties of networks of $\mathcal{IA}$ and $\mathcal{PA}$ are no longer true in the framework of $\mathcal{INDU}$.

Proposition 1. *Let $\mathcal{N}$ be a consistent $\mathcal{INDU}$ network. A 3-consistent network $\mathcal{N}'$ equivalent to $\mathcal{N}$ may not exist.*

The atomic $\mathcal{INDU}$ network depicted in Fig. 2(a) is $\diamond$-closed and consistent but it is not 3-consistent: consider the partial solution $m(V_1) = (1, 2)$, $m(V_3) = (3, 5)$. This solution cannot be extended to V_2. This network is consistent and does not admit an equivalent 3-consistent network.

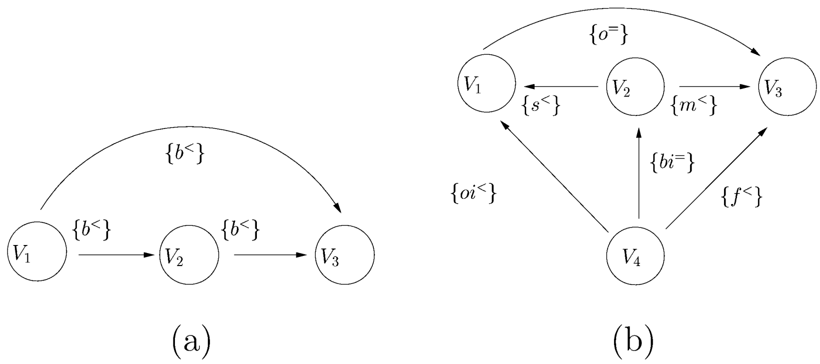

Fig. 2. (a) a consistent atomic network which is not 3-consistent, (b) an atomic network which is $\diamond$-closed but not consistent.

Moreover, there are $\diamond$-closed atomic networks which are not consistent (for example, see Fig. 2(b)).

Consequently, we can state the following property:

Proposition 2. *A $\diamond$-closed (atomic) $\mathcal{INDU}$ network which does not contain the empty constraint is not necessarily consistent.*

For $\mathcal{IA}$ it has been shown that the complexity of $\mathsf{Cons}(\mathcal{E})$, with $\mathcal{E} \in 2^{\mathsf{IA}}$, is the same as the complexity of $\mathsf{Cons}(\overline{\mathcal{E}})$, with $\overline{\mathcal{E}}$ the closure of $\mathcal{E}$ for the operations converse, intersection and composition. The proof of this result is based on the fact that from an interval network whose constraints belong to $\overline{\mathcal{E}}$ we can always build an equivalent network from $\mathcal{E}$. If we replace the composition operation by $\diamond$, we can no longer prove this property for $\mathcal{INDU}$. This is a consequence of the fact that if $x\ (r \diamond s)\ y$ then a third interval z such that $x\ r\ z$ and $z\ s\ y$ may not exist (a necessary property for building an equivalent network on $\mathcal{E}$ from the network on $\overline{\mathcal{E}}$). Nevertheless, we can prove a weaker property:

Proposition 3. *Let $\mathcal{E} \subseteq 2^{\mathsf{INDU}}$ be a subset of relations which is closed under the converse operation and which contains the atomic relation $\{eq^=\}$. Then $\mathsf{Cons}(\mathcal{E})$ is a polynomial problem (resp. a NP-complete problem) if, and only if, $\mathsf{Cons}(\mathcal{E}^*)$ is a polynomial problem (resp. a NP-complete problem), where * denotes the closure for the intersection operation.*

A proof of this proposition can be based on the fact that a constraint $x\ (r_0 \cap \cdots \cap r_i)\ y$ in $\mathcal{E}^*$ (with $r_0, \ldots, r_i \in \mathcal{E}$) can be replaced by the constraints $x\ r_0\ y$, $x\ r_1\ z_1, \ldots, x\ r_i\ z_i$, $y\ \{eq^=\}\ z_1, \ldots, y\ \{eq^=\}\ z_i$, where $z_1, \ldots, z_i$ are new variables. Hence, we have a polynomial reduction from $\mathsf{Cons}(\mathcal{E}^*)$ to $\mathsf{Cons}(\mathcal{E})$.

In the sequel of this paper, we are going to characterize several sets of $\mathcal{INDU}$ for which the consistency problem is polynomial. Several cases of tractability can be obtained in a direct way from the tractable cases of $\mathcal{IA}$. For instance, we can state the following result:

Proposition 4. *Let $\mathcal{E} \subseteq 2^{\mathsf{IA}}$ be a set for which the consistency problem is polynomial. Let $\mathcal{E}' \subseteq 2^{\mathsf{INDU}}$ be defined by $\mathcal{E}' = \{(r \times \{<, =, >\}) \cap \mathsf{INDU}\colon\ r \in \mathcal{E}\}$.*

Then $\mathsf{Cons}(\mathcal{E}')$ is polynomial.

The validity of this proposition is due to the fact that $\mathcal{E}$ and $\mathcal{E}'$ represent the same class of temporal constraints. Indeed, the $\mathcal{INDU}$ temporal constraint $x\ ((r \times \{<, =, >\}) \cap \mathsf{INDU})\ y$ (with x, y two variables and $r \in \mathcal{E}$) can be equivalently expressed by the temporal constraint of the interval algebra $x\ r\ y$. Let us now establish less trivial tractability cases.

4. Convex relations in the $\mathcal{INDU}$ calculus

4.1. Definition and representation

In this section, we introduce a lattice structure on the set of basic $\mathcal{INDU}$ relations, based on the similar structures for $\mathcal{IA}$ and $\mathcal{PA}$ [9] (see also [12]). In [9] the interval lattice

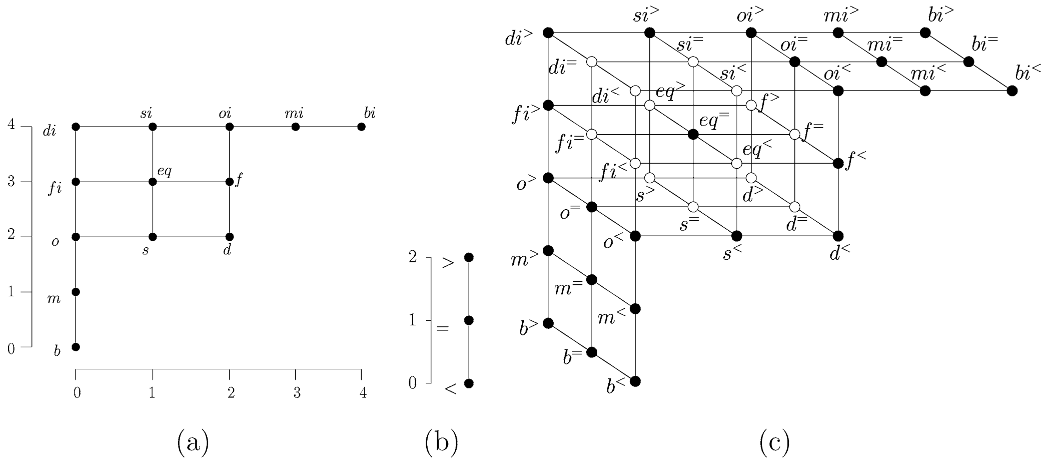

Fig. 3. (a) the interval lattice, (b) the point lattice, (c) the $\mathcal{INDU}$ lattice.

(resp. the point lattice) is defined by associating to each basic relation A a pair of integers (i_A, j_A) (resp. an integer i_A). For example, the pair $(1, 3)$ corresponds to the basic relation *eq*, while 1 corresponds to the basic relation $=$ (see Fig. 3 for a complete description). Using this correspondence, an ordering $\leqslant_{int}$ (resp. $\leqslant_{pt}$) is defined on IA (resp. on PA) by specifying that $A \leqslant_{int} B$ iff $i_A \leqslant i_B$ and $j_A \leqslant j_B$ (resp. $i_A \leqslant i_B$). We refer to the resulting structure (IA, $\leqslant_{int}$) (resp. (PA, $\leqslant_{pt}$)) as to the interval lattice (resp. the point lattice).

Once the lattice has been defined, convex relations of $\mathcal{IA}$ (resp. $\mathcal{PA}$) correspond to intervals in the lattice[2] (resp. the point lattice) (see Fig. 3 (a) and (b)). For example, the interval relation $\{o, s, fi, eq\}$ and the point relation $\{<, =\}$ are convex relations corresponding to the intervals $[o, eq]$ and $[<, =]$.

In a natural way, we define the $\mathcal{INDU}$ lattice as the Cartesian product of the interval lattice and the point lattice (see Fig. 3). This lattice is also defined on the virtual basic relations of $\mathcal{INDU}$. We define the set of convex relations of 2^{INDU} in the following way:

Definition 3. A relation $r \in 2^{\mathsf{INDU}}$ is a convex relation iff $r = [\min, \max] \cap \mathsf{INDU}$, where $[\min, \max]$ an interval in the $\mathcal{INDU}$ lattice.

For example, the relation $\{m^<, m^=, o^<, o^=\}$ is a convex relation. We denote by $\mathcal{C}$ the set of convex relations. Remark that $r \in 2^{\mathsf{INDU}}$ is convex iff $r = (s \times t) \cap \mathsf{INDU}$ with s and t convex relations of 2^{IA} and 2^{PA}. Hence from a geometrical point of view, a relation r of $\mathcal{INDU}$ is a convex relation when its geometrical representation in the plane $\mathsf{Reg}(r)$ satisfies the following equality: $\exists h \in \mathcal{H}$, $\mathsf{Reg}(r) = (Proj_1(\mathsf{Reg}(r)) \times Proj_2(\mathsf{Reg}(r))) \cap h$, where $\mathcal{H} = \{\mathsf{Reg}(\mathsf{INDU}), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{<\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{<, =\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{>\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{>, =\})), \mathsf{Reg}(\mathsf{INDU} \cap (\mathsf{IA} \times \{=\}))\}$ and where $Proj_1$ (resp. $Proj_2$) denote the projection functions on the horizontal axis (resp. vertical axis).

[2] Given a lattice $(E, \leqslant)$, an interval is either the empty set or a set $\{e \in E\colon \min \leqslant e \leqslant \max\}$ for some $\min, \max \in E$ with $\min \leqslant \max$ (this last set will be denoted by $[\min, \max]$).

In the same way as for the convex relations of $\mathcal{IA}$ and of $\mathcal{PA}$ we have the following property for the convex relation of $\mathcal{INDU}$:

Proposition 5. *A relation $r \in \mathsf{INDU}$ is a convex relation iff it can be expressed by a conjunction of unitary Horn clauses Φ such that if $u \neq v \in \Phi$ then $u \leqslant v \in \Phi$ or $v \leqslant u \in \Phi$ (where u and v denote endpoints or differences of endpoints).*

Proof. Let $r \in \mathsf{INDU}$ be a convex relation. We know that $r = (s \times t) \cap \mathsf{INDU}$ where s and t are convex relations in 2^{IA} and 2^{PA} respectively. Now s is a convex relation of 2^{IA} which expresses the constraint concerning the relative position between the two intervals. This constraint can be expressed by a conjunction of unitary Horn clauses Φ_s such that if $u \neq v \in \Phi_s$ then $u \leqslant v \in \Phi_s$ or $v \leqslant u \in \Phi_s$ (with u and v denoting endpoints). In a similar way, the duration constraint between the two intervals is expressed by t a convex relation of 2^{PA}. Again, t can be also expressed by a conjunction of unitary Horn clauses Φ_t such that if $u \neq v \in \Phi_t$ then $u \leqslant v \in \Phi_t$ or $v \leqslant u \in \Phi_t$ (with u and v denoting differences of endpoints). Let us remark that u and v denote differences of endpoints in Φ_s and endpoints in Φ_t. Finally, we can define the required conjunction Φ by $\Phi_s \wedge \Phi_t$. □

For example, consider the $\mathcal{INDU}$ convex relation $r = \{m^=, m^<, o^=, o^<\}$. We have $r = (s \times t) \cap \mathsf{INDU}$ with $s = \{m, o\}$ and $t = \{<, =\}$. The constraint $x\ r\ y$ can be expressed by $\Phi_s = y^- \leqslant x^+ \wedge x^+ \leqslant y^+ \wedge x^- \leqslant x^+ \wedge y^- \leqslant y^+ \wedge x^+ \neq y^+ \wedge x^- \neq x^+ \wedge y^- \neq y^+$. The duration constraint t can be expressed by $\Phi_t = x^+ - x^- \leqslant y^+ - y^-$. Hence, the temporal constraint $x\ r\ y$ can be expressed by the conjunction $\Phi_s \wedge \Phi_t$.

Notice that a convex relation of $\mathcal{INDU}$ cannot always be represented by a conjunction of unitary ORD Horn clauses (see the previous example). Pujari et al. enumerate 227 convex relations. Our definition, by contrast, results in 240 convex relations, this difference arising from the fact that Pujari et al. use a lattice which does not take into account the virtual basic relations. The set $\mathcal{C}$ is closed for $^{-1}$, $\cap$, but not for $\diamond$. The closure for the intersection and converse follows directly from the definition. The following example shows the instability of $\mathcal{C}$ for $\diamond$: $\{b^<\} \diamond \{d^<, o^<, o^>, o^=, s^<\}$ is the non-convex relation $\{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$. Some relations of $\mathcal{C}$ can be expressed by conjunctions of ORD Horn clauses. This set of relations, denoted by $\mathcal{C}_{IA}$, corresponds to the 83 convex relations of $\mathcal{IA}$. $\mathcal{C}_{IA}$ can be defined as follows:

Definition 4. Let r be a relation in 2^{INDU}. Then $r \in \mathcal{C}_{IA}$ iff r satisfies one of the equivalent properties:

- $r = (s \times \{<, =, >\}) \cap \mathsf{INDU}$ where s is a convex interval relation,
- $r = [a^<, b^>] \cap \mathsf{INDU}$, where $[a^<, b^>]$ is an interval of the $\mathcal{INDU}$ lattice ($a, b \in \mathsf{IA}$).

Given that the set $\mathcal{C}_{IA}$ corresponds to the set of convex interval relations, the operation $\diamond$ on $\mathcal{C}_{IA}$ corresponds to the composition operation $\circ$. As a further consequence, $\mathcal{C}_{IA}$ is a subclass.

4.2. Tractability of the convex $\mathcal{INDU}$ relations

The convex $\mathcal{INDU}$ relations can be represented by conjunctions of unitary Horn clauses; as a consequence, the consistency problem of the convex $\mathcal{INDU}$ networks ($\mathcal{INDU}$ networks whose constraints are convex) is polynomial. Indeed, we can translate this kind of network into conjunctions of Horn clauses and apply a resolution algorithm such as the algorithm proposed by Koubarakis [6]. Notice that we can also use the Simplex algorithm or Kachian's algorithm for solving these particular constraints (as a consequence of Proposition 5).

Proposition 6. $\mathsf{Cons}(\mathcal{C})$ *is a polynomial problem.*

It is well known that the path-consistency method can be used to solve the consistency problem of the convex interval networks. Hence, $\mathsf{Cons}(\mathcal{C}_{IA})$ can be decided by the $\diamond$-closure method.

5. The pre-convex $\mathcal{INDU}$ relations

The maximal tractable set of $\mathcal{IA}$ containing the 13 atomic interval relations is the set of pre-convex interval relations, which is identical with the set of ORD Horn interval relations. To define the pre-convex relations of $\mathcal{INDU}$ we use the method introduced by Ligozat [9] by extending the notions of convex closure and dimension to $\mathcal{INDU}$. For the interval algebra, the dimension of an interval relation corresponds to the dimension of the geometrical representation of this region in the plane. This dimension is the maximal dimension of the dimensions of the basic relations it contains. In a similar way, we have the following definition:

Definition 5. Let $a \in \mathsf{INDU}$. The dimension of a, denoted by $\dim(a)$, is the dimensional space of $\mathsf{Reg}(a)$. If $r \in 2^{\mathsf{INDU}}$ is a non-empty relation, $\dim(r) = \max\{\dim(a)\colon a \in r\}$.

As usual, we define $\dim(\{\})$ as -1. As an illustration, we have $\dim(m^{=}) = 0$, $\dim(s^{<}) = 1$, $\dim(b^{>}) = 2$ and $\dim(\{m^{=}, o^{<}\}) = \max\{0, 2\} = 2$.

Definition 6. The convex closure of an $\mathcal{INDU}$ relation r, denoted by $I(r)$, is the smallest convex relation of $\mathcal{C}$ containing r: $I(r) = \bigcap\{s \in \mathcal{C}\colon r \subseteq s\}$.

Notice that this definition makes sense because the set $\mathcal{C}$ is closed under intersection.

The convex closure in $\mathcal{INDU}$ can be computed from the convex closures in $\mathcal{IA}$ and $\mathcal{PA}$. Indeed, we have $I(r) = (I(r_I) \times I(r_P)) \cap \mathsf{INDU}$ for all relations $r \in 2^{\mathsf{INDU}}$. Now, we can define the pre-convex relations of $\mathcal{INDU}$:

Definition 7. Let $r \in 2^{\mathsf{INDU}}$. Then r is a pre-convex relation iff $r = \{\}$ or $\dim(I(r) \setminus r) < \dim(r)$.

We denote by $\mathcal{P}$ the set of pre-convex $\mathcal{INDU}$ relations. $\mathcal{P}$ contains 88096 relations. The set of convex relations $\mathcal{C}$ is a subset of $\mathcal{P}$. The set $\mathcal{P}$ is closed under $^{-1}$, but not closed under the operations $\cap$ and $\diamond$: consider the pre-convex $\mathcal{INDU}$ relations $r = \{eq^=, b^<, b^=, o^<\}$, $s = \{eq^=, b^>, b^=, o^>\}$, $t = \{b^<\}$ and $u = \{d^<, o^<, o^>\}$. Then the relations $r \cap s = \{eq^=, b^=\}$ and $t \diamond u = \{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$ are not pre-convex relations.

5.1. *Intractability of the pre-convex $\mathcal{INDU}$ relations*

In this section we prove that the consistency problem for $\mathcal{P}$ is NP-complete. In order to do so, we define a polynomial reduction from the 3-coloring problem of a graph [5] to $\mathsf{Cons}(\mathcal{P}^*)$.

Proposition 7. $\mathsf{Cons}(\mathcal{P}^*)$ *is a NP-complete problem.*

Proof. Let $G = (S, A)$ be a non-oriented graph, with S a set of vertices and A a set of edges between these vertices. We build an $\mathcal{INDU}$ network $\mathcal{N} = (V, C)$ in the following way: V is a set of variables corresponding to the union of $Col = \{Col_1, Col_2, Col_3\}$ and $V_S = \{S_1, \ldots, S_n\}$ with $n = |S|$. Each variable of Col is associated with a color. Each variable $S_i \in V_S$ is associated with a vertex $s_i \in S$. The constraints of $\mathcal{N}$ between the three variables of Col are given in Fig. 4(a). Those between two variables S_i and S_j such that $(s_i, s_j) \in A$ (resp. $\notin A$) are given in Fig. 4(b) (resp. (c)). We can check that these constraints belong to $\mathcal{P}^*$. For example, the relation $\{m^=, eq^=, mi^=\}$ is the intersection of the pre-convex relations $\{o^>, di^>, oi^>, m^=, eq^=, mi^=\}$ and $\{o^<, d^<, o^<, m^=, eq^=, mi^=\}$. We can prove that $G = (S, A)$ is 3-colorable iff $\mathcal{N}$ is consistent: given a solution of the 3-coloring problem for G, we assign to each variable S_i the interval corresponding to the color assigned to the vertex s_i. Conversely, to obtain a solution of the 3-coloring problem for G from a solution of $\mathcal{N}$, we assign to the vertex s_i the color corresponding to the interval assigned to S_i. $\square$

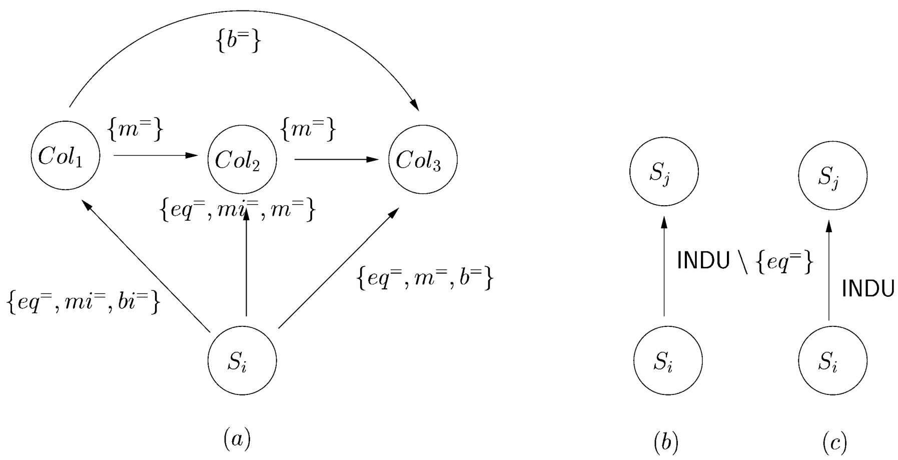

Fig. 4. The constraints of $\mathcal{N} = (V, C)$.

As a consequence of this proposition and Proposition 3 we obtain:

Theorem 1. Cons($\mathcal{P}$) *is a NP-complete problem.*

6. The strongly pre-convex $\mathcal{INDU}$ relations

Balbiani et al. have proved that for the strongly pre-convex generalized interval relations the consistency problem is polynomial [3]. Following the line of reasoning given by Balbiani et al., we define the strongly pre-convex relations of $\mathcal{INDU}$.

Definition 8. Let $r \in 2^{\mathsf{INDU}}$. Then r is strongly pre-convex iff for each convex relation $t \in \mathcal{C}$, the relation $r \cap t$ is a pre-convex relation.

The definition of the strongly pre-convex relations is guided by the desire to obtain a subset of pre-convex relations closed under the intersection operation. $\mathcal{F}$ will denote the set of strongly pre-convex relations of $\mathcal{INDU}$. It has 45792 elements.

Proposition 8. *The set $\mathcal{F}$ is closed*[3] *under the operations* $^{-1}$, $\cap$, *but not closed under* $\diamond$.

As a counter-example, consider the strongly pre-convex relations $r = \{b^<\}$ and $s = \{d^<, o^<, o^>\}$, $r \diamond s$ is the relation $\{b^<, b^>, b^=, d^<, o^<, m^<, s^<\}$ which is not a strongly pre-convex relation.

6.1. Tractability of the strongly pre-convex $\mathcal{INDU}$ relations

This section is devoted to the properties of the strongly pre-convex $\mathcal{INDU}$ relations in relation to the consistency problems.

Proposition 9. *The strongly pre-convex $\mathcal{INDU}$ relations can be represented by conjunctions of Horn clauses.*

Proof. Let $r \in \mathcal{F}$. As $I(r)$ is convex, there exists a conjunction of Horn clauses representing it. Denote by $\Phi_{I(r)}$ such a conjunction. In the general case, $\Phi_{I(r)}$ is too permissive. Indeed, each basic relation $a \in I(r) \setminus r$ is realizable w.r.t. $\Phi_{I(r)}$. We must forbid these basic relations, without forbidding the basic relations of r. For each such $a \in I(r) \setminus r$, we exhibit a Horn clause, denoted by Φ_a, such that the addition of Φ_a to Φ excludes the satisfaction of a without excluding that of the atomic relations belonging to r. Since r is pre-convex, then $\dim(I(r) \setminus r) < \dim(r)$, hence $\dim(a) < \dim(r)$. Consequently, $\dim(a) = 1$ or 0. Let us first consider the basic relations which do not impose equality on the durations.

$$\Phi_{m^<} = (x^+ \neq y^- \vee x^+ - x^- \geqslant y^+ - y^-),$$

[3] A computer-program has been used to prove this result, as well as for the future Proposition 10.

$$\Phi_{m^>} = (x^+ \neq y^- \vee x^+ - x^- \leqslant y^+ - y^-),$$
$$\Phi_{mi^<} = (y^+ \neq x^- \vee x^+ - x^- \geqslant y^+ - y^-),$$
$$\Phi_{mi^>} = (y^+ \neq x^- \vee x^+ - x^- \leqslant y^+ - y^-),$$
$$\Phi_{s^<} = (x^- \neq y^- \vee x^+ - x^- \geqslant y^+ - y^-),$$
$$\Phi_{si^>} = (x^- \neq y^- \vee x^+ - x^- \leqslant y^+ - y^-),$$
$$\Phi_{f^<} = (x^+ \neq y^+ \vee x^+ - x^- \geqslant y^+ - y^-),$$
$$\Phi_{fi^>} = (x^+ \neq y^+ \vee x^+ - x^- \leqslant y^+ - y^-).$$

Next, consider the basic relations imposing equality of the durations. These atomic relations belong to the convex relation $s = \{eq^=, b^=, bi^=, o^=, oi^=, m^=, mi^=\}$. As a consequence, Φ_a will always contain $x^+ - x^- \neq y^+ - y^-$ (except for Φ_{eq}).

$$\Phi_{b^=} = (x^+ - x^- \neq y^+ - y^- \vee x^+ \geqslant y^-),$$
$$\Phi_{bi^=} = (x^+ - x^- \neq y^+ - y^- \vee y^+ \geqslant x^-),$$
$$\Phi_{m^=} = (x^+ - x^- \neq y^+ - y^- \vee x^+ \neq y^-),$$
$$\Phi_{mi^=} = (x^+ - x^- \neq y^+ - y^- \vee y^+ \neq x^-),$$
$$\Phi_{eq^=} = (x^- \neq y^- \vee x^+ \neq y^+).$$

The cases of the basic relations $o^=$ and $oi^=$ remain to be examined. Consider the case $a = o^=$ (the case $oi^=$ is similar). Suppose that $r \cap \{b^=, m^=\} \neq \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} \neq \emptyset$. Hence $a \in I(r \cap s)$. Moreover, we know that $a \notin r$. As a consequence $\dim(I(r \cap s) \setminus (r \cap s)) \geqslant 1$. Since $r \cap s \subseteq s$ and $\dim(s) = 1$, $\dim(I(r \cap s) \setminus (r \cap s)) \leqslant 1$. Hence, $\dim(r \cap s) \leqslant \dim(I(r \cap s) \setminus (r \cap s))$ and $r \cap s$ is not pre-convex. This is a contradiction (r is a strongly pre-convex relation). Hence, only three cases have to be considered:

- $r \cap s = \emptyset$. Then $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^-$,
- $r \cap \{b^=, m^=\} \neq \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} = \emptyset$. Then $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^- \vee x^+ \leqslant y^-$,
- $r \cap \{b^=, m^=\} = \emptyset$ and $r \cap \{eq^=, mi^=, oi^=, bi^=\} \neq \emptyset$. Then $\Phi_{o^=}$ is $x^+ - x^- \neq y^+ - y^- \vee x^+ \geqslant y^+$.

Using the clauses defined above, any $r \in \mathcal{F}$ can be represented by the conjunction of Horn clauses $\Phi_{I(r)} \wedge \bigwedge_{a \in (I(r) \setminus r)} \Phi_a$. □

As a consequence, we get:

Theorem 2. $\mathsf{Cons}(\mathcal{F})$ *is a polynomial problem.*

7. The tractable subclass $\mathcal{G}$

In this section we characterize a new subset of pre-convex relations for which the $\diamond$-closure method gives a decision method for the consistency problem (contrarily to what is the case for $\mathcal{F}$). We will denote this set by $\mathcal{G}$. The definition of $\mathcal{G}$ was guided by our desire to obtain pre-convex relations forming a subclass for which the convex closures are convex interval relations.

Definition 9. Let $r \in 2^{\mathsf{INDU}}$. Then r belongs to $\mathcal{G}$ iff for each convex relation $s \in \mathcal{C}_{IA}$ $r \cap s$ is a pre-convex relation and $I(r \cap s)$ is a convex relation which belongs to the set $\mathcal{C}_{IA}$.

The set $\mathcal{G}$ forms a subclass containing 11854 relations.

Proposition 10. *The set $\mathcal{G}$ is closed for the operations $^{-1}$, $\cap$ and $\diamond$.*

Since the universal relation INDU (that is, the set $INDU$) belongs to $\mathcal{C}_{IA}$, each relation of $\mathcal{G}$ is a pre-convex relation. Moreover, we notice that some relations of $\mathcal{G}$ are not strongly pre-convex. For example, the relation $\{eq^=, d^<, di^>, o^<, o^>, oi^<, oi^>, m^<, m^>, m^=, mi^<, mi^>, mi^=\}$ belongs to $\mathcal{G}$ but is not strongly pre-convex: indeed its intersection with the convex relation $\{eq^=, o^=, oi^=, m^=, mi^=\}$ is not a pre-convex relation.

7.1. Tractability of $\mathcal{G}$

We are now in a position to prove the tractability of the consistency problem for the set $\mathcal{G}$, using the notion of maximal solution introduced by Ligozat in [9]. Given a solution m of a network $\mathcal{N} = (V, C)$, m will be said maximal if $\dim(m_{ij}) = \dim(C_{ij})$ for all $i, j \in 1, \ldots, |V|$. Intuitively, a maximal solution is a solution which involves basic relations imposing as few equalities—between endpoints and difference of endpoints—as possible. For example, given the constraint $x\ \{m^=, m^>, o^=, o^<, b^>\}\ y$, a maximal solution will satisfy $o^<$ or $b^>$ between x and y. Firstly, we prove the following result:

Proposition 11. *Let $\mathcal{N} = (V, C)$ a $\mathcal{INDU}$ network whose constraints belong to $\mathcal{C}_{IA}$ (with the exclusion of the empty relation). If $\mathcal{N}$ is $\diamond$-closed then $\mathcal{N}$ admits a maximal solution.*

Proof. $\mathcal{C}_{IA}$ corresponds to the convex relations of $\mathcal{IA}$. Let $\mathcal{N}' = (V, C')$ be the convex interval network equivalent to $\mathcal{N}$. From [9] we know that $\mathcal{N}'$ admits a solution $m_1, \ldots, m_n$ (with $n = |V|$) such that $a = b$, with a and b two endpoints of m_i and m_j, iff all basic relations belonging to C'_{ij} impose this equality (m is a maximal solution for $\mathcal{IA}$). For example, we have $m_i^- = m_j^-$ iff all basic relations of C'_{ij} impose this equality. C'_{ij} could be the relation $\{s, eq, si\}$ but could not be the relation $\{s, eq, f, d\}$ since d does not impose the equality $m_i^- = m_j^-$. We can modify m to obtain a solution s having the additional property: $s_i^+ - s_i^- = s_j^+ - s_j^-$ iff $C_{ij} = \{eq^=\}$. Consider the lower endpoint m_i^-, let l be the number of endpoints located before m_i^-. We assign to s_i^- the value $l/(1+l)$. We treat in a similar way the upper endpoints. s satisfies the properties fixed previously. Hence, we obtain a maximal solution s of $\mathcal{N} = (V, C)$. □

Proposition 12. *Let $r, s \in$ INDU such that $I(r \diamond s)$, $I(r)$ and $I(s) \in \mathcal{C}_{IA}$. We have $I(r \diamond s) \subseteq I(r) \diamond I(s)$.*

Proof. $r \subseteq I(r)$ and $s \subseteq I(s)$. Hence $r \diamond s \subseteq I(r) \diamond I(s)$. Consequently, $I(r \diamond s) \subseteq I(I(r) \diamond I(s))$. As $\mathcal{C}_{IA}$ is closed for the operation $\diamond$, $I(r) \diamond I(s)$ is a convex relation. Hence, $I(I(r) \diamond I(s)) = I(r) \diamond I(s)$. Hence, $I(r \diamond s) \subseteq I(r) \diamond I(s)$. □

Proposition 13. *Let $\mathcal{N} = (V, C)$ be a network whose constraints belong to $\mathcal{G}$. Let $\mathcal{N}^I = (V, C^I)$ be defined by $C^I_{ij} = I(C_{ij})$ for all $i, j \in \{1, \ldots, n\}$, with $n = |V|$. If $\mathcal{N}$ is $\diamond$-closed then $\mathcal{N}^I$ is $\diamond$-closed.*

Proof. Let $V_i, V_j, V_k \in V$. $C_{ij} \subseteq C_{ik} \diamond C_{kj}$, consequently, $I(C_{ij}) \subseteq I(C_{ik} \diamond C_{kj})$. We know that $\mathcal{G}$ is closed for the operation $\diamond$. Hence, $I(C_{ik} \diamond C_{kj}) \in \mathcal{C}_{IA}$. Moreover, by definition of $\mathcal{G}$, $I(C_{ik})$ and $I(C_{kj}) \in \mathcal{C}_{IA}$. From Proposition 12, we get that $I(C_{ik} \diamond C_{kj}) \subseteq I(C_{ik}) \diamond I(C_{kj})$. Using this result, we deduce that $I(C_{ij}) \subseteq I(C_{ik}) \diamond I(C_{kj})$. □

Now, we can establish the main result concerning the set $\mathcal{G}$.

Theorem 3. $\mathsf{Cons}(\mathcal{G})$ *can be decided by means of the $\diamond$-closure method.*

Proof. Let $\mathcal{N} = (V, C)$ be a network containing constraints belonging to $\mathcal{G}$. By using the $\diamond$-closure method on $\mathcal{N}$ we obtain an equivalent subnetwork $\mathcal{N}' = (V, C')$. The constraints of $\mathcal{N}'$ belong to $\mathcal{G}$ since $\mathcal{G}$ is closed for the three operations $^{-1}$, $\cap$ and $\diamond$. If $\mathcal{N}'$ contains the empty constraint, then $\mathcal{N}$ is not consistent. In the opposite case, let us show that $\mathcal{N}'$ (and consequently $\mathcal{N}$) is consistent. Let $\mathcal{N}'' = (V, C'')$ be defined by $C''_{ij} = I(C'_{ij})$. $\mathcal{N}''$ is $\diamond$-closed (Proposition 13). It admits a maximal solution m (Proposition 11). This solution m is also a maximal solution of $\mathcal{N}'$. This is due to the fact that $\dim(I(C'_{ij}) \setminus C'_{ij}) < \dim(C'_{ij})$ (see definition of $\mathcal{G}$), for each pair of variables V_i and V_j. □

Hence, we have characterized a set for which the $\diamond$-closure method is complete.

8. The atomic relations of $\mathsf{INDU}^=$

In the previous section we showed that the $\diamond$-closure solves the consistency problem $\mathsf{Cons}(\mathcal{G})$.[4] On the other hand, we know that for the general case, the $\diamond$-closure method is not complete for the consistency problem of the atomic networks of $\mathcal{INDU}$. In this section, we show that this method is complete for the atomic relations which imply the equality of the durations of the intervals. In the sequel, we will denote by $\mathsf{INDU}^=$ the subset of the basic relations of $\mathcal{INDU}$ implying the equality between the durations of two intervals, that is $\mathsf{INDU}^= = \{eq^=, b^=, bi^=, m^=, mi^=, o^=, oi^=\}$. Notice that the atomic relations defined from $\mathsf{INDU}^=$, excepted $\{eq^=\}$, are convex relations of $\mathcal{INDU}$ which do not belong to the set $\mathcal{C}_{AI}$.

Given an $\mathcal{INDU}$ constraint network $\mathcal{N} = (V, C)$, we will denote by $\mathcal{N}^{\mathsf{IA}}$ the interval constraint network (V, C^{IA}) defined as follows: for each $i, j \in 1, \ldots, |V|$, $C^{\mathsf{IA}}_{ij} = \{i\colon i^p \in$

[4] Note that the only basic $\mathcal{INDU}$ relations belonging to $\mathcal{G}$ are $\{eq^=\}$, $\{d^<\}$, $\{di^>\}$, $\{f^<\}$, $\{fi^>\}$, $\{s^<\}$, $\{si^>\}$.

$C_{ij}\}$. Similarly, we will denote by $\mathcal{N}^{\mathsf{PA}}$ the point constraint network (V, C^{PA}) defined as follows: for each $i, j \in 1, \ldots, |V|$, $C_{ij}^{\mathsf{PA}} = \{p\colon i^p \in C_{ij}\}$. The constraint networks $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are, respectively, the projection of $\mathcal{N}$ onto the Interval Algebra and the projection of $\mathcal{N}$ onto the Point Algebra. In the general case, it is clear that the consistency of $\mathcal{N}^{\mathsf{IA}}$ and the one of $\mathcal{N}^{\mathsf{PA}}$ do not imply the consistency of $\mathcal{N}$.

The projection operation retains the property of $\diamond$-closure:

Proposition 14. *Let $\mathcal{N} = (V, C)$ an $\mathcal{INDU}$ constraint network. If $\mathcal{N}$ is a $\diamond$-closed network then $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are $\diamond$-closed networks (and also $\circ$-closed networks).*

Proof. Denote by n the number of elements of the set V. Let $i, j, k \in 1, \ldots, n$. Let $a \in C_{ij}^{\mathsf{IA}}$ (resp. $b \in C_{ij}^{\mathsf{PA}}$), there exists $b \in \mathsf{PA}$ (resp. $a \in \mathsf{IA}$) such that $a^b \in C_{ij}$. By definition of the operation $\diamond$ and by $\diamond$-closure of $\mathcal{N}$, for each basic relation $a^b \in C_{ij}$, there exist $c^d \in C_{ik}$ and $e^f \in C_{kj}$ such that $a^b \in (c^d \diamond e^f)$. Moreover, $c^d \diamond e^f = ((c \circ e) \times (d \circ f)) \cap \mathsf{INDU}$. Hence $a \in (c \circ e)$ and $b \in (d \circ f)$. By projection, $c \in C_{ik}^{\mathsf{IA}}$, $e \in C_{kj}^{\mathsf{IA}}$, $d \in C_{ik}^{\mathsf{PA}}$ and $f \in C_{kj}^{\mathsf{PA}}$. Consequently, $a \in (C_{ik}^{\mathsf{IA}} \circ C_{kj}^{\mathsf{IA}})$ and $b \in (C_{ik}^{\mathsf{PA}} \circ C_{kj}^{\mathsf{PA}})$. Hence $C_{ij}^{\mathsf{IA}} \subseteq (C_{ik}^{\mathsf{IA}} \circ C_{kj}^{\mathsf{IA}})$ and $C_{ij}^{\mathsf{PA}} \subseteq (C_{ik}^{\mathsf{PA}} \circ C_{kj}^{\mathsf{PA}})$. Hence, $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are $\circ$-closed networks. For the Interval Algebra and the Point Algebra, the operations $\circ$ and $\diamond$ are the same operations, consequently, $\mathcal{N}^{\mathsf{IA}}$ and $\mathcal{N}^{\mathsf{PA}}$ are $\diamond$-closed networks too. □

Now, we are going to prove that some particular interval constraint networks admit solutions enforcing a common duration for all intervals.

Proposition 15. *Let $\mathcal{N} = (V, C)$ an atomic network of the Interval Algebra such that each constraint is formed by one basic relation belonging to the set $S = \{eq, m, mi, b, bi, o, oi\}$. If $\mathcal{N}$ is a $\circ$-closed network (and does not have the empty relation as a constraint) then $\mathcal{N}$ admits a consistent instantiation σ such that $\sigma(V_i)^+ - \sigma(V_i)^- = \sigma(V_j)^+ - \sigma(V_j)^-$ for all $V_i, V_j \in V$.*

Proof. Let $\mathcal{N} = (V, C)$ be an atomic interval constraint network such that $C_{ij} = \{A\}$ with $A \in \{eq, m, mi, b, bi, o, oi\}$. Suppose that $\mathcal{N}$ is a $\circ$-closed network. As $\mathcal{N}$ is $\circ$-closed and atomic we know that there exists a consistent instantiation σ' of $\mathcal{N}$ [11]. From this instantiation, we build a new consistent instantiation, denoted by σ, using uniquely intervals with same duration, as follows: Without loss of generality, we suppose that the variables $V = V_1, \ldots, V_n$ are such that if $\sigma_i'^+ < \sigma_j'^+$ then $i < j$, for all $i, j \in 1, \ldots, n$. Thus, if $i < j$, then, $C_{ij} \subseteq \{eq, m, b, o\}$. In the sequel of this proof we will denote by T the interval relation $\{eq, m, b, o\}$ and d will denote the difference $\sigma_1'^+ - \sigma_1'^-$. Moreover, we define the sets E_l^A, with $A \in \{eq, b, m, o, oi, mi, bi\}$ and $l \in 1, \ldots, n$, by $E_l^A = \{i \in 1, \ldots, l-1\colon C_{il} = \{A\}\}$. Let the instantiation σ defined in the following way:

- $\sigma_1 = \sigma_1'$,
- for each $k \in 2, \ldots, n$,
  - if $C_{k-1k} = \{eq\}$ then $\sigma_k = \sigma_{k-1}$,
  - if $C_{k-1k} = \{m\}$ then $\sigma_k = (\sigma_{k-1}^+, \sigma_{k-1}^+ + d)$,

– if $C_{k-1k} = \{b\}$ then $\sigma_k = (\sigma^+_{k-1} + d, \sigma^+_{k-1} + 2d)$,
– if $C_{k-1k} = \{o\}$ then
  if $E^m_k \neq \emptyset$ then $\sigma_k = (\sigma^+_i, \sigma^+_i + d)$, with $i = \min\ E^m_k$,
  else $\sigma_k = ((u + v)/2, (u + v)/2 + d)$, with $v = \sigma^+_{\min\ E^o_k}$ and $u = \sigma^+_{\max(E^b_k \cap E^o_{k-1})}$ in the case where $(E^b_k \cap E^o_{k-1}) \neq \emptyset$ else $u = \sigma^-_{k-1}$.

Let us prove that the instantiation σ satisfies the properties required. Firstly, we can remark that for all $k \in 1, \dots, n$, $\sigma^+_k - \sigma^-_k = d$. Hence, all the intervals used in the instantiation σ have the same duration. Now, we show that σ is a consistent instantiation of $\mathcal{N} = (V, C)$. Let P be the property defined by $P(k)$ (with $k \in 1, \dots, n$) be satisfied if, and only if, the partial instantiation $\sigma_1, \dots, \sigma_k$ is a consistent instantiation of the network $\mathcal{N}$. $P(1)$ is trivially true. Let $k \in 2, \dots, n$, suppose that $P(k-1)$ is true, we show that the property P is satisfied for k. We know that the constraint C_{k-1k} is a atomic relation such that $C_{k-1k} \subseteq T$. Examine all possible cases concerning the constraint C_{k-1k}.

- $C_{k-1k} = eq$. We have $\sigma_k = \sigma_{k-1}$. Hence, $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. Moreover, for each $l \in 1, \dots, k-2$, from the $\circ$-closure of $\mathcal{N}$, we have $C_{lk} \subseteq C_{lk-1} \circ C_{k-1k}$. Moreover, $C_{lk-1} \circ C_{k-1k} = C_{lk-1} \circ \{eq\} = C_{lk-1}$. Since the constraints of $\mathcal{N}$ are atomic relations, $C_{lk} = C_{lk-1}$. As $P(k-1)$ is true, $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. From the fact that $\sigma_{k-1} = \sigma_k$ and $C_{lk-1} = C_{lk}$, we deduce that $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{k-1k} = m$. Hence, $\sigma_k = (\sigma^+_{k-1}, \sigma^+_{k-1} + d)$. Consequently, we have $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. From the $\circ$-closure of $\mathcal{N}$, we have $C_{lk} \subseteq C_{lk-1} \circ C_{k-1k}$ for each $l \in 1, \dots, k-2$. Moreover, $C_{lk-1} \subseteq \{eq, m, b, o\}$ and $C_{k-1k} = \{m\}$. Since $\{eq, m, b, o\} \circ \{m\} = \{m, b\}$, we obtain the inclusion $C_{lk} \subseteq \{m, b\}$. Consider the two possible cases for C_{lk}:
  - $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, we have $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{b\} \circ \{mi\}$. Since $\{b\} \circ \{mi\} = \{b, m, o, d, s\}$ and $C_{lk-1} \subseteq T$ we can assert that $C_{lk-1} \subseteq \{b, m, o\}$. As $P(k-1)$ is satisfied, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Consequently, $\sigma^+_l < \sigma^+_{k-1}$. Moreover, remember that $\sigma^+_{k-1} = \sigma^-_k$. We deduce that $\sigma^+_l < \sigma^-_k$. We can conclude that $\sigma_l\ \{b\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
  - $C_{lk} = \{m\}$. From the $\circ$-closure of $\mathcal{N}$, we have $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{m\} \circ \{mi\}$. From the fact that $\{m\} \circ \{mi\} = \{f, fi, eq\}$ and $C_{lk-1} \subseteq T$ we get that $C_{lk-1} = \{eq\}$. Since $P(k-1)$ is satisfied, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Consequently, $\sigma_l = \sigma_{k-1}$. Moreover, remember that $\sigma^+_{k-1} = \sigma^-_k$. We conclude that $\sigma_l\ \{m\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{k-1k} = b$. Hence $\sigma_k = (\sigma^+_{k-1} + d, \sigma^+_{k-1} + 2d)$. Consequently, $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. From the $\circ$-closure of $\mathcal{N}$, for each $l \in 1, \dots, k-2$, we get $C_{lk} \subseteq C_{lk-1} \circ C_{k-1k}$. Hence, $C_{lk} \subseteq T \circ \{b\}$. From the fact that $\{eq, m, b, o\} \circ \{b\} = \{b\}$ we can deduce that $C_{lk} = \{b\}$. As $P(k-1)$ is true, $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. From the fact that $C_{lk-1} \subseteq \{eq, m, b, o\}$, we get $\sigma^+_l \leqslant \sigma^+_{k-1}$. Moreover, notice that $\sigma^+_{k-1} < \sigma^-_k$. Hence, we can assert that $\sigma^+_l < \sigma^-_k$. Consequently, $\sigma_l\ \{b\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{k-1k} = o$. We must take into account two possible cases: the case where $E^m_k \neq \emptyset$ and the case where $E^m_k = \emptyset$.

- $E_k^m \neq \emptyset$. Let $i = \min E_k^m$ (notice that $1 \leqslant i < k$ and $C_{ik} = \{m\}$). We have $\sigma_k = (\sigma_i^+, \sigma_i^+ + d)$. Let $l \in 1, \ldots, k-1$. We know that $C_{lk} \in T$. Examine all possible cases concerning C_{lk}.
  - * $C_{lk} = \{eq\}$. From the $\circ$-closure of $\mathcal{N}$, we deduce that $C_{il} \subseteq C_{ik} \circ C_{kl}$. Hence, $C_{il} \subseteq \{m\} \circ \{eq\}$. Moreover, $\{m\} \circ \{eq\} = \{m\}$. Consequently, $C_{il} = \{m\}$. Since $P(k-1)$ is satisfied, $\sigma_i m \sigma_l$. Hence, $\sigma_l^- = \sigma_i^+$. As the duration of σ_l is d, $\sigma_l = (\sigma_i^+, \sigma_i^+ + d)$. Consequently, $\sigma_l = \sigma_k$ and hence, σ_l $\{eq\}$ σ_k. We deduce that $\sigma_l\ C_{lk}\ \sigma_k$.
  - * $C_{lk} = \{m\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{il} \subseteq C_{ik} \circ C_{kl}$. Consequently, $C_{il} \subseteq \{m\} \circ \{mi\}$. From the fact $\{m\} \circ \{mi\} = \{f, fi, eq\}$ and $C_{il} \subseteq S$, we can assert that $C_{il} = \{eq\}$. Since $P(k-1)$, σ_i $\{eq\}$ σ_l. Hence, $\sigma_l = \sigma_i$. As $\sigma_k^- = \sigma_i^+$, we have $\sigma_k^- = \sigma_l^+$. Consequently, $\sigma_l\ \{m\}\ \sigma_k$ and hence, $\sigma_l\ C_{lk}\ \sigma_k$.
  - * $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, we can deduce that $C_{il} \subseteq C_{ik} \circ C_{kl}$. Consequently, $C_{il} \subseteq \{m\} \circ \{bi\}$. Moreover, $\{m\} \circ \{bi\} = \{bi, oi, mi, di, si\}$ and, we know that $C_{il} \subseteq S$. Hence, $C_{il} \subseteq \{bi, oi, mi\}$ and $C_{li} \subseteq \{b, o, m\}$. Since the property $P(k-1)$ is true, we have $\sigma_l\ C_{li}\ \sigma_i$. Consequently, we can assert that $\sigma_l^+ < \sigma_i^+$. As $\sigma_k^- = \sigma_i^+$, we get that $\sigma_k^- > \sigma_l^+$. Consequently, $\sigma_l\, b\, \sigma_k$.
  - * $C_{lk} = \{o\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{il} \subseteq C_{ik} \circ C_{kl}$. Consequently, $C_{il} \subseteq \{m\} \circ \{oi\}$. $\{m\} \circ \{oi\} = \{o, d, s\}$ and moreover, we know that $C_{il} \subseteq S$. Hence, $C_{il} = \{o\}$. As the property $P(k-1)$ is true, we have $\sigma_i\ C_{il}\ \sigma_l$. Consequently, we can assert that $\sigma_i^- < \sigma_l^- < \sigma_i^+ < \sigma_l^+$, with $\sigma_l^+ - \sigma_i^+ < d$. Moreover, we know that $\sigma_k = (\sigma_i^+, \sigma_i^+ + d)$. Hence, $\sigma_l^- < \sigma_k^- < \sigma_l^+ < \sigma_k^+$. Consequently, σ_l $\{o\}$ σ_k. Hence, $\sigma_l\ C_{lk}\ \sigma_k$.
- $E_k^m = \emptyset$. Denote by i the element corresponding to $\min E_k^o$. Notice that i exists since $C_{k-1k} = \{o\}$. Two cases must be considered: the case where $E_k^b \cap E_{k-1}^o = \emptyset$ and the case where $E_k^b \cap E_{k-1}^o \neq \emptyset$.
  - * $E_k^b \cap E_{k-1}^o = \emptyset$. Hence, $\sigma_k = ((\sigma_i^+ + \sigma_{k-1}^-)/2, (\sigma_i^+ + \sigma_{k-1}^-)/2 + d)$. From the $\circ$-closure of $\mathcal{N}$, we can deduce that $C_{ik-1} \subseteq C_{ik} \circ C_{kk-1}$. Consequently, $C_{ik-1} \subseteq \{o\} \circ \{oi\}$. From the fact that $\{o\} \circ \{oi\} = \{o, oi, d, s, f, di, si, fi, eq\}$ and $C_{ik-1} \subseteq T$ (since $i \leqslant k-1$). We deduce that $C_{ik-1} \subseteq \{o, eq\}$. Consequently,we have $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_i^+ \leqslant \sigma_{k-1}^+$, with $\sigma_{k-1}^+ - \sigma_{k-1}^- = \sigma_i^+ - \sigma_i^- = d$. As $\sigma_k = ((\sigma_i^+ + \sigma_{k-1}^-)/2, (\sigma_i^+ + \sigma_{k-1}^-)/2 + d)$ we can assert that $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_k^- < \sigma_i^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. Let $l \in 1, \ldots, k-1$. Consider all possible cases concerning the constraint C_{lk}. We know that $C_{lk} \subseteq T$, hence, we must consider four cases.
    - – $C_{lk} = \{eq\}$. From the $\circ$-closure of $\mathcal{N}$, we can deduce that $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Consequently, $C_{lk-1} \subseteq \{eq\} \circ \{oi\}$, hence, $C_{lk-1} = \{oi\}$. Consequently, $\sigma_l'^+ > {\sigma_{k-1}'}^+$. This implies that $l > k-1$. This is a contradiction.
    - – $C_{lk} = \{m\}$. By hypothesis, $E_k^m = \emptyset$. This is a contradiction.
    - – $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. We deduce that $C_{lk-1} \subseteq \{b\} \circ \{oi\}$. Moreover, $\{b\} \circ \{oi\} = \{b, o, m, d, s\}$ and we know that $C_{lk} \subseteq T$. Consequently, $C_{lk-1} \subseteq \{b, o, m\}$. In the case where $C_{lk-1} \subseteq \{b, m\}$, from the fact that $P(k-1)$ is true, we have σ_l $\{b, m\}$ σ_{k-1}. Hence, $\sigma_l^+ \leqslant \sigma_{k-1}^-$. As $\sigma_k^- > \sigma_{k-1}^-$, we have $\sigma_l^+ < \sigma_k^-$. Hence, we can assert that σ_l $\{b\}$ σ_k and

hence, $\sigma_l\ C_{lk}\ \sigma_k$. Now, suppose that $C_{lk-1} = \{o\}$ (notice that in this case $l < k-1$). Hence, $E_k^b \cap E_{k-1}^o \neq \emptyset$. This is a contradiction.

– $C_{lk} = \{o\}$. By definition of i we have $i \leqslant l$. Hence, $i \leqslant l \leqslant k-1$. Consequently, $C_{il} \subseteq T$ and $C_{lk-1} \subseteq T$. As the property $P(k-1)$ is satisfied, we have $\sigma_i\ C_{il}\ \sigma_l$ and $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_i^- \leqslant \sigma_l^- \leqslant \sigma_{k-1}^-$ and $\sigma_i^+ \leqslant \sigma_l^+ \leqslant \sigma_{k-1}^+$. Moreover, remember us that $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_k^- < \sigma_i^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. Consequently, we have $\sigma_i^- \leqslant \sigma_l^- \leqslant \sigma_{k-1}^- < \sigma_k^- < \sigma_i^+ \leqslant \sigma_l^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. We deduce that $\sigma_l\ \{o\}\ \sigma_k$. Hence, $\sigma_l\ C_{lk}\ \sigma_k$.

* $E_k^b \cap E_{k-1}^o \neq \emptyset$. We deduce that $\sigma_k = ((\sigma_i^+ + \sigma_j^+)/2, (\sigma_i^+ + \sigma_j^+)/2 + d)$, with $j = \max(E_k^b \cap E_{k-1}^o)$. As in the previous case we have $C_{ik-1} \subseteq \{o, eq\}$ and hence $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_i^+ \leqslant \sigma_{k-1}^+$ with $\sigma_{k-1}^+ - \sigma_{k-1}^- = \sigma_i^+ - \sigma_i^- = d$. Moreover, from the $\circ$-closure of $\mathcal{N}$, we deduce that $C_{ij} \subseteq C_{ik-1} \circ C_{k-1j}$. We have $C_{k-1j} = \{oi\}$ since $j \in E_{k-1}^o$. Hence, $C_{ij} \subseteq \{o, eq\} \circ \{oi\}$. Moreover, $\{o, eq\} \circ \{oi\} = \{eq, o, oi, s, si, f, fi, d, di\}$. We know that $C_{ij} \subseteq S$. Consequently, we can assert that $C_{ij} \subseteq \{eq, o, oi\}$. Similarly, from the $\circ$-closure of $\mathcal{N}$, we have $C_{ij} \subseteq C_{ik} \circ C_{kj}$. $C_{kj} = \{bi\}$ since $j \in E_k^b$ and $C_{ik} = \{o\}$ since $i \in E_k^o$. Consequently, $C_{ij} \subseteq \{o\} \circ \{bi\}$. Moreover, $\{o\} \circ \{bi\} = \{bi, oi, di, mi, si\}$ and $C_{ij} \subseteq S$. Hence, $C_{ij} \subseteq \{bi, oi, mi\}$. It results that $C_{ij} \subseteq \{eq, o, oi\} \cap \{bi, oi, mi\}$. Hence, we have $C_{ij} = \{oi\}$. As $P(k-1)$ is true, we have $\sigma_i\ C_{ij}\ \sigma_j$. It follows that $\sigma_j^- < \sigma_i^- < \sigma_j^+ < \sigma_i^+$. Moreover, $\sigma_j^- < \sigma_{k-1}^- < \sigma_j^+ < \sigma_{k-1}^+$ since $C_{jk-1} = \{o\}$. Recall that $\sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_i^+ \leqslant \sigma_{k-1}^+$. Putting these facts together, we can deduce that $\sigma_j^- < \sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_j^+ < \sigma_i^+ \leqslant \sigma_{k-1}^+$. Moreover, we know that $\sigma_j = \sigma_i = \sigma_{k-1} = d$ and $\sigma_k = ((\sigma_i^+ + \sigma_j^+)/2, (\sigma_i^+ + \sigma_j^+)/2 + d)$. Hence, $\sigma_j^- < \sigma_i^- \leqslant \sigma_{k-1}^- < \sigma_j^+ < \sigma_k^- < \sigma_i^+ \leqslant \sigma_{k-1}^+ < \sigma_k^+$. We deduce that $\sigma_{k-1}\{o\}\sigma_k$ and hence, $\sigma_{k-1}\ C_{k-1k}\ \sigma_k$. Let $l \in 1, \ldots, k-2$. Examine all possible cases concerning the constraint C_{lk}. We know that $C_{lk} \subseteq T$, hence, we must consider four possible cases.

– $C_{lk} = \{eq\}$. From the $\circ$-closure of $\mathcal{N}$, we deduce that $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{eq\} \circ \{oi\}$. Since $\{eq\} \circ \{oi\} = \{oi\}$, we have $C_{lk-1} = \{oi\}$. Consequently, $\sigma_l'^+ > \sigma_{k-1}'^+$ and hence, $l > k-1$. This is a contradiction.

– $C_{lk} = \{m\}$. By hypothesis, $E_k^m = \emptyset$. This is a contradiction.

– $C_{lk} = \{b\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{b\} \circ \{oi\}$. Moreover, $\{b\} \circ \{oi\} = \{b, o, m, d, s\}$ and $C_{lk-1} \subseteq T$. Consequently, $C_{lk-1} \subseteq \{b, o, m\}$. Consider the two following cases:
  - $C_{lk-1} = \{o\}$. We deduce that $l \in (E_k^b \cap E_{k-1}^o)$. By definition of j, we have $l \leqslant j$. Hence, $C_{lj} \subseteq T$. Moreover, $\sigma_l\ C_{lj}\ \sigma_j$. Hence, $\sigma_l^- < \sigma_j^-$ and $\sigma_l^+ < \sigma_j^+$. Since $\sigma_j^+ < \sigma_k^-$, we can assert that $\sigma_l\ \{b\}\ \sigma_k$. Consequently, $\sigma_l\ C_{lk}\ \sigma_k$.
  - $C_{lk-1} = \{b, m\}$. As the property $P(k-1)$ is true, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_l^+ \leqslant \sigma_{k-1}^-$. Moreover, $\sigma_{k-1}^- < \sigma_k^-$. Hence, we can assert that $\sigma_l^+ < \sigma_k^-$. Hence, $\sigma_l\ \{b\}\ \sigma_k$. Consequently, $\sigma_l\ C_{lk}\ \sigma_k$.

– $C_{lk} = \{o\}$. From the $\circ$-closure of $\mathcal{N}$, $C_{lk-1} \subseteq C_{lk} \circ C_{kk-1}$. Hence, $C_{lk-1} \subseteq \{o\} \circ \{oi\}$. Moreover, $\{o\} \circ \{oi\} = \{o, oi, d, s, f, di, si, fi, eq\}$ and $C_{lk-1} \subseteq S$. We deduce that $C_{lk-1} \subseteq \{o, oi, eq\}$. The case $C_{lk-1} = \{oi\}$ is not possible since $l < k-1$. We must still examine two cases:

- $C_{lk-1} = \{eq\}$. As $P(k-1)$ is true, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_l = \sigma_{k-1}$. Moreover, we know that $\sigma_{k-1}\ \{o\}\ \sigma_k$. We deduce that $\sigma_l\ C_{lk}\ \sigma_k$.
- $C_{lk-1} = \{o\}$. As the property $P(k-1)$ is true, we have $\sigma_l\ C_{lk-1}\ \sigma_{k-1}$. Hence, $\sigma_l^- < \sigma_{k-1}^- < \sigma_l^+ < \sigma_{k-1}^+$. Moreover, we know that $\sigma_{k-1}^- < \sigma_k^- < \sigma_{k-1}^+ < \sigma_k^+$. Consequently, $\sigma_l^- < \sigma_k^-$ and $\sigma_l^+ < \sigma_k^+$. Moreover, recall that the integer i is defined by $i = \min E_o^k$ and $\sigma_k^- < \sigma_i^+$. From the fact that $C_{lk} = \{o\}$ we have $l \in E_o^k$. Hence, $i \leqslant l$. Consequently, $C_{il} \subseteq T$ and, as property $P(k-1)$ is true, we can assert that $\sigma_i^+ \leqslant \sigma_l^+$. Putting everything together, we get the fact that $\sigma_k^- < \sigma_l^+$ and hence, $\sigma_l^- < \sigma_k^- < \sigma_l^+ < \sigma_k^+$. We conclude that σ_l et σ_k satisfy the relation $\{o\}$. □

We can now state the main result of this section.

Theorem 4. *Let $\mathcal{N} = (V, C)$ be a network on $\mathcal{INDU}$ whose constraints are atomic relations defined from* INDU$^=$. *If $\mathcal{N}$ is closed by $\diamond$ and does not contain the empty relation as a constraint then $\mathcal{N}$ is consistent.*

Proof. Let $\mathcal{N} = (V, C)$ be an atomic constraint network of INDU$^=$. Let us suppose that $\mathcal{N}$ is $\diamond$-closed and does not contain the empty relation. $\mathcal{N}^{\mathsf{IA}}$ is a $\circ$-closed network (Proposition 14) and moreover, we cannot have the empty relation as constraint. We can notice that the basic relations of $\mathcal{N}^{\mathsf{IA}}$ belong to the relation $S = \{eq, m, mi, b, bi, o, oi\}$. Hence, $\mathcal{N}^{\mathsf{IA}}$ admits a consistent instantiation σ which assigns to the variables intervals with common duration. σ is a consistent instantiation of $\mathcal{N}$. □

Hence, the $\diamond$-closure method is a complete method for the atomic networks on INDU$^=$.

9. Conclusions

The $\mathcal{INDU}$ calculus lacks many of the nice properties of Allen's calculus: its (weak) composition table does not define a relation algebra. Consistency does not imply 3-consistency, and neither does 3-consistency imply consistency, even for atomic networks: some four node networks are 3-consistent but not consistent. In spite of these negative results, we are able to characterize interesting tractable subsets of relations. To this end, we use both the syntactic approach (Horn classes) and the geometrical approach (convexity and pre-convexity). While the two methods yield the same class in Allen's case, they provide us with two separate tractable subsets in the case of $\mathcal{INDU}$. Following the geometrical approach, we define the set of pre-convex relations and prove that its consistency problem is NP-complete (see also Fig. 5). We then characterize two subsets of pre-convex relations: one is the subset of strongly pre-convex relations, which is tractable (for reasons pertaining to the syntactic properties of its relations), but for which consistency cannot be decided by the $\diamond$-closure method (the usual path-consistency method which uses the

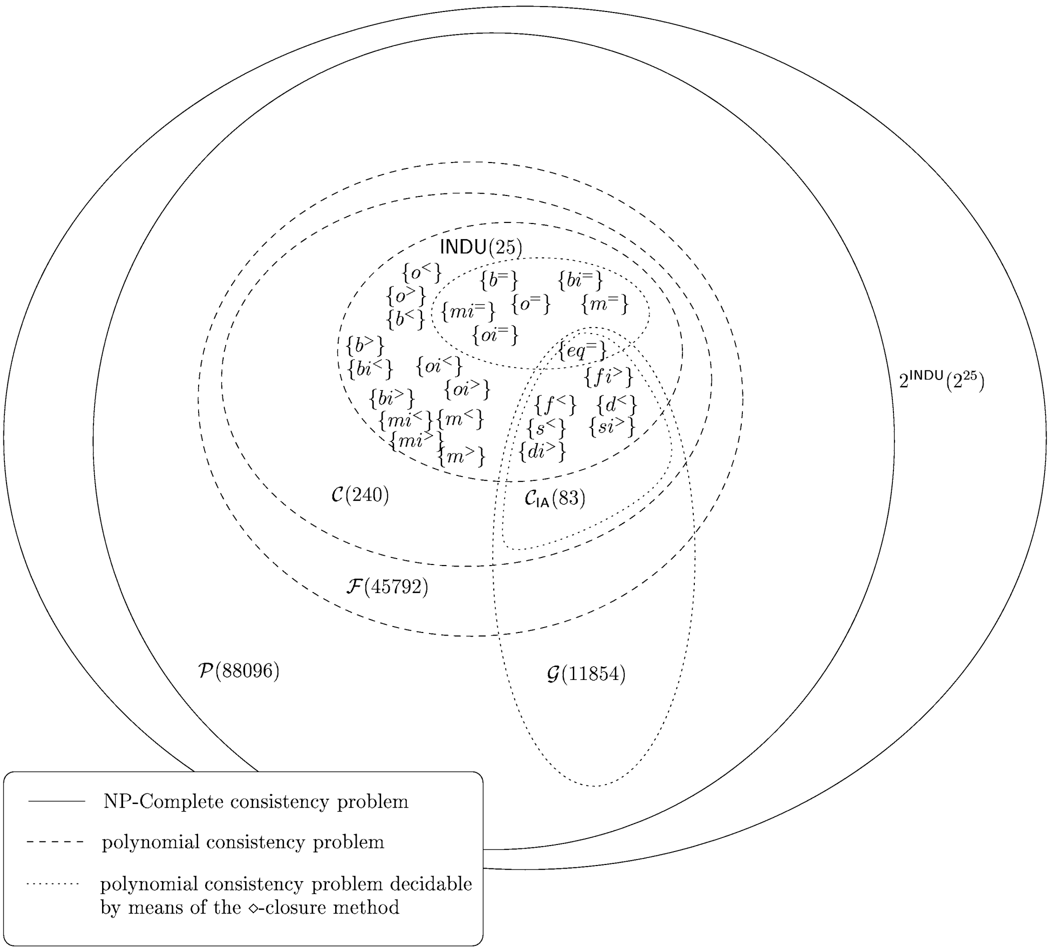

Fig. 5. Recapitulatory of the complexity results.

weak composition operation). The other (incomparable) subclass is tractable and its consistency problem can be solved by the ◇-closure method. In the general case, this method is not complete for the $\mathcal{INDU}$ atomic networks. Despite this, we prove that the ◇-closure method is also complete for the set of atomic relations of $\mathcal{INDU}$ implying that the intervals have the same duration. This paper constitutes a first fully successful exploration of the complexity properties of the $\mathcal{INDU}$ calculus.

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Merging Qualitative Constraint Networks Defined on Different Qualitative Formalisms

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Abstract. This paper addresses the problem of merging qualitative constraint networks (QCNs) defined on different qualitative formalisms. Our model is restricted to formalisms where the entities and the relationships between these entities are defined on the same domain. The method is an upstream step to a previous framework dealing with a set of QCNs defined on the same formalism. It consists of translating the input QCNs into a well-chosen common formalism. Two approaches are investigated: in the first one, each input QCN is translated to an equivalent QCN; in the second one, the QCNs are translated to approximations. These approaches take advantage of two dual notions that we introduce, the ones of *refinement* and *abstraction* between qualitative formalisms.

1 Introduction

Using a qualitative representation of information in spatial applications is needed when information is incomplete or comes from the natural language, or when quantitative information is unavailable or useless. In many spatial applications such as Geographic Information Systems (GIS), information often comes from sentences like "parcels A and B are connected" or "Paris is at north or at north-east of Toulouse", and one has to deal with such qualitative descriptions. Starting from Allen's work in the particular field of temporal reasoning [1], many approaches to deal with qualitative representation and reasoning have been proposed in the last two decades [6,18,20]. Different aspects of representation are dealt with topological relations [17] or precedence relations when orientation between entities is required [12]. These qualitative formalisms allow us to represent a set of spatial entities and their relative positions using qualitative constraint networks (QCNs).

In some applications (e.g. distributed knowledge systems) information can be gathered from several sources, and the multiplicity of sources means that combining information often leads to conflicts. Merging information has attracted much attention in the literature these past ten years. When information provided by sources is expressed by means of multi-sets of propositional formulae [14,21,10], dealing with inconsistency consists in computing a consistent propositional formula representing a global view of all input formulae.

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In Spatial Databases, knowledge about a set of spatial entities is represented by QCNs, which are provided by different sources, making them often conflicting. Consider for instance the following example: given three parcels A, B and C, one of the sources states that Parcel A is included in Parcel B, and Parcel B is disconnected from Parcel C, while another source declares that Parcels A and C are at the same location. Clearly there is a conflict and this calls for merging.

A first framework for merging QCNs has been proposed in [4], when the input QCNs are defined on the same qualitative formalism. The method computes a non-empty set of consistent scenarios which are the closest ones to all QCNs. The present work generalizes this method with QCNs defined on different qualitative formalisms. In general the sources provide QCNs based on different formalisms, although they are based on the same domain and consider similar relationships between entities. For instance, even if both RCC-5 [9] and RCC-8 [17] theories consider topological relationships between regions, the existing approach cannot deal with two QCNs defined respectively on each one of them.

In the particular context of qualitative spatial reasoning, an ultimate goal for the merging problem would be to deal with QCNs defined on heterogeneous formalisms without any restriction. Specific studies of calculi combinations have been investigated in the literature [8,11,16].

In this paper, we present an extension of the framework proposed in [4] by taking into account QCNs defined on different formalisms. We restrict our model to qualitative formalisms where the entities and the relationships between these entities are defined on the same domain. For instance, Allen's interval algebra [1] and $\mathcal{INDU}$ calculus [15] fulfill this requirement, since they both consider qualitative relationships between temporal intervals of the rational line. Our method consists of adding an upstream step to the method described in [4] by translating the input QCNs into a common formalism.

The rest of this paper is organised as follows. In Section 2 we recall some necessary background about qualitative algebras and QCNs. In Section 3 we describe the problem and the merging procedure given in [4]. Then we present in Section 4 how we can translate the input QCNs into a common *refinement* of the related qualitative algebras, using *bridges* between them. In Section 5 we give an alternative of the method proposed in the previous section by approximating the QCNs into a common *abstraction* of the qualitative algebras. Lastly we conclude and present some perspectives for further work.

2 Background

This section introduces necessary notions of qualitative algebras and definitions around qualitative constraint networks. A qualitative calculus (or qualitative algebra) considers $\mathcal{B}$, a finite set of binary relations over a domain $\mathcal{D}$, the universe of all considered entities. Each relation of $\mathcal{B}$ (called a basic relation) represents a particular qualitative position between two elements of $\mathcal{D}$. We make some initial assumptions on such a set $\mathcal{B}$. Let us first introduce the notion of *partition scheme* [13].

Definition 1 (Partition scheme). *Let $\mathcal{D}$ be a non-empty set and $\mathcal{B}$ be a set of binary relations on $\mathcal{D}$. $\mathcal{B}$ is called a* partition scheme on $\mathcal{D}$ *iff the following conditions are satisfied:*

- *The basic relations of $\mathcal{B}$ are* jointly exhaustive and pairwise disjoint*, namely any couple of $\mathcal{D}$ satisfies one and only one basic relation of $\mathcal{B}$.*
- *The* identity relation $eq = \{(a, b) \in \mathcal{D} \times \mathcal{D} \mid a = b\}$ *is one of the basic relations of $\mathcal{B}$.*
- *$\mathcal{B}$ is closed under converse, namely if r is a basic relation of $\mathcal{B}$, then so is its converse $r^{\smile} = \{(a, b) \mid (b, a) \in r\}$.*

In the rest of this paper, we will require that any considered set $\mathcal{B}$ of binary relations on $\mathcal{D}$ is a partition scheme on $\mathcal{D}$.

The set $2^{\mathcal{B}}$, the set of all subsets of $\mathcal{B}$, with the usual set-theoretic operators union ($\cup$), intersection ($\cap$), complementation ($\sim$), and weak composition ($\diamond$) [19] is called a *qualitative algebra*. Any element R of $2^{\mathcal{B}}$ is called a *relation* and represents a relation $rel(R)$ defined as $rel(R) = \bigcup\{r \mid r \in R\}$. This means that a pair of elements $(X, Y) \in \mathcal{D} \times \mathcal{D}$ satisfies a relation $R \in 2^{\mathcal{B}}$ if and only if $(X, Y) \in rel(R)$. The converse $R^{\smile}$ of a relation $R \in 2^{\mathcal{B}}$ is defined as $R^{\smile} = \{r \in \mathcal{B} \mid r^{\smile} \in R\}$.

For illustration, we consider the Cardinal Directions Algebra $2^{\mathcal{B}_{card}}$ [12], generated by the partition scheme $\mathcal{B}_{card}$ on $\mathbb{R}^2$ illustrated on Figure 1.a. The Cardinal Directions Algebra allows us to represent relative positions of points of the Cartesian plane, provided a global reference direction defined by two orthogonal lines. For each point $B = (b_1, b_2)$, the plane is divided in nine disjoint zones forming the set of basic relations $\mathcal{B}_{card}$. For example, the basic relation n corresponds to the area of points $A = (a_1, a_2)$, with $a_1 = b_1$ and $a_2 > b_2$.

Given any qualitative algebra $2^{\mathcal{B}}$, pieces of knowledge about a set of spatial or temporal entities can be represented by means of qualitative constraint networks (QCNs for short). This structure allows us to represent incomplete information

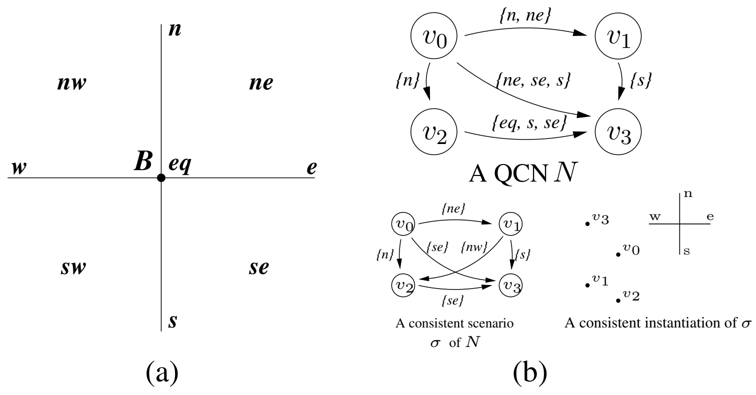


Fig. 1. The nine basic relations of $\mathcal{B}_{card}$ (a) and a QCN on $2^{\mathcal{B}_{card}}$ and one of its consistent scenarios (b)

about the relations between entities. Formally, a QCN N defined on $2^{\mathcal{B}}$ is a pair (V, C), where $V = \{v_0, \cdots, v_{n-1}\}$ is a finite set of variables representing the spatial or temporal entities and C is a mapping which associates to each pair of variables (v_i, v_j) an element R of $2^{\mathcal{B}}$. R represents the set of all possible basic relations between v_i and v_j. We write C_{ij} instead of $C(v_i, v_j)$ for short. For all $v_i, v_j \in V$, we suppose that $C_{ji} = C_{ij}^{\smile}$ and $C_{ii} = \{eq\}$.

A QCN can be represented by a graph, using some conventions: for all $v_i, v_j \in V$, we do not represent the constraint C_{ji} if C_{ij} is represented since $C_{ji} = C_{ij}^{\smile}$; we do not represent either the constraint C_{ii} since $C_{ii} = \{eq\}$; lastly when $C_{ij} = \mathcal{B}$ (i.e. no information is provided between the variables v_i and v_j), we do not represent it.

With regard to a QCN $N = (V, C)$ we have the following definitions:

Definition 2. *A consistent instantiation of N over $V' \subseteq V$ is a mapping α from V' to $\mathcal{D}$ such that $\alpha(v_i)\ C_{ij}\ \alpha(v_j)$, for all $v_i, v_j \in V'$. A solution of N is a consistent instantiation of N over V. N is a consistent QCN iff it admits a solution. A sub-network of N is a QCN$N' = (V, C')$ where $C'_{ij} \subseteq C_{ij}$ for all $v_i, v_j \in V$. A consistent scenario of N is a consistent sub-network of N in which each constraint is composed of exactly one basic relation of $\mathcal{B}$.*

$[N]$ denotes the set of all consistent scenarios of N. A QCN defined on $2^{\mathcal{B}_{card}}$ over 4 variables and one of its consistent scenarios are depicted in Figure 1.b.

3 Related Work: Merging QCNs Defined on Same Qualitative Algebras

Before summarizing a merging method for QCNs which has been proposed in the literature [4], let us introduce the merging process through an example, which will be our running example for this section.

Example 1. We consider three agents A_1, A_2 and A_3 having incomplete knowledge about the configurations of a common set $V = \{v_0, v_1, v_2, v_3\}$ of four variables. Each agent A_i provides a QCN $N_i = (V, C_i)$ defined on $2^{\mathcal{B}_{card}}$ representing the qualitative relations between pairs of V. Figure 2 depicts the three related QCNs. All consistent scenarios of each QCN are also depicted below through a qualitative representation on the plane.

The merging process takes as input a set of QCNs $\mathcal{N} = \{N_1, \ldots, N_m\}$ defined on the same qualitative algebra $2^{\mathcal{B}}$ and on the same set of variables V. Given such a set of QCNs describing different points of view about the configurations of the same entities, we would like to derive a global view of the system, taking into account each input QCN. A natural way to deal with this problem is to return as the result of merging the information on which all sources agree. For example, we can consider the set $\bigcap_{N_i \in \mathcal{N}} [N_i]$, that is the set of all consistent scenarios which belongs to each QCN N_i. However this set can be empty. It is the case in our example, since for instance the two variables v_1 and v_3 only satisfy the relation $\{w\}$ in N_1 while they only satisfy the relation $\{eq\}$ in N_3.

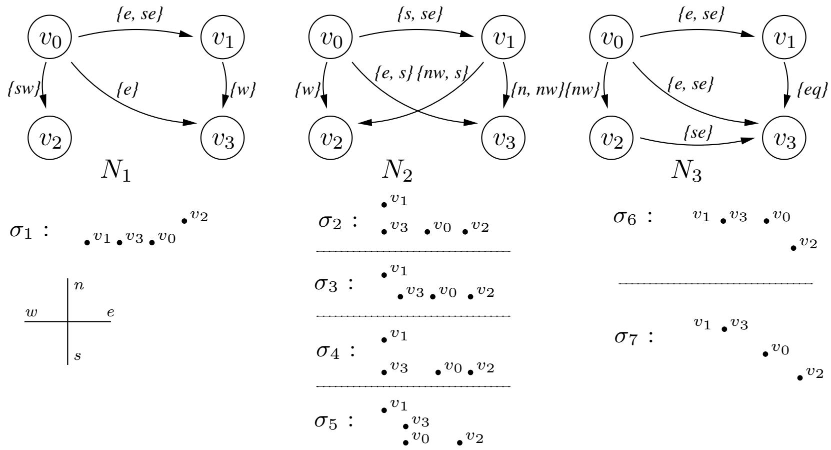


Fig. 2. Three $QCNs$ defined on V and their respective consistent scenarios

Condotta et al. [4] have proposed a parsimonious method for the merging problem. Their method is inspired from propositional merging [21,10,2]. It consists in computing a non-empty set of consistent scenarios that are the closest ones to each input QCN, with respect to a *distance*. The result is mainly based on the definition of the distance which represents the degree of closeness between scenarios and the set of QCNs. A QCN merging operator Θ is typically a mapping which associates to a set of QCNs $\mathcal{N}$ defined on $2^{\mathcal{B}}$ and V a set of consistent scenarios on V. More precisely, the merging process follows three steps:

First, we need to compute a local distance d between a scenario σ and a QCN N, which is the smallest distance between σ and each consistent scenario of N. Formally, this distance is defined as follows:

$$d(\sigma, N) = \begin{cases} \min\{d(\sigma, \sigma') \mid \sigma' \in [N]\} & \text{if } N \text{ is consistent,} \\ 0 & \text{otherwise.} \end{cases}$$

Here we need a distance d between two scenarios σ, σ'. This distance is a mapping which associates a positive number to every pair of scenarios, and satisfies the following two conditions:

$$\forall \sigma, \sigma' \text{ scenarios on } V, \quad \begin{cases} d(\sigma, \sigma') = d(\sigma', \sigma) \\ d(\sigma, \sigma') = 0 \text{ iff } \sigma = \sigma'. \end{cases}$$

A particular distance between scenarios has been proposed in [4]. It takes advantage of a notion of conceptual neighbourhood specific to the considered set of basic relations $\mathcal{B}$. Figure 3.a depicts the conceptual neighbourhood graph of the Cardinal Directions Algebra. This graph corresponds to the Hasse diagram of the corresponding lattice defined in [12]. There is an intuitive meaning behind this graph. For example, assume that two points satisfy the relation ne. Then continuously moving one of the two points can lead them to directly satisfy the relation n. Thus these two relations are considered as "close". Let us denote by $\sigma(v_i, v_j)$ the basic relation satisfied between v_i and v_j in the scenario σ. The neighbourhood distance between two scenarios is defined as follows, $\forall \sigma, \sigma'$ scenarios on V:

$$d_{NB}(\sigma, \sigma') = \sum_{i<j} d_{nb}(\sigma(v_i, v_j), \sigma'(v_i, v_j)),$$

with $d_{nb}(r_1, r_2)$ the length of the smallest path between the basic relations r_1 and r_2 in the related neighbourhood graph. We use this particular distance in our running example.

Example 1 (continued). Consider the scenarios σ_1 and σ_2 depicted in Figure 2. The neighbourhood distance between σ_1 and σ_2 is computed as follows:

$$\begin{aligned} d_{NB}(\sigma_1, \sigma_2) &= d_{nb}(\sigma_1(v_0, v_1), \sigma_2(v_0, v_1)) + d_{nb}(\sigma_1(v_0, v_2), \sigma_2(v_0, v_2)) \\ &\quad + d_{nb}(\sigma_1(v_0, v_3), \sigma_2(v_0, v_3)) + d_{nb}(\sigma_1(v_1, v_2), \sigma_2(v_1, v_2)) \\ &\quad + d_{nb}(\sigma_1(v_1, v_3), \sigma_2(v_1, v_3)) + d_{nb}(\sigma_1(v_2, v_3), \sigma_2(v_2, v_3)) \\ &= d_{nb}(e, se) + d_{nb}(sw, w) + d_{nb}(e, e) \\ &\quad + d_{nb}(sw, nw) + d_{nb}(w, n) + d_{nb}(ne, e) \\ &= 1 + 1 + 0 + 2 + 2 + 1 = 7. \end{aligned}$$

The second step of the merging process consists in aggregating local distances computed in the first step to get a global distance between a scenario and the set of QCNs $\mathcal{N}$. For example, the sum operator $\sum$ is appropriate when the result of merging has to represent the point of view of the majority of the agents. $\sum$ is so called a *majority operator* [14].

Example 1 (continued). Consider the scenario σ depicted in Figure 3.b. The global distance $d_{\sum}$ (using the majority operator $\sum$) between σ and the set of three QCNs (cf Figure 2) is computed as follows (we do not detail computations for the sake of conciseness) :

$$\begin{aligned} d_{\sum}(\sigma, \{N_1, N_2, N_3\}) &= d(\sigma, N_1) + d(\sigma, N_2) + d(\sigma, N_3) \\ &= 7 + 3 + 2 = 12. \end{aligned}$$

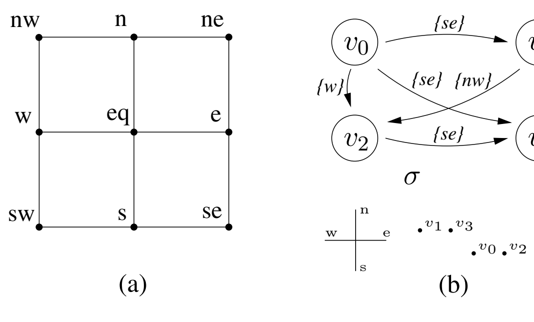


Fig. 3. The conceptual neighbourhood graph of the Cardinal Directions Algebra (a) and a consistent scenario σ (b)

The last step consists in computing the result of the merging. It is the set of all consistent scenarios which are the "closest" ones to the set $\mathcal{N}$. Formally,

$$\Theta(\mathcal{N}) = \{\sigma \mid \sigma \text{ is consistent and } d_{\sum}(\sigma, \mathcal{N}) \text{ is minimal}\}.$$

In our running example, $\Theta(\{N_1, N_2, N_3\})$ is composed of one consistent scenario, namely σ, depicted in Figure 3.b. Condotta et al. [4] have pointed out that this merging operator has a "good" logical behavior in the sense of [10,3].

Note however that the merging method described above requires the enumeration of all possible consistent scenarios on V, which makes this process hardly practicable.

4 Dealing with Different Qualitative Algebras: Toward a Common Refinement

4.1 The Model

The approach presented in the previous section permits us to deal with a set of possibly conflicting QCNs defined on the same qualitative algebra. We aim in this section at extending it by taking into account different qualitative algebras, provided that all the related partition schemes are on the same domain $\mathcal{D}$.

Formally, let $\mathcal{N} = \{N_1, \ldots, N_m\}$ be the input QCNs and $\mathcal{N} = \{\mathcal{N}_1, \ldots, \mathcal{N}_p\}$ be a partition of $\mathcal{N}$ such that $\forall k \in \{1, \ldots, p\}$, any QCN of $\mathcal{N}_k$ is defined on the qualitative algebra $2^{\mathcal{B}_k}$. Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ be the set of all related qualitative algebras. We require that each partition scheme $\mathcal{B}_k$ is on $\mathcal{D}$.

We first need to introduce the notion of *equivalence* between two QCNs. Given a QCN N defined on V, we denote by $Sol(N)$ the set of solutions of N, i.e., the possibly infinite set of consistent instantiations of N over V, different from the set of its consistent scenarios $[N]$.

Definition 3. *Let $2^{\mathcal{B}}$, $2^{\mathcal{B}'}$ be two qualitative algebras on $\mathcal{D}$, and N, N' two QCNs on V respectively defined on $2^{\mathcal{B}}$ and $2^{\mathcal{B}'}$. Then N and N' are* equivalent *iff $Sol(N) = Sol(N')$.*

The method can be summarized in two main steps:

(1) We suppose that there exists a qualitative algebra $Ref(\mathcal{A})$ such that any QCN of $\mathcal{N}$ can be translated in an equivalent QCN on $Ref(\mathcal{A})$. The set of all translated QCNs is denoted by $\mathcal{N}'$.
(2) We use a merging process of QCNs defined on the same qualitative algebra for merging the set $\mathcal{N}'$ (see Section 3).

This section is devoted to deal with the first step of this process. We show that such a qualitative algebra $Ref(\mathcal{A})$ always exists and how to define it. This algebra will be called a *refinement* of all input qualitative algebras. Let us first introduce the notion of refinement between qualitative algebras.

Definition 4. *Let $\mathcal{B}$, $\mathcal{B}'$ be two partition schemes on $\mathcal{D}$. The set $2^{\mathcal{B}'}$ is called a* refinement *of $2^{\mathcal{B}}$ iff there exists a mapping $Ref_{\mathcal{B}\to\mathcal{B}'}$ which associates to each relation of $2^{\mathcal{B}}$ a relation of $2^{\mathcal{B}'}$ such that $\forall R \in 2^{\mathcal{B}}$, $rel(Ref_{\mathcal{B}\to\mathcal{B}'}(R)) = rel(R)$. Such a mapping $Ref_{\mathcal{B}\to\mathcal{B}'}$ is called an r-bridge from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$.*

As a typical example, consider the RCC-8 algebra [17] and the RCC-5 algebra [9] which are both used to express topological relationships between regions. Then following Definition 4 the RCC-8 algebra is a refinement of the RCC-5 algebra. Let us make some remarks around this definition. First, an r-bridge $Ref_{\mathcal{B}\to\mathcal{B}'}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$ is fully characterized by its restriction to the set of singleton relations of $2^{\mathcal{B}}$ (namely the set $\{\{r\} \mid r \in \mathcal{B}\}$) to $2^{\mathcal{B}'}$. Indeed, we have for all relation $R \in 2^{\mathcal{B}}$,

$$Ref_{\mathcal{B}\to\mathcal{B}'}(R) = \bigcup\{Ref_{\mathcal{B}\to\mathcal{B}'}(\{r\}) \mid r \in R\}.$$

Moreover, let us notice that $Ref_{\mathcal{B}\to\mathcal{B}'}(\{eq\}) = \{eq\}$ and that for any relation $R \in 2^{\mathcal{B}}$, $Ref_{\mathcal{B}\to\mathcal{B}'}(R^{\smile}) = Ref_{\mathcal{B}\to\mathcal{B}'}(R)^{\smile}$. One can also state that if $2^{\mathcal{B}'}$ is a refinement of $2^{\mathcal{B}}$, there exists only one r-bridge $Ref_{\mathcal{B}\to\mathcal{B}'}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$. Lastly, $Ref_{\mathcal{B}\to\mathcal{B}'}$ is an injective function.

The definition of such an r-bridge $Ref_{\mathcal{B}\to\mathcal{B}'}$ comes from the following proposition:

Proposition 1. *$2^{\mathcal{B}'}$ is a refinement of $2^{\mathcal{B}}$ iff $\forall r' \in \mathcal{B}'$, $\forall r \in \mathcal{B}$, either $r' \subseteq r$ or $r' \cap r = \emptyset$.*

Proof

($\Rightarrow$) Let $2^{\mathcal{B}'}$ be a refinement of $2^{\mathcal{B}}$. There exists an r-bridge $Ref_{\mathcal{B}\to\mathcal{B}'}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$. Let $r \in \mathcal{B}$. Then there exists $R \in 2^{\mathcal{B}'}$ such that $rel(Ref_{\mathcal{B}\to\mathcal{B}'}(\{r\})) = r = rel(R)$. Let $r' \in \mathcal{B}'$. If $r' \in R$, then $r' \subseteq r$ since $rel(R) = r$; if $r' \notin R$, then $r' \cap rel(R) = \emptyset$ (since the basic relations of $\mathcal{B}'$ are pairwise disjoint). Thus $r' \cap r = \emptyset$.

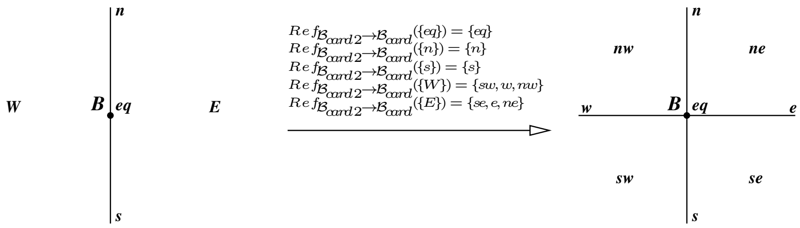
($\Leftarrow$) Suppose that $\forall r' \in \mathcal{B}'$, $\forall r \in \mathcal{B}$, either $r' \subseteq r$ or $r' \cap r = \emptyset$. Then $\forall r \in \mathcal{B}$ we get $r = \bigcup\{r' \in \mathcal{B}' \mid r' \subseteq r\}$ since basic relations of $\mathcal{B}'$ are jointly exhaustive relations on $\mathcal{D} \times \mathcal{D}$. Define the mapping $Ref_{\mathcal{B}\to\mathcal{B}'}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$ such that $\forall r \in \mathcal{B}$, $Ref_{\mathcal{B}\to\mathcal{B}'}(\{r\}) = \{r' \in \mathcal{B}' \mid r' \subseteq r\}$ and $\forall R \in 2^{\mathcal{B}}$, $Ref_{\mathcal{B}\to\mathcal{B}'}(R) = \bigcup\{Ref_{\mathcal{B}\to\mathcal{B}'}(\{r\}) \mid r \in R\}$. We can assert that $\forall R \in 2^{\mathcal{B}}$, $rel(Ref_{\mathcal{B}\to\mathcal{B}'}(R)) = rel(R)$. □

The previous proposition can be also written as follows:

$$2^{\mathcal{B}'} \text{ is a refinement of } 2^{\mathcal{B}} \text{ iff } \forall r' \in \mathcal{B}', \exists! r \in \mathcal{B}, r' \subseteq r.$$

Thus, if $2^{\mathcal{B}'}$ is a refinement of $2^{\mathcal{B}}$, we can define the restriction of the r-bridge $Ref_{\mathcal{B}\to\mathcal{B}'}$ from the singleton relations of $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$ as follows:

$$\forall r \in \mathcal{B}, Ref_{\mathcal{B}\to\mathcal{B}'}(\{r\}) = \{r' \in \mathcal{B}' \mid r' \subseteq r\}.$$



The five basic relations of $\mathcal{B}_{card2}$ The nine basic relations of $\mathcal{B}_{card}$

Fig. 4. $2^{\mathcal{B}_{card}}$ is a refinement of $2^{\mathcal{B}_{card2}}$

Example 2. Let $2^{\mathcal{B}_{card2}}$ be the qualitative algebra generated by the partition scheme $\mathcal{B}_{card2}$ on $\mathbb{R}^2$, with $\mathcal{B}_{card2} = \{eq, n, E, s, W\}$. The five basic relations of $\mathcal{B}_{card2}$ are depicted in the left-hand side of Figure 4. We point out on the same figure the r-bridge $Ref_{\mathcal{B} \to \mathcal{B}'}$ from singleton relations of $2^{\mathcal{B}_{card2}}$ to $2^{\mathcal{B}_{card}}$. Therefore $2^{\mathcal{B}_{card}}$ is a refinement of $2^{\mathcal{B}_{card2}}$.
It is now time to extend the notion of refinement to the QCNs.

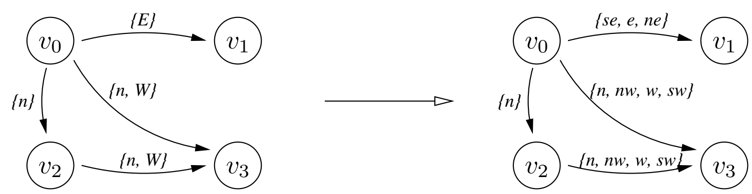
Definition 5. *Let $N = (V, C)$ be a QCN defined on $2^{\mathcal{B}}$, and $2^{\mathcal{B}'}$ be a refinement of $2^{\mathcal{B}}$. We define the QCN $Ref_{\mathcal{B} \to \mathcal{B}'}(N)$ on $2^{\mathcal{B}'}$ as the QCN (V, C') with $C'_{ij} = Ref_{\mathcal{B} \to \mathcal{B}'}(C_{ij})$ $\forall v_i, v_j \in V$.*

Proposition 2. *Given a QCN N on $2^{\mathcal{B}}$, N and $Ref_{\mathcal{B} \to \mathcal{B}'}(N)$ are equivalent QCNs.*

Proof. Given a QCN N on $2^{\mathcal{B}}$ and $Ref_{\mathcal{B} \to \mathcal{B}'}(N) = (V, C')$, for all $v_i, v_j \in V$, we have $C'_{ij} = Ref_{\mathcal{B} \to \mathcal{B}'}(C_{ij})$. Moreover, we know that $rel(C_{ij}) = rel(Ref_{\mathcal{B} \to \mathcal{B}'}(C_{ij}))$. It follows that $rel(C_{ij}) = rel(C'_{ij})$. We can conclude that for all $a, b \in \mathcal{D}$, $a\ C_{ij}\ b$ iff $a\ C'_{ij}\ b$. Hence, α is a solution of N iff α is a solution of $Ref_{\mathcal{B} \to \mathcal{B}'}(N)$. □

Example 3. Figure 5 depicts a QCN defined on $2^{\mathcal{B}_{card2}}$ and its equivalent QCN $Ref_{\mathcal{B} \to \mathcal{B}'}(N)$ defined on $2^{\mathcal{B}_{card}}$.

We now define an ordering over the qualitative algebras onto a domain $\mathcal{D}$ based on the notion of refinement.



A QCN N defined on $2^{\mathcal{B}_{card2}}$ The QCN $Ref_{\mathcal{B}_{card2} \to \mathcal{B}_{card}}(N)$ defined on $2^{\mathcal{B}_{card}}$ equivalent to N

Fig. 5. Two equivalent QCNs

Definition 6. *Given $2^{\mathcal{B}}, 2^{\mathcal{B}'}$ two qualitative algebras on $\mathcal{D}$, $2^{\mathcal{B}} \leq_{ref}^{\mathcal{D}} 2^{\mathcal{B}'}$ iff $2^{\mathcal{B}}$ is a refinement of $2^{\mathcal{B}'}$.*

Proposition 3. *$\leq_{ref}^{\mathcal{D}}$ is a weak partial ordering over qualitative algebras on $\mathcal{D}$, i.e., a reflexive, antisymmetric and transitive relation.*

Proof. Let $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$ be three partition schemes on $\mathcal{D}$.

- By defining the mapping $Ref_{\mathcal{B}\to\mathcal{B}}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}}$ by $Ref_{\mathcal{B}\to\mathcal{B}}(R) = R$ for all $R \in 2^{\mathcal{B}}$ we obtain an r-bridge from $2^{\mathcal{B}}$ to $2^{\mathcal{B}}$. Hence, $2^{\mathcal{B}} \leq_{ref}^{\mathcal{D}} 2^{\mathcal{B}}$.
- Let $\mathcal{B}$ and $\mathcal{B}'$ be two partition schemes such that $2^{\mathcal{B}}$ and $2^{\mathcal{B}'}$ are refinements of respectively $2^{\mathcal{B}'}$ and $2^{\mathcal{B}}$. Let r be a basic relation belonging to $\mathcal{B}$. Since $2^{\mathcal{B}}$ is a refinement of $2^{\mathcal{B}'}$ there exists a unique basic relation $r' \in \mathcal{B}'$ such that $r \subseteq r'$. Moreover, since $2^{\mathcal{B}'}$ is a refinement of $2^{\mathcal{B}}$ there exists a unique basic relation $r'' \in \mathcal{B}$ such that $r' \subseteq r''$. Hence, $r \cap r'' \neq \emptyset$. We can conclude that $r = r'' = r'$. Consequently, $2^{\mathcal{B}} = 2^{\mathcal{B}'}$.
- Let us suppose that $2^{\mathcal{B}''} \leq_{ref}^{\mathcal{D}} 2^{\mathcal{B}'} \leq_{ref}^{\mathcal{D}} 2^{\mathcal{B}}$ and consider the two r-bridges $Ref_{\mathcal{B}\to\mathcal{B}'}$ and $Ref_{\mathcal{B}'\to\mathcal{B}''}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$ and from $2^{\mathcal{B}'}$ to $2^{\mathcal{B}''}$ respectively. Let us define the mapping $Ref_{\mathcal{B}\to\mathcal{B}''}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}''}$ by $Ref_{\mathcal{B}\to\mathcal{B}''}(R) = Ref_{\mathcal{B}'\to\mathcal{B}''}(\ Ref_{\mathcal{B}\to\mathcal{B}'}(R))$ for all $R \in 2^{\mathcal{B}}$. We have $rel(Ref_{\mathcal{B}\to\mathcal{B}''}(R)) = rel(Ref_{\mathcal{B}'\to\mathcal{B}''}(Ref_{\mathcal{B}\to\mathcal{B}'}(R))) = rel(Ref_{\mathcal{B}\to\mathcal{B}'}(R)) = rel(R)$ for all $R \in 2^{\mathcal{B}}$. Hence, $Ref_{\mathcal{B}\to\mathcal{B}''}$ is an r-bridge from $2^{\mathcal{B}}$ to $2^{\mathcal{B}''}$. Consequently, $2^{\mathcal{B}''} \leq_{ref}^{\mathcal{D}} 2^{\mathcal{B}}$. □

Definition 7. *Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$, with $p \geq 1$, a set of qualitative algebras on $\mathcal{D}$. Then $2^{\mathcal{B}}$ is called a* common refinement *of $\mathcal{A}$ iff $2^{\mathcal{B}} \leq_{ref}^{\mathcal{D}} 2^{\mathcal{B}_k}$ for all $k \in \{1, \ldots, p\}$.*

Consider a set $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ of qualitative algebras on $\mathcal{D}$, we will define a common refinement denoted by $Ref(\mathcal{A})$.

In the sequel, we will denote by $v(k)$, with $k \in \{1, \ldots, p\}$ and $p \geq 1$, the k^{th} component of a p-tuple $v \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_p$, that is, the basic relation of v corresponding to $\mathcal{B}_k$.

Definition 8. *Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ be a set of qualitative algebras on $\mathcal{D}$. The qualitative algebra $Ref(\mathcal{A}) = 2^{\mathcal{B}_{Ref}(\mathcal{A})}$ is defined as follows :*

$$\mathcal{B}_{Ref}(\mathcal{A}) = \{ \bigcap_{1 \leq k \leq p} v(k) \mid v \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_p\} \setminus \{\emptyset\}.$$

Firstly, let us prove that $Ref(\mathcal{A})$ is well-defined.

Proposition 4. *Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ be a set of qualitative algebras on $\mathcal{D}$. $\mathcal{B}_{Ref}(\mathcal{A})$ is a partition scheme on $\mathcal{D}$.*

Proof

- We first prove that $\mathcal{B}_{Ref}(\mathcal{A})$ is a partition of $\mathcal{D} \times \mathcal{D}$. For all $(a, b) \in \mathcal{D} \times \mathcal{D}$, since each $\mathcal{B}_k$ is a partition of $\mathcal{D} \times \mathcal{D}$, there exists a unique basic relation $r_k \in \mathcal{B}_k$ such that $(a, b) \in r_k$. Thus there exists a unique p-tuple $v \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_p$ such that for each $k \in \{1, \ldots, p\}$, $(a, b) \in v(k)$ (v is defined by $v(k) = r_k$). Hence, there exists a unique relation $r \in \mathcal{B}_{Ref}(\mathcal{A})$ such that $(a, b) \in r$.

- For all $2^{\mathcal{B}_k} \in \mathcal{A}$, $eq \in \mathcal{B}_k$. Consequently, we can assert that the identity relation eq onto $\mathcal{D}$ is an element of $\mathcal{B}_{Ref}(\mathcal{A})$.
- Let $r \in \mathcal{B}_{Ref}(\mathcal{A})$ and v the p-tuple of $\mathcal{B}_1 \times \cdots \times \mathcal{B}_p$ such that $r = \bigcap_k v(k)$. By defining the p-tuple $v' \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_p$ by $v'(k) = (v(k))^{\smile}$ for all $k \in \{1, \ldots, p\}$. We have $\bigcap_k v'(k)$ which is a relation belonging to $\mathcal{B}_{Ref}(\mathcal{A})$ and which is the converse of r. □

Let us now prove that the built refinement is the greatest common refinement of $Ref(\mathcal{A})$ w.r.t. $\leq^{\mathcal{D}}_{ref}$.

Proposition 5. *Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ be a set of qualitative algebras on $\mathcal{D}$. $Ref(\mathcal{A})$ is the greatest common refinement of $\mathcal{A}$* w.r.t. $\leq^{\mathcal{D}}_{ref}$.

Proof

- First we show that $\forall 2^{\mathcal{B}_i} \in \mathcal{A}$, $Ref(\mathcal{A}) = 2^{\mathcal{B}_{Ref}(\mathcal{A})}$ is a refinement of $2^{\mathcal{B}_i}$. Let $2^{\mathcal{B}_i} \in \mathcal{A}$, and r be a basic relation of $\mathcal{B}_i$. Consider an element $(a, b) \in r$. Denote by r_k the basic relation of $\mathcal{B}_k$ containing (a, b) for each $k \in \{1, \ldots, p\}$. Note that $r_i = r$. Let $r' = \bigcap_k r_k$. We have r' which is a basic relation belonging to $\mathcal{B}_{Ref}(\mathcal{A})$, moreover $r \subseteq r'$. From Proposition 1, we can assert that $Ref(\mathcal{A}) \leq^{\mathcal{D}}_{ref} 2^{\mathcal{B}_i}$.
- Let $2^{\mathcal{B}}$ be a common refinement of $\mathcal{A}$. We now show that $Ref(\mathcal{A}) \leq^{\mathcal{D}}_{ref} 2^{\mathcal{B}}$. Let $r \in \mathcal{B}$. Since $2^{\mathcal{B}}$ is a common refinement of $\mathcal{A}$, for all $i \in \{1, \ldots, p\}$ there exists $r_i \in \mathcal{B}_i$ such that $r \subseteq r_i$. Let $r' = \bigcap_k r_k$. We have r' which is a basic relation belonging to $\mathcal{B}_{Ref}(\mathcal{A})$, moreover $r \subseteq r'$. We can conclude that $2^{\mathcal{B}}$ is a refinement of $Ref(\mathcal{A})$, that is, $2^{\mathcal{B}} \leq^{\mathcal{D}}_{ref} Ref(\mathcal{A})$. □

Thus, for each $2^{\mathcal{B}_i} \in \mathcal{A}$ there exists an r-bridge $Ref_{\mathcal{B}_i \to \mathcal{B}_{Ref}(\mathcal{A})}$ from $2^{\mathcal{B}_i}$ to $Ref(\mathcal{A})$ defined for each basic relation $r \in \mathcal{B}_i$ as $Ref_{\mathcal{B}_i \to \mathcal{B}_{Ref}(\mathcal{A})}(\{r\}) = \{r' \in \mathcal{B}_{Ref}(\mathcal{A}) \mid r' \subseteq r\}$, and for each relation $R \in 2^{\mathcal{B}_i}$ as $Ref_{\mathcal{B}_i \to \mathcal{B}_{Ref}(\mathcal{A})}(R) = \bigcup\{Ref_{\mathcal{B}_i \to \mathcal{B}_{Ref}(\mathcal{A})}(\{r\}) \mid r \in R\}$. Then, for each QCN $N_i \in \mathcal{N}$ defined on $2^{\mathcal{B}_j} \in \mathcal{A}$, we translate N_i into an equivalent QCN $N'_i = Ref_{\mathcal{B}_i \to \mathcal{B}_{Ref}(\mathcal{A})}(N_i)$ on $Ref(\mathcal{A})$ (cf Definition 5). We then get a set of QCNs $\mathcal{N}' = \{N'_1, \ldots, N'_m\}$ defined on the same qualitative algebra $Ref(\mathcal{A})$, and we can use a merging process for QCNs defined on the same qualitative algebra, as described in Section 3.

4.2 Instantiating our Framework on the Star Algebra

The Star algebra [20] is a generalization of two kinds of algebras around cardinal directions distinguished by Frank [7], the coned-shaped directions and the projection-based directions [12]. It considers relations of the domain $\mathcal{D} = \mathbb{R}^2$ and it is set by a level of granularity m. The parameter m specifies the number of lines which intersect in a reference point B. Each line j, $1 \leq j \leq m$ forms an angle δ_j with the reference direction. The plane is split into $4m + 1$ zones ($2m$ half-lines, $2m$ two-dimensional zones, and the relation eq), forming the partition scheme $\mathcal{STAR}_m[\delta_1, \ldots, \delta_m](\delta_1)$ on $\mathbb{R}^2$. We call such a partition scheme a *Star partition scheme*. When all two-dimensional zones are of equal size, we rather write $\mathcal{STAR}_m(\delta_1)$, with δ_1 the angle formed by the first line w.r.t. the reference

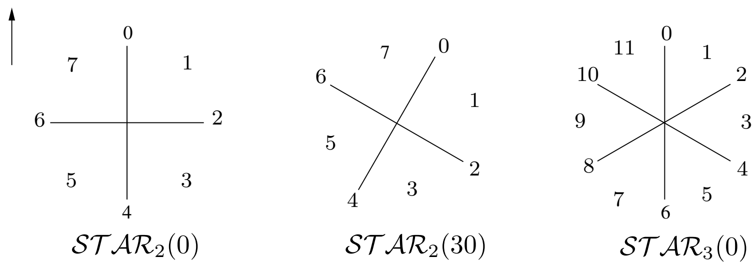


Fig. 6. Three Star partition schemes

direction. The basic relations of a Star partition scheme are numbers from 0 to $4m-1$ identifying a zone, and the relation eq.

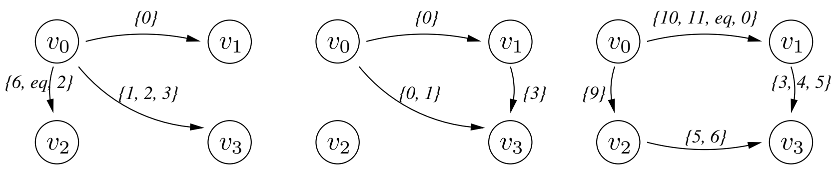
Figure 6 depicts three Star partition schemes $\mathcal{STAR}_2(0)$, $\mathcal{STAR}_2(30)$ and $\mathcal{STAR}_3(0)$. Notice that $\mathcal{STAR}_2(0)$ corresponds to the set $\mathcal{B}_{card}$.

Let $\mathcal{A} = \{2^{\mathcal{B}_1}, 2^{\mathcal{B}_2}, 2^{\mathcal{B}_3}\}$ with $\mathcal{B}_1 = \mathcal{STAR}_2(0)$, $\mathcal{B}_2 = \mathcal{STAR}_2(30)$ and $\mathcal{B}_3 = \mathcal{STAR}_3(0)$.

Let us now suppose that we have to merge a set of QCNs $\mathcal{N} = \{N_1, N_2, N_3\}$. The QCNs N_1, N_2, N_3, respectively defined on $2^{\mathcal{B}_1}$, $2^{\mathcal{B}_2}$ and $2^{\mathcal{B}_3}$, are depicted in Figure 7.

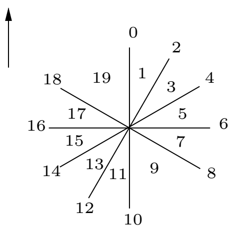
We aim to get a set $\mathcal{N}'$ of QCNs equivalent to those of $\mathcal{N}$ and defined on the same qualitative algebra. First, we define the qualitative algebra $Ref(\mathcal{A})$ which is the greatest common refinement of $\mathcal{A}$. This qualitative algebra is generated by the partition scheme $\mathcal{B}_{Ref}(\mathcal{A})$ defined as $\mathcal{B}_{Ref}(\mathcal{A}) = \{v(1) \cap v(2) \cap v(3) \mid v \in \mathcal{B}_1 \times \mathcal{B}_2 \times \mathcal{B}_3, (v(1) \cap v(2) \cap v(3)) \neq \emptyset\}$. We obtain from this definition the partition scheme depicted in Figure 8.

For each $2^{\mathcal{B}_i} \in \mathcal{A}$, recall that the bridge from $2^{\mathcal{B}_i}$ to $Ref(\mathcal{A})$ is defined for each basic relation $r \in \mathcal{B}_i$ as $Ref_{\mathcal{B}_i \rightarrow \mathcal{B}_{Ref}(\mathcal{A})}(\{r\}) = \{r' \in \mathcal{B}_{Ref}(\mathcal{A}) \mid r' \subseteq r\}$, and for each relation $R \in 2^{\mathcal{B}_i}$ by $Ref_{\mathcal{B}_i \rightarrow \mathcal{B}_{Ref}(\mathcal{A})}(R) = \bigcup\{Ref_{\mathcal{B}_i \rightarrow \mathcal{B}_{Ref}(\mathcal{A})}(\{r\}) \mid r \in R\}$. Then for each $\mathcal{B}_i \in \mathcal{A}$, using the bridge $Ref_{\mathcal{B}_i \rightarrow \mathcal{B}_{Ref}(\mathcal{A})}$ from $2^{\mathcal{B}_i}$ to $Ref(\mathcal{A})$ we translate each QCN $N_i \in \mathcal{N}$ on $2^{\mathcal{B}_i}$ into an equivalent QCN $N'_i = Ref_{\mathcal{B}_i \rightarrow \mathcal{B}_{Ref}(\mathcal{A})}(N_i)$ on $Ref(\mathcal{A})$ (cf Definition 5). Figure 9 depicts the three QCNs N'_1, N'_2 and N'_3 on $Ref(\mathcal{A})$.



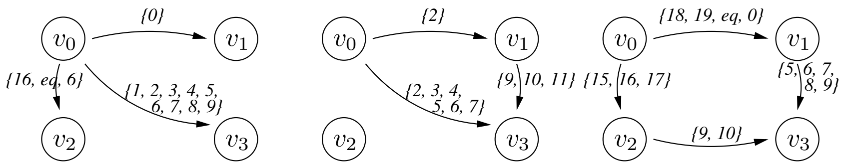
N_1 (defined on $2^{\mathcal{B}_1}$, with $\mathcal{B}_1 = \mathcal{STAR}_2(0)$) N_2 (defined on $2^{\mathcal{B}_2}$, with $\mathcal{B}_2 = \mathcal{STAR}_2(30)$) N_3 (defined on $2^{\mathcal{B}_3}$, with $\mathcal{B}_3 = \mathcal{STAR}_3(0)$)

Fig. 7. The set $\mathcal{N}$ of three QCNs defined on different Star partition schemes



$\mathcal{B}_{Ref}(\mathcal{A})$

Fig. 8. The partition scheme $\mathcal{B}_{Ref}(\mathcal{A})$



N'_1 (defined on $Ref(\mathcal{A})$) N'_2 (defined on $Ref(\mathcal{A})$) N'_3 (defined on $Ref(\mathcal{A})$)

Fig. 9. The three QCNs N'_1, N'_2 and N'_3 defined on $Ref(\mathcal{A})$

We get a set $\mathcal{N}' = \{N'_1, N'_2, N'_3\}$ of QCNs defined on the same qualitative algebra, with each N'_i equivalent to $N_i \in \mathcal{N}$. Thus we can use a merging process for QCNs defined on the same qualitative algebra as the one described in Section 3.

Let us now point out an interesting property when elements of $\mathcal{A}$ are all Star algebras. Consider again the partition scheme $Ref(\mathcal{A})$ on Figure 8, and recall that $Ref(\mathcal{A})$ is a common refinement of three Star algebras. We can notice that $\mathcal{B}_{Ref}(\mathcal{A})$ is the Star partition scheme $\mathcal{STAR}_5[0, 30, 60, 90, 120](0)$. More generally, the following proposition holds:

Proposition 6. *Let $\mathcal{A}$ be a set of Star algebras. Then $Ref(\mathcal{A})$ is a Star algebra.*

For lack of space, we will omit the proofs of the propositions in the sequel.

5 Another Alternative: Toward a Common Abstraction

5.1 The Model

In the previous section we explained how to associate to each input QCN on a qualitative algebra $2^{\mathcal{B}_i} \in \mathcal{A}$ a QCN which is defined on the greatest common refinement $Ref(\mathcal{A})$. The merging method described in [4] can now be used since each QCN is translated into an equivalent QCN on the same qualitative algebra. However the number of basic relations of $\mathcal{B}_{Ref}(\mathcal{A})$ can be large and this number has an important role in the next step of the merging process (see Section 3). In this section, instead of translating the QCNs into a common refinement of $\mathcal{A}$, we look at the consequences of using a common abstraction of $\mathcal{A}$, a notion we first have to define. The notion of abstraction is dual to the notion of refinement and is given by the following definition.

Definition 9. *Let $2^{\mathcal{B}}$ and $2^{\mathcal{B}'}$ be two qualitative algebras on $\mathcal{D}$. $2^{\mathcal{B}'}$ is an* abstraction *of $2^{\mathcal{B}}$ iff $2^{\mathcal{B}}$ is a refinement of $2^{\mathcal{B}'}$.*

Assume $2^{\mathcal{B}'}$ to be an abstraction of $2^{\mathcal{B}}$. Recall that there exists a unique r-bridge $Ref_{\mathcal{B}' \to \mathcal{B}}$ from $2^{\mathcal{B}'}$ to $2^{\mathcal{B}}$ which is an injective, not necessary bijective function (in case of a bijective function, we have $2^{\mathcal{B}'} \leq^{\mathcal{D}}_{ref} 2^{\mathcal{B}}$ and $2^{\mathcal{B}} \leq^{\mathcal{D}}_{ref} 2^{\mathcal{B}'}$, namely $2^{\mathcal{B}} = 2^{\mathcal{B}'}$ since $\leq^{\mathcal{D}}_{ref}$ is an ordering relation). Thus in the general case we cannot find an r-bridge $Ref_{\mathcal{B} \to \mathcal{B}'}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$. This means that while it is always possible to translate a QCN on $2^{\mathcal{B}'}$ to an equivalent QCN on $2^{\mathcal{B}}$ (cf Proposition 2), in general the inverse is not possible.

However we can define a "weaker" bridge from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$: we consider the mapping $Abs_{\mathcal{B} \to \mathcal{B}'}$ from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$ such that for each basic relation $r \in \mathcal{B}$, $Abs_{\mathcal{B} \to \mathcal{B}'}(\{r\})$ is the relation $\{r'\}$ of $2^{\mathcal{B}'}$ such that $r' \supseteq r$, and for each relation R of $2^{\mathcal{B}}$, $Abs_{\mathcal{B} \to \mathcal{B}'}(R) = \bigcup \{Abs_{\mathcal{B} \to \mathcal{B}'}(\{r\}) \mid r \in R\}$. We will call such a mapping $Abs_{\mathcal{B} \to \mathcal{B}'}$ an a-bridge from $2^{\mathcal{B}}$ to $2^{\mathcal{B}'}$.

Example 4. Consider again the qualitative algebras $2^{\mathcal{B}_{card}}$ and $2^{\mathcal{B}_{card2}}$ (cf Example 2) and recall that $2^{\mathcal{B}_{card}} \leq_{ref} 2^{\mathcal{B}_{card2}}$. For example, we have $Abs_{\mathcal{B}_{card} \to \mathcal{B}_{card2}}(\{nw\}) = \{W\}$ and $Abs_{\mathcal{B}_{card} \to \mathcal{B}_{card2}}(\{sw, s\}) = \{W, s\}$.

Definition 10. *Let $N = (V, C')$ be a QCN defined on $2^{\mathcal{B}}$, and $2^{\mathcal{B}'}$ be an abstraction of $2^{\mathcal{B}}$. We define the QCN $Abs_{\mathcal{B} \to \mathcal{B}'}(N) = (V, C)$ on $2^{\mathcal{B}'}$ as $\forall v_i, v_j \in V$, $C_{ij} = Abs_{\mathcal{B} \to \mathcal{B}'}(C'_{ij})$.*

For any QCN N on $2^{\mathcal{B}}$, its translation $Abs_{\mathcal{B} \to \mathcal{B}'}(N)$ on $2^{\mathcal{B}'}$ is not an equivalent QCN in the general case. Nevertheless, we have the following weaker property.

Proposition 7. *Let N be a QCN defined on $2^{\mathcal{B}}$, and $2^{\mathcal{B}'}$ be an abstraction of $2^{\mathcal{B}}$. We have $Sol(N) \subseteq Sol(Abs_{\mathcal{B} \to \mathcal{B}'}(N))$.*

This means that the translation $Abs_{\mathcal{B} \to \mathcal{B}'}(N)$ of a QCN N on $2^{\mathcal{B}}$ can be viewed as an approximation of N.

Now similarly to Definition 5, we define the notion of *common abstraction*.

Definition 11. *Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ be a set of qualitative algebras on $\mathcal{D}$. Then $2^{\mathcal{B}}$ is called a* common abstraction *of $\mathcal{A}$ iff $\forall 2^{\mathcal{B}_k} \in \mathcal{A}$, $2^{\mathcal{B}_k} \leq_{ref} 2^{\mathcal{B}}$.*

We claim that a common abstraction of a set $\mathcal{A}$ of qualitative algebras on $\mathcal{D}$ always exists. Indeed consider the partition scheme $\mathcal{B}_{\neq}$ on $\mathcal{D}$ with $\mathcal{B}_{\neq} = \{eq, (\mathcal{D} \times \mathcal{D}) \setminus eq\}$. Then it is easy to see that $2^{\mathcal{B}_{\neq}}$ is an abstraction of any qualitative algebra on $\mathcal{D}$.

Definition 12. *Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ be a set of qualitative algebras on $\mathcal{D}$. The qualitative algebra $Abs(\mathcal{A}) = 2^{\mathcal{B}_{Abs}(\mathcal{A})}$ is defined as follows:*

$$\begin{aligned}\mathcal{B}_{Abs}(\mathcal{A}) = \{r \mid\ & \forall \mathcal{B}_k \in \mathcal{A}, \exists R_k \in 2^{\mathcal{B}_k} : Rel(R_k) = r \\ & \textit{and } \nexists r' \subset r : \forall \mathcal{B}_k \in \mathcal{A}, \exists R'_k \in 2^{\mathcal{B}_k} : Rel(R'_k) = r'\}.\end{aligned}$$

We claim that $Abs(\mathcal{A})$ is well defined and is the least common abstraction of $\mathcal{A}$. Similarly to Propositions 4 and 5, the following proposition holds:

Proposition 8. *Let $\mathcal{A} = \{2^{\mathcal{B}_1}, \ldots, 2^{\mathcal{B}_p}\}$ be a set of qualitative algebras on $\mathcal{D}$.*

- *$\mathcal{B}_{Abs}(\mathcal{A})$ is a partition scheme on $\mathcal{D}$.*
- *$Abs(\mathcal{A})$ is the least common abstraction of $\mathcal{A}$.*

Therefore, the process consists in translating each QCN N_i defined on $2^{\mathcal{B}_j} \in \mathcal{A}$ in a QCN $N'_i = Abs_{\mathcal{B}_j \to \mathcal{B}_{Abs}(\mathcal{A})}(N_i)$ on $Abs(\mathcal{A})$, using the a-bridge $Abs_{\mathcal{B}_j \to \mathcal{B}_{Abs}(\mathcal{A})}$ from $\mathcal{B}_j$ to $Abs(\mathcal{A})$. We get a new set $\mathcal{N}'$ of QCNs defined on $Abs(\mathcal{A})$ which are approximations of the QCNs of $\mathcal{N}$. Although we do not get equivalent QCNs, the counterpart is that since the size of the partition scheme $\mathcal{B}_{Abs}(\mathcal{A})$ is much smaller than the size of $\mathcal{B}_{Ref}(\mathcal{A})$, we reduce the complexity of the following step of the merging process, namely, merging the set of the translated QCNs defined on the same qualitative algebra. Furthermore, since this method provides a set of QCNs $\mathcal{N}'$ which are approximations of the initial set of QCNs $\mathcal{N}$, each translated QCN $N'_i \in \mathcal{N}'$ admits a larger set of solutions than its corresponding QCN $N_i \in \mathcal{N}$. Then in some cases, even if the QCNs of the initial set $\mathcal{N}$ are conflicting (namely, if they do not admit any common solution), the translated QCNs of $\mathcal{N}'$ can be not conflicting. Thus, this method can be viewed as a first step to deal with contradictions.

5.2 An Example on the Star Algebra

Similarly to Proposition 6, notice that the following proposition holds:

Proposition 9. *Let $\mathcal{A}$ be a set of Star algebras. Then $Abs(\mathcal{A})$ is a Star algebra.*

Now consider the three Star partition schemes $\mathcal{STAR}_3[0,45,90](0)$, $\mathcal{STAR}_4[0,60,90,150](0)$ and $\mathcal{STAR}_4[0,30,90,120](0)$ depicted in Figure 10 and forming the set $\mathcal{A}$. The partition scheme of the least common abstraction $Abs(\mathcal{A})$ is depicted in this same figure.

Let $\mathcal{N} = \{N_1, N_2, N_3\}$ be the three QCNs respectively defined on $2^{\mathcal{B}_1}$, $2^{\mathcal{B}_2}$ and $2^{\mathcal{B}_3}$, with $\mathcal{B}_1 = \mathcal{STAR}_3[0,45,90](0)$, $\mathcal{B}_2 = \mathcal{STAR}_4[0,60,90,150](0)$ and $\mathcal{B}_3 = \mathcal{STAR}_4[0,30,90,120](0)$ (these three partition schemes are depicted in Figure 10). Figure 11 depicts the three QCNs of $\mathcal{N}$ and their translation to their least common refinement $Abs(\mathcal{A})$ (with $\mathcal{B}_{Abs}(\mathcal{A}) = \mathcal{STAR}_2(0)$), forming the set $\mathcal{N}' = \{N'_1, N'_2, N'_3\}$.

Let R_1 be the relation of $2^{\mathcal{B}_1}$ with $R_1 = \{1,2\}$, and R_2 be the relation of $2^{\mathcal{B}_2}$ with $R_2 = \{2,3\}$. We can notice that the constraint between v_0 and v_1 is assigned to R_1 in N_1 and this same constraint is assigned to R_2 in N_2. Since $rel(R_1) \cap rel(R_2) = \emptyset$ (see Figure 10), there does not exist any solution of both N_1 and N_2. Thus the QCNs of $\mathcal{N}$ are conflicting. Nevertheless, the QCNs of $\mathcal{N}'$ are not conflicting, since there exists a consistent scenario of all QCNs of $\mathcal{N}'$. This consistent scenario is depicted in Figure 12. In this example there is no need to apply a merging process of the set $\mathcal{N}'$ *a posteriori*.

Let $QA_{\mathcal{D}}$ be the set of all qualitative algebras on $\mathcal{D}$. From Propositions 5 and 11, there is a particular structure induced by the ordering $\leq^{\mathcal{D}}_{ref}$ on $QA_{\mathcal{D}}$:

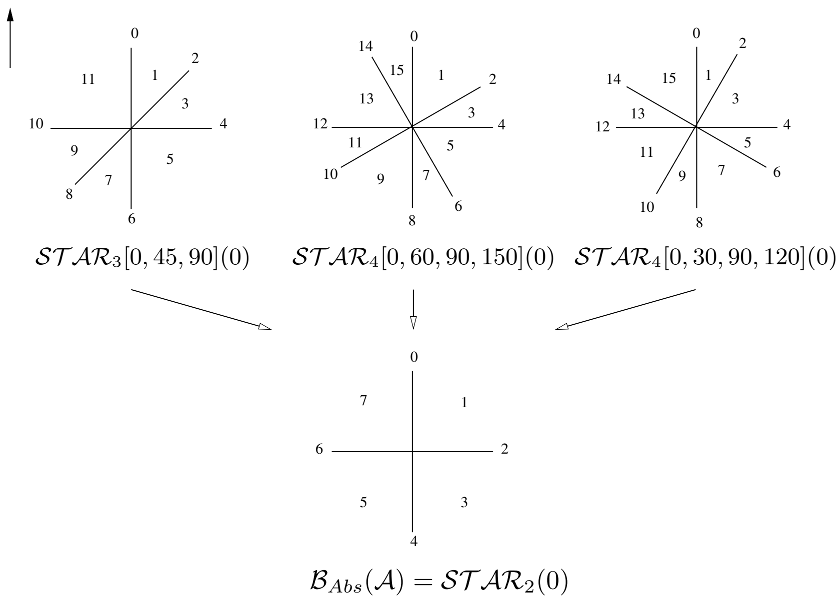


Fig. 10. Three Star Algebras and the partition scheme $\mathcal{B}_{Abs}(\mathcal{A})$ of their least common abstraction $Abs(\mathcal{A})$

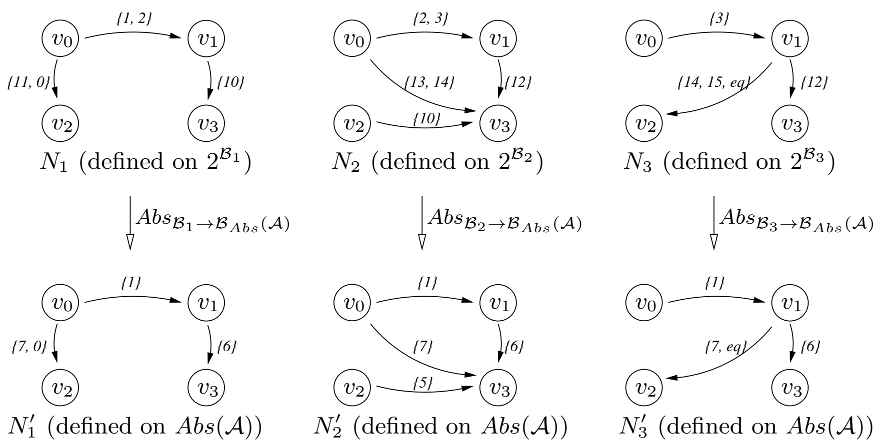
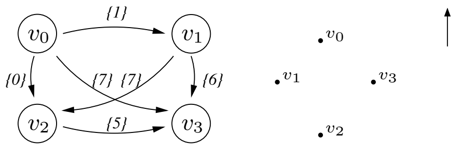


Fig. 11. Three QCNs and their translation to their least common abstraction $Abs(\mathcal{A})$

Lemma 1. *$(QA_{\mathcal{D}}, \leq^{\mathcal{D}}_{ref})$ is a lattice, namely, $\forall 2^{\mathcal{B}}, 2^{\mathcal{B}'} \in QA_{\mathcal{D}}$, $2^{\mathcal{B}}$ and $2^{\mathcal{B}'}$ have a least upper bound $Abs(2^{\mathcal{B}}, 2^{\mathcal{B}'})$ and a greatest lower bound $Ref(2^{\mathcal{B}}, 2^{\mathcal{B}'})$.*

Notice that the lattice is not bounded, since for any qualitative algebra $2^{\mathcal{B}}$ of $QA_{\mathcal{D}}$, we can find a qualitative algebra $2^{\mathcal{B}'}$ of $QA_{\mathcal{D}}$ such that $2^{\mathcal{B}'} \leq^{\mathcal{D}}_{ref} 2^{\mathcal{B}}$ and not $2^{\mathcal{B}} \leq^{\mathcal{D}}_{ref} 2^{\mathcal{B}'}$. However, the lattice has an upper bound, which corresponds to the qualitative algebra $2^{\mathcal{B}_{\neq}}$, with $\mathcal{B}_{\neq} = \{eq, (\mathcal{D} \times \mathcal{D}) \setminus eq\}$.



$\sigma \in ([N_1] \cap [N_2] \cap [N_3])$

Fig. 12. A consistent scenario of all QCNs $N'_i \in \mathcal{N}'$

6 Conclusion and Perspectives

In this paper, we addressed the problem of merging qualitative constraint networks (QCNs) when defined on different qualitative formalisms. The extension proposed here consists of translating the input QCNs into a common formalism. Therefore it is an upstream process to existing methods dealing with QCNs which are defined on the same formalism. We established a structure for all qualitative algebras which consider relations defined on the same domain. This structure is given by a weak partial ordering relation and forms a lattice over the qualitative algebras. The relation is based on the dual notions of *refinement* and *abstraction*, allowing us to define bridges between these qualitative algebras.

We are currently implementing the structure into QAT (Qualitative Algebra Toolkit) [5], a JAVA library allowing us to handle QCNs. An experimental study has to be made to measure the interest of using common abstractions instead of common refinements for the QCNs merging problem.

A future work will pursue study about the properties of this structure and about the links between qualitative algebras. This could allow us to increase the range of this work. For example, we will investigate the correspondences (in terms of consistency) between a QCN and its "translation" using the bridges involved by the structure. Indeed, translating an inconsistent QCN into another one using a well-chosen bridge could lead to restore the consistency of the QCN. We also intend to include subalgebras in the structure in order to define some bridges (in terms of refinements/abstractions) between qualitative algebras and their subalgebras (in particular their tractable subclasses), and study the consequences on the QCNs. We will also study how to define a bridge between heterogeneous formalisms, depending on a degree of compatibility between the related relations. In order to deal with this very general setting, we will have to think about new and possibly weaker structures.

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A syntactical approach to qualitative constraint networks merging

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A Syntactical Approach to Qualitative Constraint Networks Merging

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Abstract. We address the problem of merging qualitative constraint networks (QCNs) representing agents local preferences or beliefs on the relative position of spatial or temporal entities. Two classes of merging operators which, given a set of input QCNs defined on the same qualitative formalism, return a set of qualitative configurations representing a global view of these QCNs, are pointed out. These operators are based on local distances and aggregation functions. In contrast to QCN merging operators recently proposed in the literature, they take account for each constraint from the input QCNs within the merging process. Doing so, inconsistent QCNs do not need to be discarded at start, hence agents reporting locally consistent, yet globally inconsistent pieces of information (due to limited rationality) can be taken into consideration.

1 Introduction

Qualitative representation of time and space arises in many domains of Artificial Intelligence such as language processing, computer vision, planning. One needs to take advantage of a qualitative formalism when the available information about a set of spatial or temporal entities is expressed in terms of non-numerical relationships between these entities (e.g., when information comes primarily from natural language sentence). Starting from Allen's formalism [1] basically used to represent relative positions of temporal intervals, many other qualitative formalisms have been put forward in the literature these last three decades [24,19,15,2,8,20]. Besides temporal and spatial aspects, these formalisms also constitute powerful representation settings for a number of applications of Artificial Intelligence, such as reasoning about preferences [9] or multiple taxonomies [23].

When we are asked to express a set of preferred or believed relationships between entities, we are generally more willing to provide local relations about a small number of entities from which the underlying set of preferred or possible configurations about the whole set of entities can be deduced. Consider for example a student, William, who expresses his preferences on the schedule of four courses (Operating Systems, Algebra, Analysis, Programming). William prefers to learn Analysis after Algebra. Assume William would also like to learn Programming after Analysis and wants to start learning Programming before Algebra finishes. Then no schedule can satisfy all his preferences, since satisfying

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two of his wishes implies the third one to be discarded. Obviously, conflicts can also arise in the case when several students are asked to express their preferences on a common schedule.

In this paper we address the problem where several agents express their preferences / beliefs on relative positions of (spatial or temporal) entities. This information is represented, for each agent, by means of a qualitative constraint network (QCN). A procedure for merging QCN has been proposed in [6], directly adapted from a "model-based" method for merging propositional knowledge bases [13,14]. This procedure is generic in the sense that it does not depend on a specific qualitative formalism. It consists in defining a merging operator which associates with a finite set of QCNs a set of consistent (spatial or temporal) information representing a global view of the input QCNs. While this method represents a starting point in the problem of merging QCNs, it has however some limitations. First, a QCN to be merged is reduced to its global possible configurations; therefore inconsistent QCNs are discarded. As we will show in the paper, even if a QCN is inconsistent, it may however contain relevant information which deserves to be considered in the merging process. Secondly, this approach is expensive from a computational point of view as it requires the computation of all possible configurations about the whole set of entities. This paper aims at overcoming the above limitations. We propose a syntactical approach for merging QCNs in which each constraint from the input QCNs participates in the merging process. We define two classes of QCN merging operators, where each operator associates with a finite set of QCNs defined on the same qualitative formalism and the same set of entities a set of consistent qualitative configurations representing a global view of the input set of QCNs. Each operator is based on distances between relations of the underlying qualitative formalism and on two aggregation functions.

The rest of the paper is organized as follows. The next section recalls necessary preliminaries on qualitative constraint networks, distances between relations of a qualitative formalim and aggregation functions. In Section 3, we address the problem of dealing with conflicting QCNs. We introduce a running example and give some postulates that QCN merging operators are expected to satisfy. In Section 4, we define the two proposed classes of QCN merging operators and discuss their logical properties. We give some hints to choose a QCN merging operator in Section 5, and also give some comparisons with related works. We conclude in the last section and present some perspectives for further research. For space reasons, proofs are not provided; they are available in the longer version of the paper, http://www.cril.univ-artois.fr/∼marquis/lpar2010longversion.pdf.

2 Preliminaries

2.1 Qualitative Formalisms and Qualitative Constraint Networks

A qualitative formalism considers a finite set B of basic binary relations defined on a domain D. The elements of D represent the considered (spatial or temporal) entities. Each basic relation $b \in \mathsf{B}$ represents a particular relative position between two elements of D. The set B is required to be a *partition scheme* [16], i.e., it satisfies the following properties: (i) B forms a partition of $\mathsf{D} \times \mathsf{D}$, namely any pair of $\mathsf{D} \times \mathsf{D}$ satisfies one and only one basic relation of B; (ii) the identity relation on D, denoted by eq, belongs to B; lastly, (iii) if b is a basic relation of B, then its converse, denoted by b^{-1}, also belongs to B.

For illustration we consider a well-known qualitative formalism introduced by Allen, called Interval Algebra [1]. This formalism considers a set $\mathsf{B}_{\mathsf{int}}$ of thirteen basic relations defined on the domain of non-punctual (durative) intervals over the rational numbers: $\mathsf{D}_{\mathsf{int}} = \{(x^-, x^+) \in \mathbb{Q} \times \mathbb{Q} : x^- < x^+\}$. An interval typically represents a temporal entity. The basic relations of $\mathsf{B}_{\mathsf{int}} = \{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$ are depicted in Figure 1. Each one of them represents a particular situation between two intervals. For example, the relation $m = \{((x^-, x^+), (y^-, y^+)) \in \mathsf{D}_{\mathsf{int}} \times \mathsf{D}_{\mathsf{int}} : x^+ = y^-\}$ represents the case where the upper bound of the first interval and the lower bound of the second one coincide.

Relation	Symbol	Inverse	Illustration
precedes	p	pi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

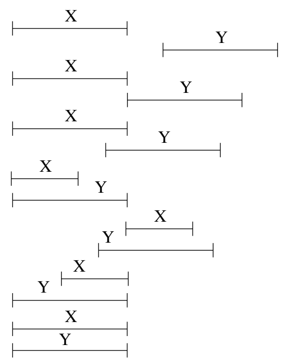


Fig. 1. The basic relations of Interval Algebra

Given a set B of basic relations, a *complex relation* is the union of basic relations and is represented by the set of the basic relations it contains. In the following we omit the qualifier "complex". For instance, considering Interval Algebra, the set $\{m, d\}$ represents the union of the basic relations m and d. The set of all relations is denoted by 2^{B}.

Pieces of information about the relative positions of a set of (spatial or temporal) entities can be represented by means of qualitative constraint networks (QCNs for short). Formally, a QCN (on B) is defined as follows:

Definition 1 (Qualitative constraint network). *A* QCN N *is a pair* (V, C) *where:*

- $V = \{v_1, \ldots, v_n\}$ *is a finite set of variables representing the entities,*
- C *is a mapping which associates with each pair of variables* (v_i, v_j) *a relation* $N[i, j]$ *of* 2^{B}. C *is such that* $N[i, i] = \{eq\}$ *and* $N[i, j] = N[j, i]^{-1}$ *for every pair of variables* $v_i, v_j \in V$.

Given a QCN $N = (V, C)$, a *consistent instantiation* of N over $V' \subseteq V$ is a mapping α from V' to D such that for every pair $(v_i, v_j) \in V' \times V'$, $(\alpha(v_i), \alpha(v_j))$ satisfies $N[i, j]$, i.e., there exists a basic relation $b \in N[i, j]$ such that $(\alpha(v_i), \alpha(v_j)) \in b$ for every $v_i, v_j \in V'$. A *solution* of N is a consistent instantiation of N over V. N is *consistent* iff it admits a solution. A *sub-network* N' of N is a QCN (V, C') such that $N'[i, j] \subseteq N[i, j]$, for every pair of variables v_i, v_j. A *scenario* σ is a QCN such that each constraint is defined by a singleton relation of 2^{B}, i.e., a relation containing exactly one basic relation. Let σ be a scenario, the basic relation specifying the constraint between two variables v_i and v_j is denoted by σ_{ij}. A scenario σ of N is a sub-network of N. In the rest of this paper, $\langle N \rangle$ denotes the set of scenarios of N and $[N]$ the set of its consistent scenarios. Two QCNs N and N' are said to be *equivalent*, denoted by $N \equiv N'$, iff $[N] = [N']$. N^V_{All} denotes the QCN on V such that for each pair of variables (v_i, v_j), $N^V_{All}[i, j] = \{eq\}$ if $v_i = v_j$, $N^V_{All}[i, j] = \mathsf{B}$ otherwise. N^V_{All} represents the complete lack of information about the relative positions of the variables.

Figures 2(a), 2(b) and 2(c) represent respectively a QCN N of Interval Algebra defined on the set $V = \{v_1, v_2, v_3, v_4\}$, an inconsistent scenario σ of N and a consistent scenario σ' of N. A solution α of σ' is represented in Figure 2(d). In order to alleviate the figures, for each pair of variables (v_i, v_j), we do not represent the constraint $N[i, j]$ when $N[i, j] = \mathsf{B}$, when $N[j, i]$ is represented or when $i = j$.

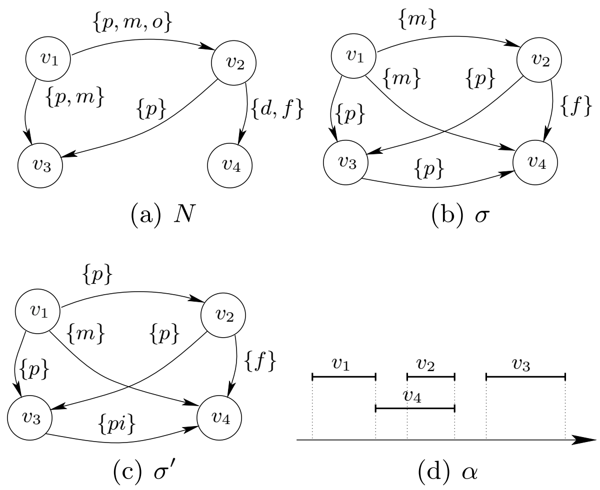


Fig. 2. A QCN N, an inconsistent scenario σ of N, a consistent scenario σ' of N and a solution α of σ'

2.2 Basic Distances and Aggregation Functions

In the following, we consider two classes of QCN merging operators parameterized by a distance between basic relations of B called *basic distance*, and by aggregation functions.

Basic distances. A basic distance associates with a pair of basic relations of B a positive number representing their degree of closeness [6].

Definition 2 (Basic distance). *A basic distance* d_{B} *is a pseudo-distance, i.e., a mapping from* $\mathsf{B} \times \mathsf{B}$ *to* $\mathbb{R}_0^+$ *such that* $\forall b, b' \in \mathsf{B}$*, we have:*

$$\begin{cases} d_{\mathsf{B}}(b, b') = d_{\mathsf{B}}(b', b) & \textit{(symmetry)} \\ d_{\mathsf{B}}(b, b') = 0 \textit{ iff } b = b' & \textit{(identity of indiscernibles)} \\ d_{\mathsf{B}}(b, b') = d_{\mathsf{B}}(b^{-1}, (b')^{-1}). & \end{cases}$$

For instance, the drastic distance d_D is equal to 1 for every pair of distinct basic relations, 0 otherwise.

In the context of qualitative algebras, two distinct basic relations can be more or less close from each other. This intuition takes its source in works of Freksa [7] who defined different notions of conceptual neighborhood between basic relations of Interval Algebra. By generalizing his definition, it is natural to state that two basic relations $b, b' \in \mathsf{B}$ are conceptually neighbors if a continuous transformation on the elements of the domain leads to two entities which satisfy the basic relation b and also directly satisfy the basic relation b' without satisfying any other basic relation. A conceptual neighborhood defines a binary relation on elements of B. This relation can be represented by an undirected connected graph in which every vertice is an element of B. In such a graph, called *conceptual neighborhood graph*, two vertices connected by an edge are conceptual neighbors. For example, in a context where a continuous transformation between two intervals corresponds to moving only one of the four possible bounds, we get the conceptual neighborhood graph $\mathsf{GB}_{\mathsf{int}}$ depicted in Figure 3.

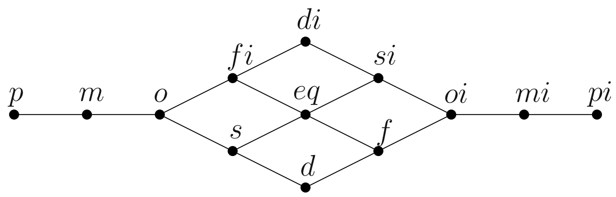


Fig. 3. The conceptual neighborhood graph $\mathsf{GB}_{\mathsf{int}}$ of Interval Algebra

Using conceptual neighborhood graphs a specific basic distance has been defined in the context of QCNs in [6]. The so-called *conceptual neighborhood distance* is formally defined as follows:

Definition 3 (Conceptual neighborhood distance). *Let* GB *be a conceptual neighborhood graph on* B*. The conceptual neighborhood distance* $d_{\mathsf{GB}}(a, b)$ *between two basic relations* $a, b \in \mathsf{B}$ *is the length of the shortest chain leading from* a *to* b *in* GB*.*

In the following examples, we will use the conceptual neighborhood distance $d_{\mathsf{GB}_{\mathsf{int}}}$ defined from the graph $\mathsf{GB}_{\mathsf{int}}$. For instance, $d_{\mathsf{GB}_{\mathsf{int}}}(m, di) = 4$. Notice that $d_{\mathsf{GB}_{\mathsf{int}}}$ is a basic distance in the sense of Definition 2.

Aggregation functions. An aggregation function [18,11,12] typically combines in a given manner several numerical values into a single one.

Definition 4 (Aggregation function). *An aggregation function f associates with a vector of non-negative real numbers a non-negative real number verifying the following properties:*

$$\begin{cases} \textit{if } x_1 \leq x'_1, \ldots, x_p \leq x'_p, \\ \qquad\qquad \textit{then } f(x_1, \ldots, x_p) \leq f(x'_1, \ldots, x'_p) & \textit{(non-decreasingness)} \\ x_1 = \cdots = x_p = 0 \quad \textit{iff} \quad f(x_1, \ldots, x_p) = 0 & \textit{(minimality).} \end{cases}$$

Many aggregation functions have been considered so far in various contexts. For instance, $\sum$ (sum), Max (maximum), $Leximax$[1] are often considered in the belief merging setting [17,21,13,11,14]. We give some additional properties on aggregation functions.

Definition 5 (Properties on aggregation functions). *Let f and g be two aggregation functions.*

- *f is* symmetric *iff for every permutation τ from $\mathbb{R}_0^p$ to $\mathbb{R}_0^p$, p being a positive integer, $f(x_1, \ldots, x_p) = f(\tau(x_1), \ldots, \tau(x_p))$.*
- *f is* associative *iff*

$$f(f(x_1, \ldots, x_p), f(y_1, \ldots, y_{p'})) = f(x_1, \ldots, x_p, y_1, \ldots, y_{p'}).$$

- *f is* strictly non-decreasing *iff if $x_1 \leq x'_1, \ldots, x_p \leq x'_p$ and $\exists i \in \{1, \ldots, p\}$, $x_i < x'_i$, then $f(x_1, \ldots, x_p) < f(x'_1, \ldots, x'_p)$.*
- *f* commutes with *g (or f and g are* commuting aggregation functions*) iff* $f(g(x_{1,1}, \ldots, x_{1,q}), \ldots, g(x_{p,1}, \ldots, x_{p,q})) = g(f(x_{1,1}, \ldots, x_{p,1}), \ldots, f(x_{1,q}, \ldots, x_{p,q}))$.

For example, the aggregation function $\sum$ is symmetric and associative, hence it commutes with itself, as well as the aggregation function Max. Symmetry means that the order of the aggregated values does not affect the result, associativity means that the aggregation of values can be factorized into partial aggregations. In the following, aggregation functions are supposed to be symmetric, i.e., they aggregate multi-sets of numbers instead of vectors of numbers. In [22], the authors focus on commuting aggregation functions since such functions play a significant role in any two-step merging process for which the result should not depend on the order of the aggregation processes. In the end of Section 4.2 we stress the influence of commuting aggregation functions in our merging procedures.

[1] Stricto sensu the *Leximax* function returns the input vector sorted decreasingly w.r.t the standard lexicographic ordering; it turns out that we can associate to *Leximax* an aggregation function in the sense of Definition 4, leading to the same vector ordering as *Leximax* (see [11], Definition 5.1); for this reason, slightly abusing words, we also call this function "*Leximax*".

3 The Merging Issue

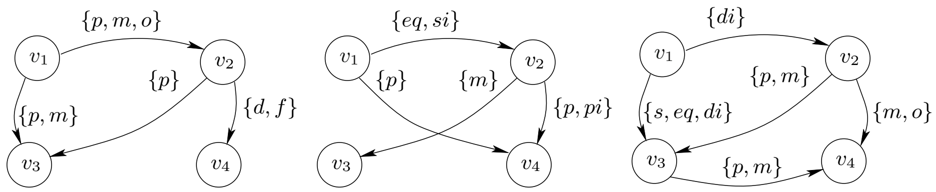
3.1 Problem and Example

Let $V = \{v_1, \ldots, v_n\}$ be a set of variables and $\mathcal{N} = \{N^1, \ldots, N^m\}$ be a multiset of QCNs defined over V. $\mathcal{N}$ is called a *profile*. Every input QCN $N^k \in \mathcal{N}$ stems from a particular agent k providing her own preferences or beliefs about the relative configurations over V. Every constraint $N^k[i,j]$ corresponds to the set of basic relations that agent k considers as possibly satisfied by (v_i, v_j). In such a setting, two kinds of inconsistency are likely to appear. On the one hand, a QCN N^k may be inconsistent since the agent expresses local preferences over pairs of variables. Therefore an inconsistency may arise without the agent being necessarily aware of that. On the other hand, the multiplicity of sources makes that the underlying QCNs are generally conflicting when combined. For example, in case of preferences representation, a single conflict of interest between two agents about the same pair of variables is sufficient to introduce inconsistency.

Consider a group of three students expressing their preferences about the schedule of fours common courses: Operating Systems (OS), Algebra, Analysis and Programming. Every student of the group provides a set of binary relations between these courses. The variables we consider here are four temporal entities v_1, v_2, v_3, v_4 that respectively correspond to OS, Algebra, Analysis, Programming and that form the set V. We consider Interval Algebra to model qualitative relations between these courses. For exemple, the first student prefers to start learning OS before the beginning of Algebra and to finish studying OS before the end of Algebra. This can be expressed by the relation $v_1\ \{p, m, o\}\ v_2$. The three students provide the QCNs N^1, N^2, N^3 depicted in Figure 4 and forming the profile $\mathcal{N}$. Notice that the conflict occurring in the example sketched in the introduction is represented in the QCN N^3, indeed there does not exist any consistent instantiation of N^3 over $\{v_2, v_3, v_4\}$.

3.2 Rationality Postulates for QCN Merging Operators

Given a profile $\mathcal{N} = \{N^1, \ldots, N^m\}$ defined on V representing local preferences or beliefs of a set of agents, we want to get as result of the merging operation a non-empty set of *consistent* information representing $\mathcal{N}$ in a global way. Formally, this calls for a notion of merging operator. In [5] a set of rationality postulates has been proposed for QCN merging operators. These postulates are



(a) N^1 (b) N^2 (c) N^3

Fig. 4. Three QCNs N^1, N^2 and N^3 to be merged

the direct counterparts in the QCN setting of the postulates from [13] characterizing merging operators for propositional logic. We separate these postulates into two classes: the first one defines the QCN merging operators, the second one provides additional properties that QCN merging operators should satisfy to exhibit a rational behaviour.

Definition 6 (QCN merging operator). *An operator Δ is a mapping which associates with a profile $\mathcal{N}$ a set $\Delta(\mathcal{N})$ of consistent scenarios. Let $\mathcal{N}$ be a profile. Δ is a* QCN *merging operator iff it satisfies the following postulates:*

(N1) $\Delta(\mathcal{N}) \neq \emptyset$.
(N2) *If* $\bigcap\{[N^k] \mid N^k \in \mathcal{N}\} \neq \emptyset$, *then* $\Delta(\mathcal{N}) = \bigcap\{[N^k] \mid N^k \in \mathcal{N}\}$.

(N1) ensures that the result of the merging is non-trivial; (N2) requires $\Delta(\mathcal{N})$ to be the set of consistent scenarios shared by all $N^k \in \mathcal{N}$, when this set is non-empty.

Before giving the additional postulates, we need to define the notion of equivalence between profiles. Two profiles $\mathcal{N}$ and $\mathcal{N}'$ are said to be *equivalent*, denoted by $\mathcal{N} \equiv \mathcal{N}'$, iff there exists a one-to-one correspondence f between $\mathcal{N}$ and $\mathcal{N}'$ such that $\forall N^k \in \mathcal{N}$, $f(N^k) \equiv N^k$. We use $\sqcup$ to denote the union operator for multisets.

Definition 7 (postulates (N3) - (N6)). *Let $\mathcal{N}, \mathcal{N}_1$ and $\mathcal{N}_2$ be three profiles, and let N, N' be two consistent* QCNs.

(N3) *If* $\mathcal{N}_1 \equiv \mathcal{N}_2$, *then* $\Delta(\mathcal{N}_1) = \Delta(\mathcal{N}_2)$.
(N4) *If* $\Delta(\{N, N'\}) \cap [N] \neq \emptyset$, *then* $\Delta(\{N, N'\}) \cap [N'] \neq \emptyset$.
(N5) $\Delta(\mathcal{N}_1) \cap \Delta(\mathcal{N}_2) \subseteq \Delta(\mathcal{N}_1 \sqcup \mathcal{N}_2)$.
(N6) *If* $\Delta(\mathcal{N}_1) \cap \Delta(\mathcal{N}_2) \neq \emptyset$, *then* $\Delta(\mathcal{N}_1 \sqcup \mathcal{N}_2) \subseteq \Delta(\mathcal{N}_1) \cap \Delta(\mathcal{N}_2)$.

(N3) is the syntax-irrelevance principle for QCNs. It states that if two profiles are equivalent, then merging independently each profile should lead to the same result. (N4) is an equity postulate, it requires the QCN merging operator not to exploit any hidden preference between two QCNs to be merged. (N5) and (N6) together ensure that when merging independently two profiles leads both results to share a non-empty set of consistent scenarios, let us say E, then merging the joint profiles should return E as result.

4 Two Classes of QCN Merging Operators

In this section, we define two classes of QCN merging operators. Operators from the first and second class are respectively denoted by Δ_1 and Δ_2. These operators associate with a profile $\mathcal{N}$ a set of consistent scenarios that are the “closest” ones to $\mathcal{N}$ in terms of “distance”. The difference between Δ_1 and Δ_2 is inherent to the definition of such a distance.

For $i \in \{1, 2\}$, a QCN merging operator Δ_i is characterized by a triple $(d_{\mathsf{B}}, f_i, g_i)$ where d_{B} is a basic distance on B and f_i and g_i are two symmetric aggregation functions. Δ_i is then denoted by $\Delta_i^{d_{\mathsf{B}}, f_i, g_i}$. The set of consistent scenarios $\Delta_i^{d_{\mathsf{B}}, f_i, g_i}(\mathcal{N})$ is the result of a two-step process.

4.1 Δ_1 Operators

The first step consists in computing a *local distance* d_{f_1} between every consistent scenario on V, i.e., every element of $[N^V_{All}]$ and each QCN of the profile $\mathcal{N}$. For this purpose, the basic distance d_B and the aggregation function f_1 are used to define the distance d_{f_1} between two scenarios σ and σ' of N^V_{All}, as follows:

$$d_{f_1}(\sigma,\sigma') = f_1\{d_\mathsf{B}(\sigma_{ij},\sigma'_{ij}) \mid v_i, v_j \in V, i < j\}.$$

Therefore the distance between two scenarios results from the aggregation of distances at the constraints level. The definition of d_{f_1} is extended in order to compute a distance between a consistent scenario σ of N^V_{All} and a QCN N^k of $\mathcal{N}$ as follows:

$$d_{f_1}(\sigma, N^k) = \min\{d_{f_1}(\sigma,\sigma') \mid \sigma' \in \langle N^k\rangle\}.$$

Therefore the distance between a scenario σ and a QCN N^k is the minimal distance (w.r.t. d_{f_1}) between σ and a scenario of N^k.

The choice of the aggregation function f_1 depends on the context. For example, $f_1 = Max$ is appropriate when only the greatest distance over all constraints between a scenario and a QCN is important, whatever their number. However, by instantiating $f_1 = \sum$, the distances d_B over all constraints are summed up, thus all of them are taken into account.

Example (continued). For the sake of conciseness, we represent a scenario as the list of its constraints following the lexicographical order over (v_i, v_j), $i < j$. For instance, the consistent scenario σ_1 depicted in Figure 5(a) is specified by the list $(\{fi\}, \{m\}, \{p\}, \{m\}, \{p\}, \{m\})$. Let σ'' be the (inconsistent) scenario of N^1 (see Figure 4(a)) defined by $(\{o\}, \{m\}, \{p\}, \{p\}, \{d\}, \{m\})$. We use here the basic distance $d_{\mathsf{GB}_{\mathsf{int}}}$ and will do so for the next examples. We consider $f_1 = \sum$. Then we have:

$$\begin{aligned} d_\sum(\sigma_1, N^1) &= \min\{d_\sum(\sigma_1,\sigma') \mid \sigma' \in \langle N^1\rangle\} = d_\sum(\sigma_1,\sigma'') \\ &= \sum\{d_{\mathsf{GB}_{\mathsf{int}}}(fi, o), d_{\mathsf{GB}_{\mathsf{int}}}(m, m), d_{\mathsf{GB}_{\mathsf{int}}}(p, p), \\ &\qquad d_{\mathsf{GB}_{\mathsf{int}}}(m, p), d_{\mathsf{GB}_{\mathsf{int}}}(p, d), d_{\mathsf{GB}_{\mathsf{int}}}(m, m)\} \\ &= 1 + 0 + 0 + 1 + 4 + 0 = 6. \end{aligned}$$

Similarly we get $d_\sum(\sigma_1, N^2) = 1$ and $d_\sum(\sigma_1, N^3) = 4$.

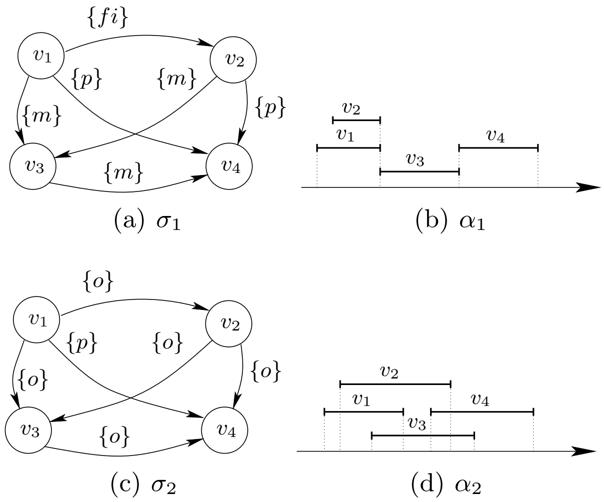
The second step of the merging process consists in taking advantage of the aggregation function g_1 to aggregate the local distances $d_{f_1}(\sigma, N^k)$ for every QCN $N^k \in \mathcal{N}$; the resulting value can be viewed as a *global distance* d_{g_1} between σ and the profile $\mathcal{N}$. This distance is defined as follows:

$$d_{g_1}(\sigma, \mathcal{N}) = g_1\{d_{f_1}(\sigma, N^k) \mid N^k \in \mathcal{N}\}.$$

For the arbitration function $g_1 = Max$, the global distance represents a consensual value w.r.t. all sources [21]; with $g_1 = \sum$, it reflects the majority point of view of the sources [17].

Example (continued). Consider here $g_1 = Max$. We have :

$$d_{Max}(\sigma_1, \mathcal{N}) = \max\{d_\sum(\sigma_1, N^k) \mid N^k \in \mathcal{N}\} = \max\{6, 1, 4\} = 6.$$



(a) σ_1 (b) α_1

(c) σ_2 (d) α_2

Fig. 5. Two consistent scenarios σ_1 and σ_2 of N_{All}^V, and two consistent instantiations α_1 and α_2 of σ_1 and σ_2

The set $\Delta_1^{d_{\mathsf{B}},f_1,g_1}(\mathcal{N})$ is the set of the consistent scenarios of N_{All}^V having a minimal global distance d_{g_1}. Formally,

$$\Delta_1^{d_{\mathsf{B}},f_1,g_1}(\mathcal{N}) = \{\sigma \in [N_{All}^V] \mid \nexists \sigma' \in [N_{All}^V], d_{g_1}(\sigma',\mathcal{N}) < d_{g_1}(\sigma,\mathcal{N})\}.$$

Example (continued). Consider the consistent scenario σ_2 depicted in Figure 5(c). We can compute its global distance similarly as for σ_1. We then have $d_{Max}(\sigma_2,\mathcal{N}) = 5$. Since $d_{Max}(\sigma_2,\mathcal{N}) < d_{Max}(\sigma_1,\mathcal{N})$, we can conclude that the consistent scenario σ_1 does not belong to the set $\Delta_1^{d_{\mathsf{GB}_{\mathsf{int}}},\sum,Max}(\mathcal{N})$.

Proposition 1. $\Delta_1^{d_{\mathsf{B}},f_1,g_1}$ *is a* QCN *merging operator in the sense of Definition 6, i.e., it satisfies postulates (N1) and (N2). Moreover, if* g_1 *is an associative aggregation function, then* $\Delta_1^{d_{\mathsf{B}},f_1,g_1}$ *satisfies (N5), and if* g_1 *is an associative and strictly non-decreasing aggregation function, then* $\Delta_1^{d_{\mathsf{B}},f_1,g_1}$ *satisfies (N6). It does not satisfy (N3) and (N4).*

4.2 Δ_2 Operators

An operator from the Δ_2 family is defined in two steps as follows. The first step consists in computing a *local distance* d_{f_2} between every basic relation of B and the multiset $\mathcal{N}[i,j] = \{N^k[i,j] \mid N^k \in \mathcal{N}\}$, for every pair (v_i, v_j), $i < j$. The definition of the basic distance d_{B} between two basic relations of B is extended to the basic distance between a basic relation $b \in \mathsf{B}$ and a relation $R \in 2^{\mathsf{B}}$, $R \neq \emptyset$. It corresponds to the minimal basic distance between b and every basic relation of R. Formally we write:

$$d_{\mathsf{B}}(b,R) = \min\{d_{\mathsf{B}}(b,b') \mid b' \in R\}.$$

The aggregation function f_2 is used to compute the local distance between every basic relation of B and the multiset of constraints $\mathcal{N}[i,j] = \{N^k[i,j] \mid N^k \in \mathcal{N}\}$ as follows:

$$d_{f_2}(b, \mathcal{N}[i,j]) = f_2\{d_{\mathsf{B}}(b, N^k[i,j]) \mid N^k[i,j] \in \mathcal{N}[i,j]\}.$$

The choice of f_2 is motivated in the same way as that of g_1 for Δ_1 operators. Depending on the context, we opt for a majority function $\sum$ [17], or for an arbitration function Max [21]. Here the aggregation step relates the constraints $N^k[i,j]$ of the QCNs $N^k \in \mathcal{N}$, for a given pair of variables (v_i, v_j), $i < j$.

Example (continued). Consider the multiset $\mathcal{N}[1,2] = \{\{p,m,o\},\{eq,si\},\{di\}\}$ (see Figure 4). We consider $d_{\mathsf{GB}_{\mathsf{int}}}$ as the basic distance and $f_2 = Max$. The distance between the basic relation fi and the multiset $\mathcal{N}[1,2]$ is defined as follows:

$$\begin{aligned} d_{Max}(fi, \mathcal{N}[1,2]) &= \max\{d_{\mathsf{GB}_{\mathsf{int}}}(fi, \{p,m,o\}), \\ &\qquad d_{\mathsf{GB}_{\mathsf{int}}}(fi, \{eq, si\}), d_{\mathsf{GB}_{\mathsf{int}}}(fi, \{di\})\} \\ &= \max\{d_{\mathsf{GB}_{\mathsf{int}}}(fi, o), d_{\mathsf{GB}_{\mathsf{int}}}(fi, eq), d_{\mathsf{GB}_{\mathsf{int}}}(fi, di)\} \\ &= \max\{1,1,1\} = 1. \end{aligned}$$

The second step consists in aggregating the local distances computed in the previous step for all pairs (v_i, v_j), $i < j$, in order to compute a *global distance* d_{g_2} between a scenario σ of N^V_{All} and the profile $\mathcal{N}$. This distance is computed using the aggregation function g_2 as follows:

$$d_{g_2}(\sigma, \mathcal{N}) = g_2\{d_{f_2}(\sigma_{ij}, \mathcal{N}[i,j]) \mid v_i, v_j \in V, i < j\}.$$

The choice of g_2 is motivated in the same way as the aggregation function f_1 for Δ_1 operators.

Example (continued). Consider again the consistent scenario σ_1 (see Figure 5(a)) and choose $g_2 = \sum$. We get:

$$\begin{aligned} d_{\sum}(\sigma_1, \mathcal{N}) &= \textstyle\sum\{d_{Max}(\sigma_1(1,2), \mathcal{N}[1,2]), \ldots, d_{Max}(\sigma_1(3,4), \mathcal{N}[3,4])\} \\ &= 1+2+0+1+4+0 = 8. \end{aligned}$$

Similarly to Δ_1 operators, the result of the merging process over the profile $\mathcal{N}$ using $\Delta_2^{d_{\mathsf{B}}, f_2, g_2}$ corresponds to the set of consistent scenarios of N^V_{All} that minimize the global distance d_{g_2}. Formally,

$$\Delta_2^{d_{\mathsf{B}}, f_2, g_2}(\mathcal{N}) = \{\sigma \in [N^V_{All}] \mid \nexists \sigma' \in [N^V_{All}], d_{g_2}(\sigma', \mathcal{N}) < d_{g_2}(\sigma, \mathcal{N})\}.$$

Example (continued). Consider again the consistent scenario σ_2 depicted in Figure 5(c). Its global distance to $\mathcal{N}$, computed similarly to the one of σ_1, is $d_{\sum}(\sigma_2, \mathcal{N}) = 8$. Notice that the consistent scenarios σ_1 and σ_2 have the same global distance to $\mathcal{N}$. We can then conclude that $\sigma_1 \in \Delta_2^{d_{\mathsf{GB}_{\mathsf{int}}}, Max, \sum}(\mathcal{N})$ iff $\sigma_2 \in \Delta_2^{d_{\mathsf{GB}_{\mathsf{int}}}, Max, \sum}(\mathcal{N})$.

One can prove that Δ_2 operators typically satisfies less expected postulates than the Δ_1 ones:

Proposition 2. *$\Delta_2^{d_B,f_2,g_2}$ is a* QCN *merging operator in the sense of Definition 6, i.e., it satisfies the postulates (N1) and (N2). The postulates (N3) - (N6) are not satisfied.*

That Δ_1 and Δ_2 are syntactical operators is reflected by the fact that they do not satisfy the syntax-independence postulate (N3) (see Propositions 1 and 2). Similarly in [10] several syntax-sensitive propositional merging operators have been investigated, none of them satisfying the counterpart of (N3) in the propositional setting. We give some conditions under which Δ_1 and Δ_2 operators are equivalent.

Proposition 3. *If $f_1 = g_2$, $f_2 = g_1$ and f_1 and f_2 are commuting aggregation functions, then $\Delta_1^{d_B,f_1,g_1}(\mathcal{N}) = \Delta_2^{d_B,f_2,g_2}(\mathcal{N})$.*

Consequently, when $f_1 = g_2$, $f_2 = g_1$ and for instance when $(f_1, f_2) \in \{(\sum, \sum),$ $(Max, Max)\}$, then choosing a Δ_1 operator rather than a Δ_2 one (or conversely) has no impact on the result. However, $\sum$ and Max are not commuting aggregation functions, so for such choices using Δ_1 or Δ_2 can lead to different results.

4.3 Computational Complexity

Beyond logical postulates, complexity considerations can be used as choice criteria for a QCN merging operator. Clearly enough, the merging result may be of exponential size in the worst case, just like representation of the merging result in the propositional case [3,11,12]. As discussed in [6], a set of consistent scenarios cannot always be represented by a single QCN. In [6] a basic construction of a QCN N_S is given from a set S of consistent scenarios leading to $S = [N_S]$ when possible. Nevertheless, computing explicitly the merging result (as a set of consistent scenarios in our setting, as a propositional formula in the propositional framework) is not mandatory to reason with [3,11,12]; often it is enough to be able to determine whether a given scenario belongs to it. This is why we focus on the following MEMBERSHIP problem (MS for short): given $i \in \{1, 2\}$, d_B a basic distance, f_i, g_i two aggregation functions, $\mathcal{N}$ a profile and σ_* a scenario, does σ_* belong to $\Delta_i^{d_B,f_i,g_i}(\mathcal{N})$? The following proposition provides an upper bound of complexity for the MS problem.

Proposition 4. *If f_i, g_i are computed in polynomial time, then* $\mathsf{MS} \in$ **coNP**.

Interestingly, usual aggregation functions like $\sum$ or Max can be computed in polynomial time. For the merging procedure proposed in [6], MS is likely harder, i.e., falls to a complexity class above **coNP** in the polynomial hierarchy. Indeed for both Δ_1 and Δ_2 operators, the global distance between a scenario and a profile is computed in polynomial time. In comparison, for the merging operator proposed in [6], computing the global distance between a scenario and a profile requires the computation of all consistent scenarios of every QCN of the profile, which are exponentially many in the worst case.

5 Comparison between Δ_1, Δ_2 and Related Works

5.1 When to Choose a Δ_1 Operator

Given a profile $\mathcal{N}$, opting for an operator Δ_1 is appropriate when the sources are independent of one another, i.e., when information provided by each QCN of the profile should be treated independently. Indeed the first aggregation step is "local" to a particular QCN, while the second one is an "inter-source" aggregation. In this respect, Δ_1 operators are close to QCN merging operators Θ proposed in [6] and propositional merging operators $\mathbf{DA}^2$ studied in [11,12]. In [6] the QCN merging operators Θ consider like Δ_1 operators a profile $\mathcal{N}$ as input and return a set of consistent scenarios following a similar two-step process, with only $f_1 = \sum$. However, while Δ_1 operators consider the sets of scenarios of the QCNs of $\mathcal{N}$ in the computation of the local distance d_{f_1}, Θ operators consider the sets of their *consistent* scenarios. Doing so, neither inconsistent QCNs of $\mathcal{N}$ are taken into account by Θ operators, nor the basic relations of the constraints of the QCNs which do not participate in any consistent scenario of this QCN. In [11,12] the authors define a class $\mathbf{DA}^2$ of propositional knowledge bases merging operators, based on a distance between interpretations and two aggregation functions. A profile corresponds in this case to a multiset of knowledge bases, each one expressed as a finite set of propositional formulas. A first step consists in computing a local distance between an interpretation ω and a knowledge base K through the aggregation of the distances between ω and every propositional formula of K. A second step then consists in aggregating the local distances to combine all knowledge bases of the profile. In the context of QCN merging, the Δ_1 operators typically follow the same merging principle.

5.2 When to Choose a Δ_2 Operator

Δ_2 operators are suited to the context when a global decision should be made *a priori* for every pair of variables (v_i, v_j). In this case every pair of variables is considered as a "criterion' or "topic" on which a mutual agreement has to be found as a first step. The second step then can be viewed as a relaxation of the independence criteria which are combined in order to find consistent global configurations. Δ_2 operators consider a local distance d_{f_2} which coincides with the one proposed in [4]. In this work, the authors use this local distance d_{f_2} to define a *constraint merging operator*. Such an operator associates with a multiset $\mathcal{R}$ of relations the set of basic relations for which the distance d_{f_2} to $\mathcal{R}$ is minimal. In this framework, a QCN merging operator, denoted by Ω, associates with a profile $\mathcal{N}$ a single QCN $\Omega(\mathcal{N})$. Similarly to Δ_2 operators, Ω operators take into consideration inconsistent QCNs and consider every basic relation of all constraints of the input QCNs as a relevant piece of information in the merging process. However, Ω operators require to be given a fixed total ordering $<_V$ on the pairs of variables (v_i, v_j). Following this ordering, the constraint of the QCN $\Omega(\mathcal{N})$ bearing on (v_i, v_j) is affected using the constraint merging operator on the constraints of the QCNs of $\mathcal{N}$ bearing on (v_i, v_j). At each step, $\Omega(\mathcal{N})$ is kept consistent. Though the computation of $\Omega(\mathcal{N})$ is efficient, the choice of $<_V$ leads to specific results, while Δ_2 operators - which do not require $<_V$ to be specified - do not suffer from this drawback.

6 Conclusion

In this paper, we have defined two classes Δ_1 and Δ_2 of operators for merging qualitative constraint networks (QCNs) defined on the same qualitative formalism. We have studied their logical properties and we have also considered the problem of deciding whether a given scenario belongs to the result of the merging. From a methodology point of view, we have addressed the problem of choosing such a merging operator. Compared with previous merging operators, Δ_1 and Δ_2 operators achieve a good compromise to QCN merging. Indeed, (i) they take into account fine-grained information provided by the input sources in the sense that each constraint from the input QCNs participates in the merging process (in particular inconsistent scenarios are not excluded); (ii) the computational complexity of query answering for those operators is not very high; (iii) they are QCN merging operators since rationality postulates (N1) and (N2) hold. Interestingly, our operators do not trivialize when applied to a single inconsistent QCN; as such, they can also be viewed as consistency restoring operators.

As a matter for further research, we plan to investigate in depth the complexity issues for all classes of operators defined so far.

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A class of $\diamond_f$-consistencies for qualitative constraint networks

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A Class of $\overset{\diamond}{f}$-consistencies for Qualitative Constraint Networks

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Abstract

In this paper, we introduce a new class of local consistencies, called $\overset{\diamond}{f}$-consistencies, for qualitative constraint networks. Each consistency of this class is based on weak composition ($\diamond$) and a mapping f that provides a covering for each relation. We study the connections existing between some properties of mappings f and the relative inference strength of $\overset{\diamond}{f}$-consistencies. The consistency obtained by the usual closure under weak composition is shown to be the weakest element of the class, and new promising perspectives are shown to be opened by $\overset{\diamond}{f}$-consistencies stronger than weak composition. We also propose a generic algorithm that allows us to compute the closure of qualitative constraint networks under any "well-behaved" consistency of the class. The experimentation that we have conducted on qualitative constraint networks from the Interval Algebra shows the interest of these new local consistencies, in particular for the consistency problem.

Introduction

Qualitative Spatial-Temporal Reasoning (QSTR) is an area of computer science dealing with qualitative information about configurations of spatial/temporal entities. A calculus in QSTR introduces particular elements for representing the entities and a finite set of base relations on these elements. Each base relation is an abstraction of concrete metric information about the relative position of entities. For applications in domains such as e.g. geographic information systems and natural language understanding, a qualitative description can reveal to be far more appropriate than a metric description, in particular when precise information is not necessary or simply not available. In the past twenty years, numerous QSTR formalisms have been proposed and studied; see e.g. (Randell, Cui, and Cohn 1992; Pujari, Kumari, and Sattar 1999; Renz and Nebel 2007).

In QSTR, Qualitative Constraint Networks (QCNs) are typically used to express information on spatial/temporal situations. A constraint represents a set of acceptable qualitative configurations between some variables (entities), and is then defined by a set of base relations. Given a QCN, the main problem is to determine whether the information contained in the QCN is consistent. In the general case, this problem is NP-hard. However, because the worst-case only arises within a limited range of situations, many studies have been led to develop efficient practical approaches to solve this problem.

One such approach is backtrack search combined with a constraint propagation mechanism based on tractable subclasses of relations and the closure of QCNs under weak composition, which is an operation denoted by $\diamond$ and related to path consistency (Mackworth 1977). More precisely, at each step of search, a constraint is split into relations belonging to a tractable class and closure under weak composition is an inference method applied to filter the search space (i.e. to reduce its size) by removing some inconsistent base relations. This effective approach, initiated by Nebel, has been adopted by most of the qualitative constraint solvers (Condotta, Saade, and Ligozat 2006; Gantner, Westphal, and Wolfl 2008), and in particular by GQR*, which is currently the fastest solver. On the other hand, some recent approaches (Pham, Thornton, and Sattar 2006; Westphal and Wölfl 2009; Li, Huang, and Renz 2009) translate the consistency problem of QCNs into CSP (Constraint Satisfaction Problem) or SAT (propositional satisfiability) instances. Published results indicate that these approaches are promising.

Closure under weak composition is at the heart of the various approaches that directly handle qualitative constraint networks. It was the first inference method used to address the consistency problem of the temporal QCNs in the well-known Interval Algebra (Allen 1981). Weak composition is currently recognized as an operation that offers a good balance between the execution overhead and the filtering benefit. Besides, it has been shown to be a complete approach for most of the identified tractable classes. Nevertheless, for the hardest QCNs it may be worthwhile to consider operations stronger than $\diamond$, i.e. stronger forms of local consistency.

In this paper, we propose a new class of local consistencies adapted to qualitative calculi. Each of them is defined from $\diamond$ and a mapping f that associates with every relation r of a qualitative calculus a set of sub-relations of r forming a covering of r. Intuitively, a QCN is $\overset{\diamond}{f}$-consistent if and only if after substituting any sub-relation defined by f for the relation associated with a constraint of the QCN, the

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obtained QCN is closed under $\diamond$. We prove that $\diamond$ corresponds to the weakest consistency of the class whereas a local consistency similar to SAC, Singleton Arc Consistency (Debruyne and Bessiere 1997) introduced for CSP, is the strongest one. Other consistencies of the class are situated between these two bounds since the class forms a complete lattice. We also characterize an important subset of the class of $\overset{\diamond}{f}$-consistencies that contains consistencies under which closure of QCNs exists, and we propose a general-purpose algorithm to enforce any of them. A preliminary experimentation carried out using the Interval Algebra shows promising results.

Preliminaries

A qualitative calculus is defined from a finite set B of base relations on a domain D. Without any loss of generality, we will only consider binary relations. The elements of D represent temporal or spatial entities, and the elements of B represent all possible configurations between two entities. B is a set that satisfies the following properties (Ligozat and Renz 2004): B forms a partition of $\mathsf{D} \times \mathsf{D}$, B contains the identity relation Id, and B is closed under the converse operation ($^{-1}$). A (complex) relation is the union of some base relations, but it is customary to represent a relation as the set of base relations contained in it. Hence, the set 2^{B} will represent the set of relations of the qualitative calculus. The set 2^{B} is equipped with the weak composition operation, denoted by $\diamond$ and defined by: $\forall a, b \in \mathsf{B}, a \diamond b = \{c \in \mathsf{B} : \exists x, y, z \in \mathsf{D} \mid x\ a\ z \wedge z\ b\ y \wedge x\ c\ y\}$; $\forall r, s \in 2^{\mathsf{B}}, r \diamond s = \bigcup_{a \in r, b \in s}\{a \diamond b\}$. Note that $r \diamond s$ is the smallest relation of 2^{B} including the usual relational composition $r \circ s = \{(x, y) \in \mathsf{D} \times \mathsf{D} : \exists z \in \mathsf{D} \mid x\ r\ z \wedge z\ s\ y\}$. In some qualitative algebras (e.g. the Interval Algebra introduced below), $r \circ s$ and $r \diamond s$ are identical.

A well known temporal qualitative formalism is the Interval Algebra, also called Allen's calculus (Allen 1981). The domain $\mathsf{D}_{\mathsf{int}}$ of this calculus is the set $\{(x^-, x^+) \in \mathbb{Q} \times \mathbb{Q} : x^- < x^+\}$ since temporal entities are intervals of the rational line. The set $\mathsf{B}_{\mathsf{int}}$ of this calculus is the set $\{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$ of thirteen binary relations representing all orderings of the four endpoints of two intervals; see Figure 1.

Relation	Symbol	Inverse	Meaning
precedes	p	pi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

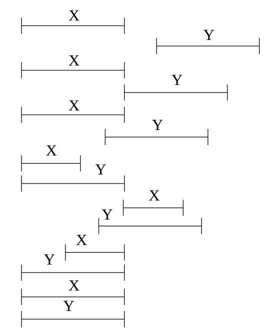


Figure 1: Base relations of the Interval Algebra.

A Qualitative Constraint Network (QCN) is a pair composed of a set of variables and a set of constraints. Each variable represents a spatial/temporal entity of the system that is modelled. Each constraint represents a set of acceptable qualitative configurations between two variables and is defined by a relation. Formally, a QCN is defined as follows:

Definition 1 *A* QCN *is a pair* $\mathcal{N} = (V, C)$ *where:*

- $V = \{v_1, \dots, v_n\}$ *is a finite set of n variables;*
- C *is a mapping that associates a relation* $C(v_i, v_j) \in 2^{\mathsf{B}}$, *also denoted by* C_{ij} *or* $\mathcal{N}[i, j]$, *with each pair* (v_i, v_j) *of* $V \times V$. C *is such that* $C_{ii} \subseteq \{\mathsf{Id}\}$ *and* $C_{ij} = C_{ji}^{-1}$.

A *partial solution* of $\mathcal{N}$ on $V' \subseteq V$ is a mapping σ defined from V' to D such that for every pair (v_i, v_j) of variables in V', $(\sigma(v_i), \sigma(v_j))$ satisfies C_{ij}, i.e. there exists a base relation $b \in C_{ij}$ such that $(\sigma(v_i), \sigma(v_j)) \in b$. A *solution* of $\mathcal{N}$ is a partial solution of $\mathcal{N}$ on V. $\mathcal{N}$ is *consistent* iff it admits a solution. Two QCNs are *equivalent* iff they admit the same set of solutions. A *sub*QCN $\mathcal{N}'$ of $\mathcal{N}$, denoted by $\mathcal{N}' \subseteq \mathcal{N}$, is a QCN (V, C') such that $C'_{ij} \subseteq C_{ij}$, for every pair (v_i, v_j) of variables. An *atomic* QCN is a QCN such that each constraint is defined by a base relation. A *scenario* $\mathcal{S}$ of $\mathcal{N}$ is an atomic consistent subQCN of $\mathcal{N}$. A base relation for C_{ij} is *inconsistent* iff there does not exist any scenario $\mathcal{S}$ of $\mathcal{N}$ such that $\mathcal{S}[i, j] = \{b\}$.

A QCN $\mathcal{N} = (V, C)$ is said to be *$\diamond$-consistent* or *closed under weak composition* if and only if $C_{ij} \subseteq C_{ik} \diamond C_{kj}$ $\forall v_i, v_j, v_k \in V$. The *closure* under weak composition of $\mathcal{N}$, denoted by $\diamond(\mathcal{N})$, is the greatest (w.r.t. $\subseteq$) $\diamond$-consistent subQCN of $\mathcal{N}$; $\diamond(\mathcal{N})$ is equivalent to $\mathcal{N}$. This (sub)QCN can be obtained by iterating the triangulation operation:

$$C_{ij} \leftarrow C_{ij} \cap (C_{ik} \diamond C_{kj}), \forall v_i, v_j, v_k \in V$$

until a fixed point is reached. This method can be implemented by an algorithm running in $O(n^3)$ time. Weak composition admits the following properties:

- $\diamond(\mathcal{N}) \subseteq \mathcal{N}$ ($\diamond$ is contracting),
- $\diamond(\diamond(\mathcal{N})) = \diamond(\mathcal{N})$ ($\diamond$ is idempotent),
- $\mathcal{N} \subseteq \mathcal{N}' \Rightarrow \diamond(\mathcal{N}) \subseteq \diamond(\mathcal{N}')$ ($\diamond$ is monotonic).

$\mathcal{N}_{[i,j]/r}$, with $v_i, v_j \in V$ and $r \in 2^{\mathsf{B}}$, is the QCN (V, C') defined by $C'_{ij} = r$, $C'_{ji} = r^{-1}$ and $C'_{kl} = C_{kl}$ $\forall (v_k, v_l) \in V \times V \setminus \{(v_i, v_j), (v_j, v_i)\}$. The union of two QCNs $\mathcal{N} = (V, C)$ and $\mathcal{N}' = (V, C')$ is the QCN $\mathcal{N} \cup \mathcal{N}' = (V, C'')$ such that $\forall (v_i, v_j) \in V$, $C''_{ij} = C_{ij} \cup C'_{ij}$.

The Class of $\overset{\diamond}{f}$-consistencies

In this section, we introduce (for qualitative constraint networks) a general class of local consistencies, called $\overset{\diamond}{f}$-consistencies, where f is a mapping that associates a set of relations of 2^{B} with each relation of 2^{B}. Intuitively, a QCN is said to be $\overset{\diamond}{f}$-consistent iff for any constraint C_{ij} of the QCN, after substituting any element r' of $f(r)$ for the relation r associated with C_{ij} and computing the closure under weak composition, the relation r' associated with C_{ij} is let unchanged. Before proposing a formal definition of $\overset{\diamond}{f}$-consistencies, we introduce a set $\mathcal{F}$ that exactly contains the mappings f considered hereafter. More precisely, $\mathcal{F}$ is the set of mappings f defined from 2^{B} to $2^{2^{\mathsf{B}}}$ associating a set of relations $f(r) \in 2^{2^{\mathsf{B}}}$ with each relation $r \in 2^{\mathsf{B}}$ such that $\bigcup f(r) = r$, and $\emptyset \notin f(r)$ when $r \neq \emptyset$. Note that $f(r)$ is a covering of r and $f(\{b\}) = \{\{b\}\} \forall b \in \mathsf{B}$. Moreover, we have $f(\emptyset) = \{\emptyset\}$, which will be always implicitly assumed whenever we introduce a mapping later.

Definition 2 *Let f be an element of $\mathcal{F}$. A QCN $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent iff for every pair (v_i, v_j) of variables of $\mathcal{N}$ and for every $s \in f(\mathcal{N}[i,j])$, $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$.*

We obtain a new class (or family) of local consistencies since each mapping $f \in \mathcal{F}$ determines a new consistency denoted by $\overset{\diamond}{f}$. The class (set) of all $\overset{\diamond}{f}$-consistencies that can be built from elements of $\mathcal{F}$ is denoted by $\overset{\diamond}{\mathcal{F}}$. The following result shows the practical interest of the new class of consistencies: when a QCN is not $\overset{\diamond}{f}$-consistent, some base relations said to be $\overset{\diamond}{f}$-inconsistent can be identified and safely removed.

Proposition 1 *Let f be an element of $\mathcal{F}$, $\mathcal{N}$ be a QCN, (v_i, v_j) be a pair of variables of $\mathcal{N}$ and $s \in f(\mathcal{N}[i,j])$. Any base relation b in $s \setminus \diamond(\mathcal{N}_{[i,j]/s})[i,j]$ is inconsistent for C_{ij}.*

Proof. Let $\mathcal{S}$ be a scenario of $\mathcal{N}$ and b' be the base relation in $\mathcal{S}[i,j]$. Either we have $b' \notin s$ or $b' \in s$. If $b' \notin s$, necessarily $b' \neq b$. On the other hand, if $b' \in s$ then $b' \in \diamond(\mathcal{N}_{[i,j]/s})[i,j]$ because $\diamond$ preserves scenarios. By hypothesis, $b \notin \diamond(\mathcal{N}_{[i,j]/s})[i,j]$, which proves that $b' \neq b$. We conclude that $v_i \; b \; v_j$ cannot be true in any scenario. $\dashv$

The following mappings will be useful to illustrate our purpose. $\forall r \in 2^{\mathsf{B}} \setminus \{\emptyset\}$:

- f_{B} associates the set $f_{\mathsf{B}}(r) = \{\{b\} : b \in r\}$ with r.
- $f_{\neq}$ associates the set $f_{\neq}(r) = \{r \setminus \{b\} : b \in r\}$ with r iff $|r| > 1$; $f_{\neq}(r) = \{r\}$ otherwise.
- $f_{\diamond}$ associates the set $f_{\diamond}(r) = \{r\}$ with r.

For example, if $r = \{p, m, o\}$, then $f_{\mathsf{B}}(r) = \{\{p\}, \{m\}, \{o\}\}$ and $f_{\neq}(r) = \{\{p, m\}, \{p, o\}, \{m, o\}\}$. Moreover, given a partition $P = \{r_1, \ldots, r_k\}$ of B, the mapping f_P is defined as follows: for every relation $r \in 2^{\mathsf{B}}$, $f(r) = \{r \cap r_i : i \in \{1, \ldots, k\}\} \setminus \{\emptyset\}$. Note that $\overset{\diamond}{f_{\mathsf{B}}}$ is a consistency that can be related to SAC (introduced for CSP) but $\overset{\diamond}{f_{\neq}}$ and partition-based consistencies $\overset{\diamond}{f_P}$ (as well as many other $\overset{\diamond}{f}$-consistencies) have no CSP counterpart.

We will consider later the following (representative) partitions of $\mathsf{B}_{\mathsf{int}}$:

- $P_1 = \{\{p, m, o, fi, s, d\}, \{pi, mi, oi, f, si, di, eq\}\}$
- $P_2 = \{\{p, m, o\}, \{fi, s, d\}, \{pi, mi, oi\}, \{f, si, di, eq\}\}$
- $P_3 = \{\{p\}, \{m, o\}, \{fi\}, \{s, d\}, \{pi\}, \{mi, oi\}, \{f, eq\}, \{si, di\}\}$

In order to compare the inference capability of different consistencies, we need to introduce a preorder. Let ϕ and ψ be two consistencies in $\overset{\diamond}{\mathcal{F}}$, *$\phi$ is stronger than ψ*, denoted by $\phi \trianglerighteq \psi$, iff whenever ϕ holds on a QCN $\mathcal{N}$ (i.e. $\mathcal{N}$ is ϕ-consistent), ψ also holds on $\mathcal{N}$; ϕ is *strictly stronger than* ψ, denoted by $\phi \triangleright \psi$, iff ϕ is stronger than ψ and there exists at least one QCN $\mathcal{N}$ such that ψ holds on $\mathcal{N}$ but not ϕ. Finally, ϕ and ψ are *equivalent*, denoted by $\phi \approx \psi$, iff both $\phi \trianglerighteq \psi$ and $\psi \trianglerighteq \phi$.

First, we can show that a QCN $\mathcal{N}$ is $\overset{\diamond}{f_{\diamond}}$-consistent if, and only if, $\mathcal{N}$ is closed under weak composition.

Proposition 2 *The consistency $\overset{\diamond}{f_{\diamond}}$ is equivalent to $\diamond$.*

Proof. $\mathcal{N}$ is $\diamond$-consistent $\Leftrightarrow \diamond(\mathcal{N}) = \mathcal{N} \Leftrightarrow$ for every pair (v_i, v_j) of variables of $\mathcal{N}$, $\diamond(\mathcal{N})[i,j] = \mathcal{N}[i,j] \Leftrightarrow$ for every pair (v_i, v_j) of variables of $\mathcal{N}$ and for every $s \in f_{\diamond}(\mathcal{N}[i,j])$, $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$ (because $f_{\diamond}(r) = \{r\}$ for each relation $r \in 2^{\mathsf{B}}$) $\Leftrightarrow \mathcal{N}$ is $\overset{\diamond}{f_{\diamond}}$-consistent $\dashv$

The finer the coverings of relations by an element f of $\mathcal{F}$ are, the stronger the consistency $\overset{\diamond}{f}$ is. In particular, to relate $\overset{\diamond}{f}$-consistencies, we have the following result:

Proposition 3 *Let f, f' be two elements of $\mathcal{F}$. If for every $r \in 2^{\mathsf{B}}$ and for every $s' \in f'(r)$, there exists a set of relations $S \subseteq f(r)$ such that $s' = \bigcup S$, then $\overset{\diamond}{f} \trianglerighteq \overset{\diamond}{f'}$.*

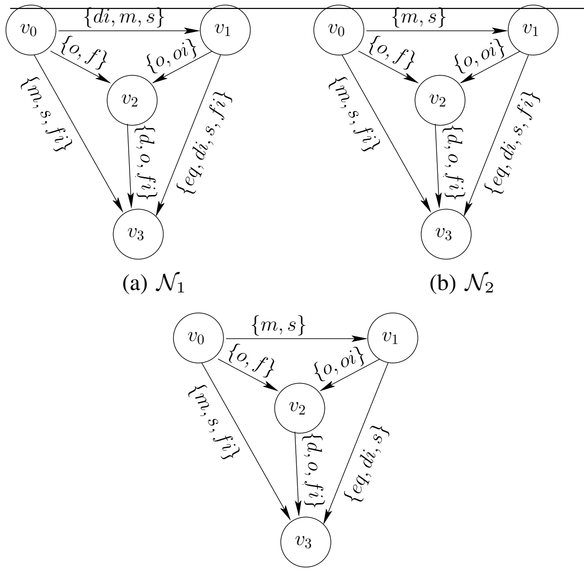
Proof. We suppose that we have a QCN $\mathcal{N}$ that is $\overset{\diamond}{f}$-consistent. Let v_i, v_j be two variables of $\mathcal{N}$, $r = \mathcal{N}[i,j]$ and s' be an element of $f'(r)$. By hypothesis, there exists a set of relations $S \subseteq f(r)$ such that $s' = \bigcup S$. For every relation $s \in S$ we have $s \subseteq s'$, and because $\mathcal{N}_{[i,j]/s} \subseteq \mathcal{N}_{[i,j]/s'}$ and $\diamond$ is monotonic, we have $\diamond(\mathcal{N}_{[i,j]/s})[i,j] \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$. We can deduce that $\bigcup\{\diamond(\mathcal{N}_{[i,j]/s})[i,j] : s \in S\} \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$. Since $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent (by hypothesis), for every relation $s \in S$, we have $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$. Hence, $\bigcup S \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$, and as $s' = \bigcup S$, we obtain $s' \subseteq \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$. On the other hand, we also know that $\diamond(\mathcal{N}_{[i,j]/s'})[i,j] \subseteq s'$ because $\diamond$ is contracting. We can conclude that $s' = \diamond(\mathcal{N}_{[i,j]/s'})[i,j]$ and consequently that $\mathcal{N}$ is $\overset{\diamond}{f'}$-consistent. $\dashv$

For example, for the Interval Algebra, we have $\overset{\diamond}{f_{\mathsf{B}}} \trianglerighteq \overset{\diamond}{f_{P_3}} \trianglerighteq \overset{\diamond}{f_{P_2}} \trianglerighteq \overset{\diamond}{f_{P_1}} \trianglerighteq \overset{\diamond}{f_{\diamond}}$. The following corollary stipulates that $\overset{\diamond}{f_{\mathsf{B}}}$ is the strongest consistency (of $\overset{\diamond}{\mathcal{F}}$) and $\overset{\diamond}{f_{\diamond}}$ is the weakest one.

Corollary 1 *For every element $f \in \mathcal{F}$, $\overset{\diamond}{f_{\mathsf{B}}} \trianglerighteq \overset{\diamond}{f} \trianglerighteq \overset{\diamond}{f_{\diamond}}$.*

From this result, we can deduce in particular that $\overset{\diamond}{f_{\mathsf{B}}} \trianglerighteq \overset{\diamond}{f_{\neq}} \trianglerighteq \overset{\diamond}{f_{\diamond}}$. Now, let us consider the three QCNs of the Interval Algebra depicted in Figure 2. On each of these graphs, a variable is represented by a node, and a constraint by an arc labelled with the associated relation; note that, for simplicity, there is no arc going from v_i to v_j when either there is already an arc going from v_j to v_i or $i = j$. We can check that $\mathcal{N}_1$ is $\overset{\diamond}{f_{\diamond}}$-consistent but not $\overset{\diamond}{f_{\neq}}$-consistent because $di \notin \diamond(\mathcal{N}_{1[0,1]/\{di,m\}})[0,1]$, $\mathcal{N}_2$ is $\overset{\diamond}{f_{\neq}}$-consistent but not $\overset{\diamond}{f_{\mathsf{B}}}$-consistent because $\diamond(\mathcal{N}_{2[1,3]/\{fi\}})[1,3] = \emptyset$, and $\mathcal{N}_3$ is $\overset{\diamond}{f_{\mathsf{B}}}$-consistent.

From Corollary 1 and QCNs $\mathcal{N}_1$ and $\mathcal{N}_2$, we deduce that (for the Interval Algebra) $\overset{\diamond}{f_{\mathsf{B}}} \triangleright \overset{\diamond}{f_{\neq}} \triangleright \overset{\diamond}{f_{\diamond}}$ (note the strict order).



(a) $\mathcal{N}_1$ (b) $\mathcal{N}_2$ (c) $\mathcal{N}_3$

Figure 2: $\mathcal{N}_3 \subset \mathcal{N}_2 \subset \mathcal{N}_1$.

The equivalence classes of $\approx$ form a partition of $\diamond_{\mathcal{F}}$; the set of all equivalence classes is denoted by $\diamond_{\mathcal{F}}|^{\approx}$. Note that $\diamond_{\mathcal{F}}|^{\approx}$ is a finite set since the set B of base relations is considered to be finite. The relation $\trianglerighteq^{\approx}$ defined on $\diamond_{\mathcal{F}}|^{\approx}$ by $\forall[\phi], [\psi] \in \diamond_{\mathcal{F}}|^{\approx}$, $[\phi] \trianglerighteq^{\approx} [\psi]$ iff $\phi \trianglerighteq \psi$ where ϕ and ψ are any representatives (elements) in $[\phi]$ and $[\psi]$, is a partial order. We have the following result:

Proposition 4 *$(\diamond_{\mathcal{F}}|^{\approx}, \trianglerighteq^{\approx})$ is a complete lattice with $[\diamond_{f_{\mathsf{B}}}]$ as greatest element and $[\diamond_{f_\diamond}]$ as least element.*

Proof.
(Existence of binary joins) Let $\diamond_{f_1}$ and $\diamond_{f_2}$ be two elements of $\diamond_{\mathcal{F}}$, and let us define f as $\forall r \in 2^{\mathsf{B}}$, $f(r) = f_1(r) \cup f_2(r)$. First, we can observe that $f \in \mathcal{F}$ by construction. From Proposition 3, we deduce that $\diamond_f \trianglerighteq \diamond_{f_1}$ and $\diamond_f \trianglerighteq \diamond_{f_2}$. Now, suppose that there exists $f' \in \mathcal{F}$ such that $\diamond_{f'} \trianglerighteq \diamond_{f_1}$ and $\diamond_{f'} \trianglerighteq \diamond_{f_2}$. By definition, any $\diamond_{f'}$-consistent QCN $\mathcal{N}$ is $\diamond_{f_1}$-consistent and $\diamond_{f_2}$-consistent. Hence, for every pair (v_i, v_j) of variables of $\mathcal{N}$, $s \in f_1(\mathcal{N}[i,j]) \Rightarrow \diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$ and $s \in f_2(\mathcal{N}[i,j]) \Rightarrow \diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$. So, for every $s \in f_1(\mathcal{N}[i,j]) \cup f_2(\mathcal{N}[i,j])$, $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$. We deduce that $\mathcal{N}$ is $\diamond_f$-consistent, and $\diamond_{f'} \trianglerighteq \diamond_f$. $[\diamond_f]$ is the least upper bound of $[\diamond_{f_1}]$ and $[\diamond_{f_2}]$.
(Existence of binary meets) Let $\diamond_{f_1}$ and $\diamond_{f_2}$ be two elements of $\diamond_{\mathcal{F}}$, and let us define the set E as $E = \{f' \in \mathcal{F} : \diamond_{f_1} \trianglerighteq \diamond_{f'} \wedge \diamond_{f_2} \trianglerighteq \diamond_{f'}\}$. Note that $E \neq \emptyset$ since $f_\diamond \in E$. Next, let us define f as $\forall r \in 2^{\mathsf{B}}$, $f(r) = \bigcup\{f'(r) : f' \in E\}$. From this definition and Proposition 3, we deduce that $\diamond_f \trianglerighteq \diamond_{f'}$ for every $f' \in E$. We now prove by contradiction that $\diamond_{f_1} \trianglerighteq \diamond_f$ and $\diamond_{f_2} \trianglerighteq \diamond_f$. Let us suppose that $\diamond_{f_1} \trianglerighteq \diamond_f$ does not hold. This means that there exists a $\diamond_{f_1}$-consistent QCN $\mathcal{N}$ that is not $\diamond_f$-consistent. Hence, there exist two variables v_i, v_j of $\mathcal{N}$ such that $\diamond(\mathcal{N}_{[i,j]/s})[i,j] \neq s$ with $s \in f(\mathcal{N}[i,j])$. From construction of f, we know that there exists a mapping $f' \in E$ such that $s \in f'(\mathcal{N}[i,j])$. Hence, $\mathcal{N}$ is not $\diamond_{f'}$-consistent. On the other hand, as $f' \in E$ we have $\diamond_{f_1} \trianglerighteq \diamond_{f'}$, and $\mathcal{N}$ $\diamond_{f'}$-consistent since $\mathcal{N}$ is $\diamond_{f_1}$-consistent. This is a contradiction, so $\diamond_{f_1} \trianglerighteq \diamond_f$ does hold. Similarly, we can show that $\diamond_{f_2} \trianglerighteq \diamond_f$. $[\diamond_f]$ is the greatest lower bound of $[\diamond_{f_1}]$ and $[\diamond_{f_2}]$. $\dashv$

To conclude this section, let us prove the following result for atomic QCNs.

Proposition 5 *Let f be an element of $\mathcal{F}$, and $\mathcal{N}$ be an atomic* QCN. *If $\mathcal{N}$ is consistent then $\mathcal{N}$ is $\diamond_f$-consistent.*

Proof. For any element f of $\mathcal{F}$ and any base relation b, we know that $f(\{b\}) = \{\{b\}\}$. It means that all consistencies in $\diamond_{\mathcal{F}}$ are equivalent when restricted to atomic QCNs. As $\diamond_{f_\diamond}$ is equivalent to $\diamond$ (see Proposition 2) and as it is known that an atomic consistent QCN is necessarily closed under weak composition (i.e. $\diamond$-consistent), we deduce that an atomic consistent QCN is necessarily $\diamond_f$-consistent, whatever f is. $\dashv$

Closure of QCNs under $\diamond_f$-consistencies

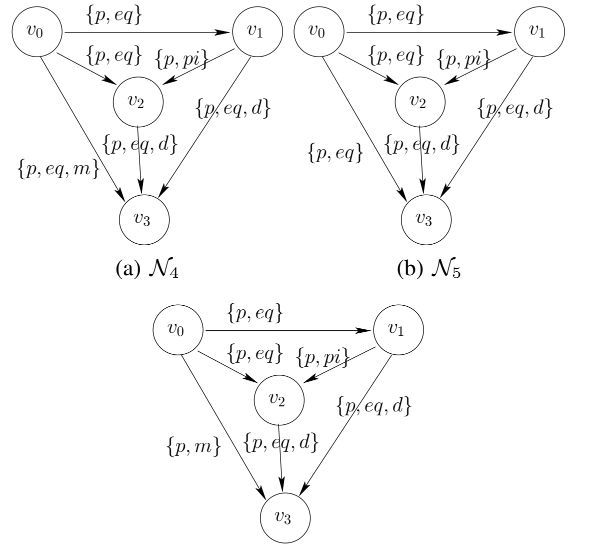
A consistency ϕ is *well-behaved* iff for any QCN $\mathcal{N}$, there exists a (unique) largest ϕ-consistent QCN $\mathcal{N}'$ smaller than or equal to $\mathcal{N}$ (w.r.t. $\subseteq$). $\mathcal{N}'$ is called the ϕ-closure of $\mathcal{N}$, and denoted by $\phi(\mathcal{N})$. In this section, we are concerned with the closure of QCNs under $\diamond_f$-consistencies. Are $\diamond_f$-consistencies well-behaved? In other words, given a QCN $\mathcal{N}$ and a consistency $\diamond_f$ in $\diamond_{\mathcal{F}}$, does the $\diamond_f$-closure of $\mathcal{N}$ exist? We first show that this is not always the case with an example taken from the Interval Algebra. We consider $f \in \mathcal{F}$ such that $f(\{p, eq, m\}) = \{\{p, eq, m\}, \{eq\}\}$ and $f(r) = \{r\}$ for every relation $r \in 2^{\mathsf{B}_{\mathsf{int}}} \setminus \{\{p, eq, m\}\}$. Figure 3 shows three distinct QCNs. The first QCN $\mathcal{N}_4$ is not $\diamond_f$-consistent because $\diamond(\mathcal{N}_{4[0,3]/\{eq\}})[0,3] = \emptyset$. Now let us turn to the two distinct QCNs $\mathcal{N}_5$ and $\mathcal{N}_6$: both QCNs are $\diamond_f$-consistent and (strictly) smaller than $\mathcal{N}_4$. Observing that there does not exist any $\diamond_f$-consistent QCN strictly greater than $\mathcal{N}_5$ and $\mathcal{N}_6$ and smaller than $\mathcal{N}_4$, we have just proved that $\diamond_f$ is not well-behaved.

Nevertheless, there exist some mappings f for which the consistencies $\diamond_f$ are guaranteed to be well-behaved. This is the case for the elements of the set $\mathcal{F}^*$ introduced below. Roughly speaking, for every relation r, $f(r)$ cannot be finer than the set of $f(r')$ with r' contained in r.

Definition 3 *$\mathcal{F}^*$ is the set of mappings f in $\mathcal{F}$ such that for every $r, r' \in 2^{\mathsf{B}}$ with $r' \subset r$ and for every $s \in f(r)$, we have $s \cap r' \neq \emptyset \Rightarrow \exists S \subseteq f(r')$ such that $s \cap r' = \bigcup S$.*

For example, all mappings mentioned in our previous illustrations belong to $\mathcal{F}^*$, except the last one that has been introduced above to prove that some $\diamond_f$-consistencies are not well-behaved. We first show the following result.

Proposition 6 *Let f be a mapping of $\mathcal{F}^*$. If $\mathcal{N}_1$ and $\mathcal{N}_2$ are two $\diamond_f$-consistent* QCN*s defined on the same set of variables, then $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$ is a $\diamond_f$-consistent* QCN.



(a) $\mathcal{N}_4$ (b) $\mathcal{N}_5$ (c) $\mathcal{N}_6$

Figure 3: $\mathcal{N}_4 = \mathcal{N}_5 \cup \mathcal{N}_6$.

Proof. Let v_i, v_j be two variables of $\mathcal{N}$ (and consequently of $\mathcal{N}_1$ and $\mathcal{N}_2$), $r = \mathcal{N}[i,j]$ and $s \in f(r)$. We have to show that $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$. Let $r_1 = \mathcal{N}_1[i,j]$, $r_2 = \mathcal{N}_2[i,j]$ and let s_1 and s_2 be the two relations defined as $s_1 = s \cap r_1$ and $s_2 = s \cap r_2$. As $s \in f(r)$, we have $s \subseteq r$ and as $\mathcal{N} = \mathcal{N}_1 \cup \mathcal{N}_2$, we have $r = r_1 \cup r_2$. We can deduce $s = s_1 \cup s_2$, and also $\mathcal{N}_{1[i,j]/s_1} \subseteq \mathcal{N}_{[i,j]/s}$ and $\mathcal{N}_{2[i,j]/s_2} \subseteq \mathcal{N}_{[i,j]/s}$. Because $\diamond$ is monotonic, we have $\diamond(\mathcal{N}_{1[i,j]/s_1}) \subseteq \diamond(\mathcal{N}_{[i,j]/s})$ and $\diamond(\mathcal{N}_{2[i,j]/s_2}) \subseteq \diamond(\mathcal{N}_{[i,j]/s})$.

On the other hand, as $f \in \mathcal{F}^*$ there exist $S_1 \subseteq f(r_1)$ and $S_2 \subseteq f(r_2)$ such that $\bigcup S_1 = s_1$ and $\bigcup S_2 = s_2$ (if we assume that $s_1 \neq \emptyset$ and $s_2 \neq \emptyset$). From $\mathcal{N}_1$ and $\mathcal{N}_2$ being $\overset{\diamond}{f}$-consistent, we deduce that $\diamond(\mathcal{N}_{1[i,j]/s'_1})[i,j] = s'_1$, $\forall s'_1 \in S_1$ and $\diamond(\mathcal{N}_{2[i,j]/s'_2})[i,j] = s'_2$, $\forall s'_2 \in S_2$. Moreover, because $\diamond$ is monotonic, we have $\diamond(\mathcal{N}_{1[i,j]/s'_1})[i,j] \subseteq \diamond(\mathcal{N}_{1[i,j]/s_1})[i,j]$, $\forall s'_1 \in S_1$ and $\diamond(\mathcal{N}_{2[i,j]/s'_2})[i,j] \subseteq \diamond(\mathcal{N}_{2[i,j]/s_2})[i,j]$, $\forall s'_2 \in S_2$. From this, we obtain $s'_1 \subseteq \diamond(\mathcal{N}_{1[i,j]/s_1})[i,j]$, $\forall s'_1 \in S_1$ and $s'_2 \subseteq \diamond(\mathcal{N}_{2[i,j]/s_2})[i,j]$, $\forall s'_2 \in S_2$. Consequently, $s_1 \subseteq \diamond(\mathcal{N}_{1[i,j]/s_1})[i,j]$ and $s_2 \subseteq \diamond(\mathcal{N}_{2[i,j]/s_2})[i,j]$. As $\diamond$ is contracting, we also have $\diamond(\mathcal{N}_{1[i,j]/s_1})[i,j] \subseteq s_1$ and $\diamond(\mathcal{N}_{2[i,j]/s_2})[i,j] \subseteq s_2$. Finally, $\diamond(\mathcal{N}_{1[i,j]/s_1})[i,j] = s_1$ and $\diamond(\mathcal{N}_{2[i,j]/s_2})[i,j] = s_2$.

From what precedes, we obtain $s_1 \subseteq \diamond(\mathcal{N}_{[i,j]/s})[i,j]$ and $s_2 \subseteq \diamond(\mathcal{N}_{[i,j]/s})[i,j]$. So, $s = s_1 \cup s_2 \subseteq \diamond(\mathcal{N}_{[i,j]/s})[i,j]$. The same result can be obtained when $s_1 = \emptyset$ or $s_2 = \emptyset$. Moreover, we also know that $\diamond(\mathcal{N}_{[i,j]/s})[i,j] \subseteq s$ because $\diamond$ is contracting. We can conclude that $\diamond(\mathcal{N}_{[i,j]/s})[i,j] = s$, and consequently that $\mathcal{N}$ is $\overset{\diamond}{f}$-consistent. $\dashv$

From the previous result, we can show that for every QCN $\mathcal{N}$ and every f in $\mathcal{F}^*$, the QCN $\bigcup\{\mathcal{N}' : \mathcal{N}' \subseteq \mathcal{N}$ and $\mathcal{N}'$ is $\overset{\diamond}{f}$-consistent$\}$ is the largest $\overset{\diamond}{f}$-consistent subQCN of $\mathcal{N}$, i.e. the $\overset{\diamond}{f}$-closure of $\mathcal{N}$.

Corollary 2 *If $\overset{\diamond}{f}$ is a consistency in $\overset{\diamond}{\mathcal{F}^*}$, then $\overset{\diamond}{f}$ is well-behaved.*

Observing that $f_{\neq}$ and f_{B} do belong to $\mathcal{F}^*$, we can show that the QCNs from Figure 2 are such that $\overset{\diamond}{f_{\neq}}(\mathcal{N}_1) = \mathcal{N}_2$ and $\overset{\diamond}{f_{\mathsf{B}}}(\mathcal{N}_2) = \mathcal{N}_3$.

Importantly, every consistency in $\overset{\diamond}{\mathcal{F}^*}$ preserves the set of scenarios (the proof is omitted due to lack of space). This is not very surprising since Proposition 1 already indicates that identified base $\overset{\diamond}{f}$-inconsistent relations can be safely discarded.

Proposition 7 *Let f be an element of $\mathcal{F}^*$. For every* QCN *$\mathcal{N}$, $\overset{\diamond}{f}(\mathcal{N})$ is equivalent to $\mathcal{N}$.*

Proof. Suppose that there exist two variables (v_i, v_j) of $\mathcal{N}$ and a base relation $b \in \mathsf{B}$ such that $b \in \mathcal{N}[i,j]$, $b \notin \overset{\diamond}{f}(\mathcal{N})[i,j]$ and a scenario $\mathcal{S}$ of $\mathcal{N}$ with $\mathcal{S}[i,j] = \{b\}$. From Proposition 5, we know that $\mathcal{S}$ is $\overset{\diamond}{f}$-consistent, and from $\mathcal{S}$ being a scenario of $\mathcal{N}$, we know that $\mathcal{S} \subseteq \mathcal{N}$. Hence, by closure definition, we have $\mathcal{S} \subseteq \overset{\diamond}{f}(\mathcal{N})$. This leads to a contradiction since $\mathcal{S}[i,j] = \{b\}$ and $b \notin \overset{\diamond}{f}(\mathcal{N})[i,j]$. $\dashv$

Generic Algorithm

In this section, we present a basic $\overset{\diamond}{f}$-algorithm, that is to say an algorithm that allows us to compute the $\overset{\diamond}{f}$-closure $\overset{\diamond}{f}(\mathcal{N})$ of any given QCN $\mathcal{N}$. Such a closure is guaranteed to exist since f is assumed to belong to $\mathcal{F}^*$; see Corollary 2. We introduce a constraint-oriented propagation scheme for enforcing the consistency $\overset{\diamond}{f}$. The constraint-oriented propagation scheme is characterized by revision of constraints that are successively picked from a dedicated set Q called the queue of the propagation.

The revision of a constraint C_{ij} removes from C_{ij} some base relations that are $\overset{\diamond}{f}$-inconsistent (if any). A revision is said to be effective if it removes at least one base relation. This is the role of function df-revise. For each element s of $f(C_{ij})$, a $\overset{\diamond}{f}$-check on s is performed, that is to say, $\mathcal{N}' = \diamond(\mathcal{N}_{[i,j]/s})$ is computed (line 3), which enables the identification of $\overset{\diamond}{f}$-inconsistent base relations, those in $s \setminus \mathcal{N}'[i,j]$ (line 4). The variable r collects $\overset{\diamond}{f}$-inconsistent base relations from C_{ij}, and if r is not empty, C_{ij} is updated (line 7) and $true$ is returned.

The main function, called df-closure, performs one or several turns (passes) of the main loop. At each pass, all constraints are revised in turn: constraints are iteratively selected from Q (line 7) and df-revise is called to perform revisions (line 8). When an inference is performed (i.e. a revision is effective), the Boolean variable $modified$ is set to $true$, which determines that a next pass is necessary. The algorithm stops when no inference is performed during a pass, or when an inconsistency is detected (lines 2 and 10). Note that when a QCN $\mathcal{N}$ is trivially inconsistent because there exists an empty constraint in $\mathcal{N}$, we note $\mathcal{N} = \bot$. Initially (line 1), and after each effective revision (line 9), $\diamond$ (closure under weak composition) is applied on $\mathcal{N}$. This

Function df-revise (C_{ij}) : Boolean

in/out : C_{ij}, a constraint of the QCN $\mathcal{N}$
output: $true$ iff the revision of C_{ij} is effective

1 $r \leftarrow \emptyset$
2 **foreach** *relation* $s \in f(C_{ij})$ **do**
3 $\quad \mathcal{N}' \leftarrow \diamond(\mathcal{N}_{[i,j]/s})$ // $\overset{\diamond}{f}$-check on s
4 $\quad r \leftarrow r \cup (s \setminus \mathcal{N}'[i,j])$ // $\overset{\diamond}{f}$-inconsistent base relations are collected
5 **if** $r \neq \emptyset$ **then**
6 $\quad r' \leftarrow \mathcal{N}_{[i,j]} \setminus r$
7 $\quad \mathcal{N} \leftarrow \mathcal{N}_{[i,j]/r'}$ // C_{ij} becomes r'
8 $\quad$ **return** *true*
9 **else return** *false*

Function df-closure ($\mathcal{N}$,f) : Boolean

in/out : $\mathcal{N} = (V, C)$, a QCN
in : f, an element of $\mathcal{F}^*$
output: $true$ iff $\overset{\diamond}{f}(\mathcal{N}) \neq \bot$

1 $\mathcal{N} \leftarrow \diamond(\mathcal{N})$ // $\diamond$ enforced
2 **if** $\mathcal{N} = \bot$ **then return** *false*
3 **repeat**
4 $\quad modified \leftarrow false$
5 $\quad Q \leftarrow \{C_{ij} \in C \mid i < j \wedge |f(C_{ij})| > 1\}$
6 $\quad$ **while** $Q \neq \emptyset$ **do**
7 $\quad\quad$ select and remove a constraint C_{ij} from Q
8 $\quad\quad$ **if** *df-revise*(C_{ij}) **then**
9 $\quad\quad\quad \mathcal{N} \leftarrow \diamond(\mathcal{N})$ // $\diamond$ maintained
10 $\quad\quad\quad$ **if** $\mathcal{N} = \bot$ **then return** *false*
11 $\quad\quad\quad modified \leftarrow true$
12 **until** $\neg modified$
13 **return** *true*

is sound because we know that for any $f \in \mathcal{F}^*$, $\overset{\diamond}{f}(\mathcal{N}) \subseteq \overset{\diamond}{f_\diamond}(\mathcal{N}) = \diamond(\mathcal{N})$. When initializing Q (line 5), a constraint C_{ji} with $i < j$ is ignored because it can be deducted from C_{ij} by means of the inverse operation. Also, a constraint C_{ij} such that $|f(C_{ij})| = 1$ is ignored because it is necessarily $\overset{\diamond}{f}$-consistent (recall that closure under weak composition is maintained during search).

We can prove that the algorithm df-closure is correct, i.e. enforce $\overset{\diamond}{f}$. Indeed, the algorithm is sound because every base relation removed in df-revise is $\overset{\diamond}{f}$-inconsistent. On the other hand, the algorithm is complete because, as soon as an inference is performed, a new pass is run (and all constraints are revised). However, it is important to note that the function df-revise removes at least one $\overset{\diamond}{f}$-inconsistent base relation from a given $\overset{\diamond}{f}$-inconsistent constraint. Consequently, this guarantess completeness although the function df-revise does not systematically render the given constraint $\overset{\diamond}{f}$-consistent.

The worst-case time complexity of the function df-revise is $O(\mathsf{s}\lambda)$ where s is the greatest size (cardinality) of sets in $\{f(r) : r \in 2^B\}$ and λ the worst-case time complexity of enforcing $\diamond$, i.e. $O(n^3)$ for binary relations. Indeed, at most s $\overset{\diamond}{f}$-checks are performed. At each pass of the function df-closure, the number of calls to df-revise is $O(n^2)$, so the worst-case time complexity of one pass of df-closure is $O(\mathsf{s}\lambda n^2)$. Although the number of passes is bounded by $O(|\mathsf{B}|n^2)$ (only one base relation removed at each pass), we think that it is a small number in practice (this will be confirmed in our experimentations). Besides, we believe that the basic algoritm presented here can be refined so as to make it incremental (similarly to what is done for SAC (Bessiere and Debruyne 2008)).

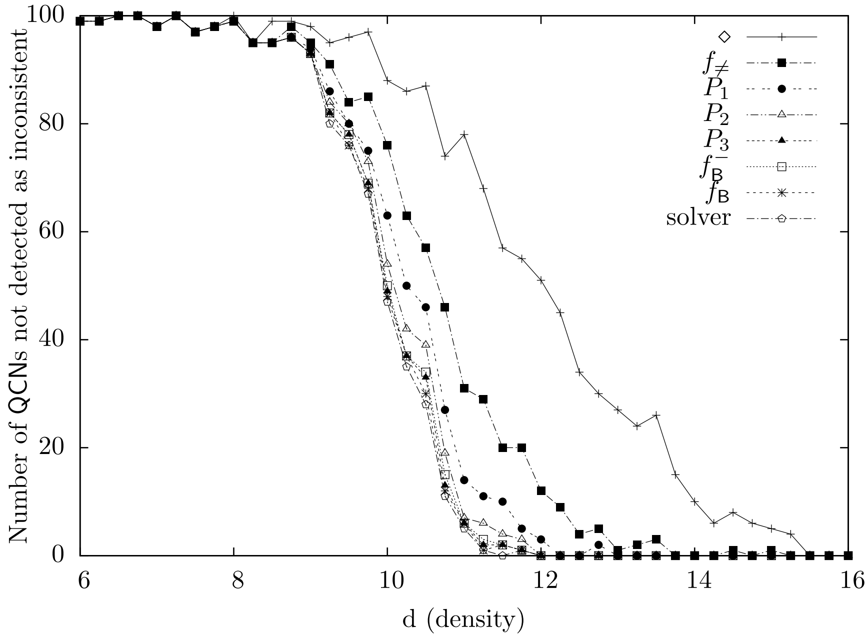
Experiments

In our experimentation, we have focused on qualitative constraint networks from the Interval Algebra, randomly generated following Model A (Nebel 1996). This model involves the generation of QCNs according to three parameters: n the number of variables, d the density and s the average number of base relations in each constraint. The set (or series) of QCNs that can be generated from n, d and s is denoted by $A(n, d, s)$. The experimental results presented in this section concern QCN instances from series $A(75, d, 6.5)$ and $A(100, d, 6.5)$ for d varying from 2 to 24 with a step of 0.25. For these series, the hardest instances are located in a region where the density ranges from 8 to 11. For each series, we generated 100 instances.

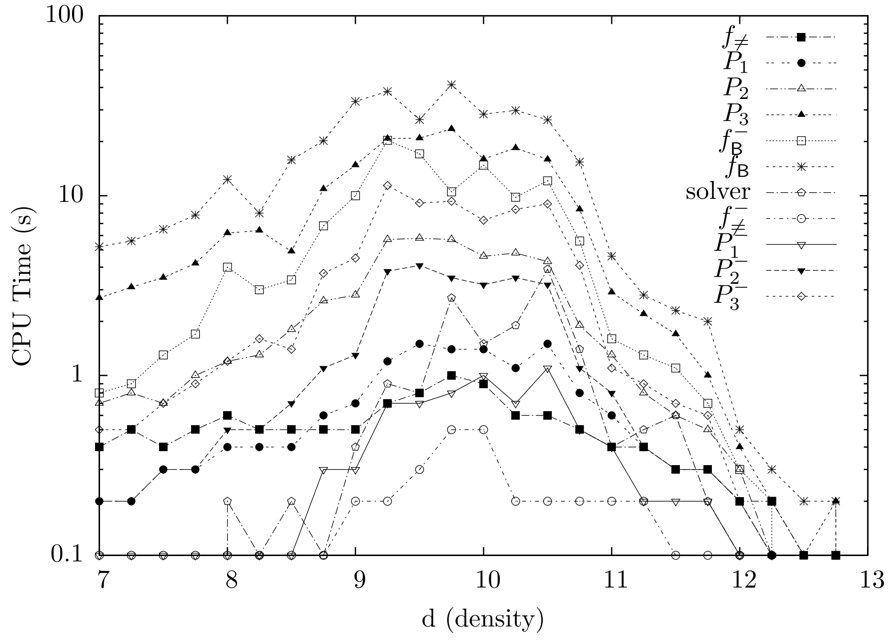
The main objective of our experimental study is to compare both the filtering strength and the time efficiency of some $\overset{\diamond}{f}$-algorithms (those based on consistencies introduced in previous sections). The first criterion used for our comparisons is the number of QCNs detected as inconsistent, within the phase transition. This informs us about the relative filtering strength[1] of different consistencies. Note that the exact number of inconsistent QCNs will be computed using a complete solving method: this represents the ideal filtering capability for a consistency. This method, called $solver$ afterwards, is the solver proposed in (Nebel 1996). Basically, it performs search by successively reducing each constraint relation to a tractable one (using a splitting of the initial relations) and maintaining the QCN closed under weak composition. In our context, we used the tractable sets of the Ord-Horn relations as split elements, and sought the best control parameters of $solver$ to solve our instances. The second criterion used for our comparisons is the CPU time (given in seconds) taken by $\overset{\diamond}{f}$-algorithms (and $solver$).

When enforcing $\overset{\diamond}{f}$-consistencies, we may decide to ignore universal constraints so as to limit the computation effort of the algorithms. This means that when there is a constraint between two variables v_i, v_j such that $C_{ij} = \mathsf{B}$ then no check on C_{ij} is performed by means of f. Pragmatically, for every mapping f, we can introduce a related so-called *reduced* mapping f^- defined as: $f^-(r) = f(r)$ if $r \neq \mathsf{B}$, and $f^-(r) = \{\mathsf{B}\}$ otherwise. Intuitively, we may expect to

[1] We could also assess the filtering strength of a given local consistency ϕ in terms of the number of base relations deleted when applying ϕ, but this information is closely related to our first criterion.

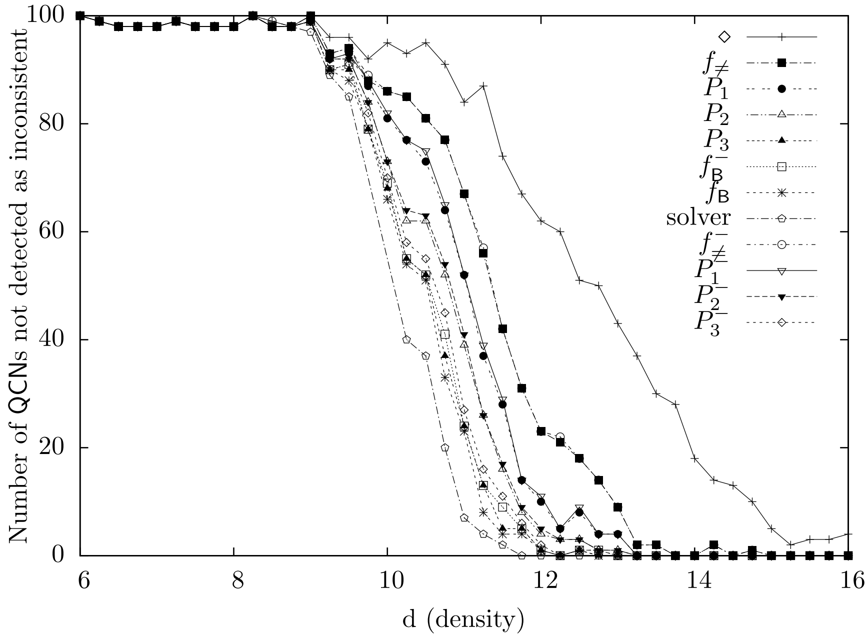


(a) Filtering Strength for $A(75, d, 6.5)$

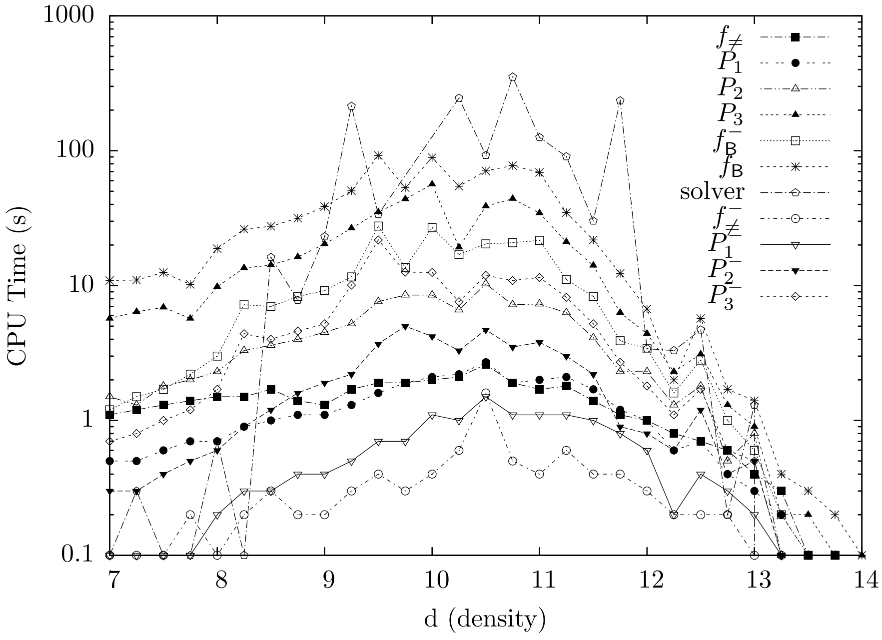


(b) CPU Time for $A(75, d, 6.5)$

Figure 4: Experimental results for series $A(75, d, 6.5)$.



(a) Filtering Strength for $A(100, d, 6.5)$



(b) CPU Time for $A(100, d, 6.5)$

Figure 5: Experimental results for series $A(100, d, 6.5)$.

save time whereas limiting the loss of inferences due to the universal nature of these constraints.

Figures 4(a) and 5(a) show the filtering capabilities of various $\overset{\diamond}{f}$-consistencies on series $A(75, d, 6.5)$ and $A(100, d, 6.5)$, respectively. A first observation is that consistencies based on reduced mappings are quite close to unreduced ones. For $n = 75$ variables, this was so striking that we decided (for clarity reasons) to not plot the curves corresponding to reduced mappings (except for f_{B}^{-}). For $n = 100$ variables, a small difference is visible. This means that for a a given mapping f, the mapping f^{-} allows us to detect almost the same number of inconsistent QCNs. However, we conjecture that this is less ans less true when n increases. A second observation is that $\overset{\diamond}{f_{\mathsf{B}}}$ (theoretically shown to be the strongest local consistency in $\overset{\diamond}{\mathcal{F}}$) is very effective as it almost detects all inconsistent QCNs (as identified by $solver$) from series $A(75, d, 6.5)$, and a lot of them from series $A(100, d, 6.5)$. Other consistencies stronger than $\diamond = \overset{\diamond}{f_{\diamond}}$ are, in order, P_3, P_2, P_1 and $\overset{\diamond}{f_{\neq}}$. Interestingly, there is even so a significant gap between $\overset{\diamond}{f_{\neq}}$ and $\diamond$, which motivates us to further study each of these new consistencies. Finally, note that the number of passes executed by the function df-closure is very limited (around 3.5 on average).

Figures 4(b) and 5(b) show the CPU time taken by the $\overset{\diamond}{f}$-algorithms on the same series. Note the use of a log scale on the y-axis in order to better distinguish between the behaviour of all algorithms. For $n = 75$ variables, the use of the strongest consistencies such as $\overset{\diamond}{f_{\mathsf{B}}}$, $\overset{\diamond}{f_{P_3}}$ and $\overset{\diamond}{f_{P_2}}$ involve a large overhead with respect to $solver$, but when reduced mappings are used the $\overset{\diamond}{f}$ algorithms are far faster. For $n = 100$ variables, $solver$ becomes clearly slower than all other algorithms, but recall that $solver$ performs a complete search whereas $\overset{\diamond}{f}$-algorithms are incomplete since they can only perform some inferences. However, it is fair to compare $solver$ and $\overset{\diamond}{f}$-algorithms on instances shown to be inconsistent by both approaches. This is the case for most of the instances of series $A(100, d, 6.5)$ with d around 11.75 or higher; see Figure 5(a). For such instances, algorithms such as $\overset{\diamond}{f_{P_1}}$ and $\overset{\diamond}{f_{\neq}}$ (and their reduced variants) are about two orders of magnitude faster than $solver$. Finally, $\overset{\diamond}{f_{\diamond}} = \diamond$ is clearly the fastest algorithm as it is usually enforced within 0.1s (we did not plot its CPU curves because this flattens the figures) but remember that it is far weaker than other introduced $\overset{\diamond}{f}$-consistencies as shown in Figures 4(a) and 5(a). Besides, $\overset{\diamond}{P_1^{-}}$ and $\overset{\diamond}{f_{\neq}^{-}}$ are also cheap to enforce.

To summarize, our (preliminary) experimentation shows how promising $\overset{\diamond}{f}$-consistencies may be, and in particular those based on reduced mappings that offer a good compromise between time overhead and filtering capability. Maintaining such consistencies during search is a perspective that we envision using a fast solver like GQR*.

Conclusion

In this paper, we have introduced the class of $\overset{\diamond}{f}$-consistencies for qualitative constraint networks. This class forms a complete lattice and contains original local consistencies (even when considering their CSP counterparts) such as $\overset{\diamond}{f_{\neq}}$, all being stronger than weak composition. Looking for the $\overset{\diamond}{f}$-consistency that is the most appropriate to solve hard instances (from different qualitative algebras) is a pragmatic perspective of this work. On the other hand, we may imagine additional new classes built from coverings where $\diamond$ is substituted by another local consistency. Studying the connections between all these consistencies and the problems of (global) consistency and minimality of QCNs is an exciting theoretical perspective.

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A framework for decision-based consistencies

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A Framework for Decision-Based Consistencies

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Abstract. Consistencies are properties of constraint networks that can be enforced by appropriate algorithms to reduce the size of the search space to be explored. Recently, many consistencies built upon taking decisions (most often, variable assignments) and stronger than (generalized) arc consistency have been introduced. In this paper, our ambition is to present a clear picture of decision-based consistencies. We identify four general classes (or levels) of decision-based consistencies, denoted by S^{ϕ}_{Δ}, E^{ϕ}_{Δ}, B^{ϕ}_{Δ} and D^{ϕ}_{Δ}, study their relationships, and show that known consistencies are particular cases of these classes. Interestingly, this general framework provides us with a better insight into decision-based consistencies, and allows us to derive many new consistencies that can be directly integrated and compared with other ones.

1 Introduction

Consistencies are properties of constraint networks that can be used to make inferences. Such inferences are useful to filter the search space of problem instances. Most of the current constraint solvers interleave inference and search. Typically, they enforce generalized arc consistency (GAC), or one of its partial form, during the search of a solution. One avenue to make solvers more robust is to enforce strong consistencies, i.e., consistencies stronger than GAC. Whereas GAC corresponds to the strongest form of local reasoning when constraints are treated separately, strong consistencies necessarily involve several constraints (e.g., path inverse consistency [12], max-restricted path consistency [8] and their adaptations [20] to non-binary constraints) or even the entire constraint network (e.g., singleton arc consistency [9]).

A trend that emerges from recent works on strong consistencies is the resort to taking decisions before enforcing a well-known consistency (typically, GAC) and making some deductions. Among such decision-based consistencies, we find SAC (singleton arc consistency), partition-k-AC [2], weak-k-SAC [22], BiSAC [4], and DC (dual consistency) [15]. Besides, a partial form of SAC, better known as shaving, has been introduced for a long time [6,18] and is still an active subject of research [17,21]; when shaving systematically concerns the bounds of each variable domain, it is called BoundSAC [16]. What makes decision-based consistencies particularly attractive is that they are (usually) easy to define and

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understand, and easy to implement since they are mainly based on two concepts (decision, propagation) already handled by constraint solvers. The increased interest perceived in the community for decision-based consistencies has motivated our study.

In this paper, our ambition is to present a clear picture of decision-based consistencies that can derive nogoods of size up to 2; i.e., inconsistent values or inconsistent pairs of values. The only restriction we impose is that decisions correspond to unary constraints. The four classes (or levels) of consistencies, denoted by S^{ϕ}_{Δ}, E^{ϕ}_{Δ}, B^{ϕ}_{Δ} and D^{ϕ}_{Δ}, that we introduce are built on top of a consistency ϕ and a so-called decision mapping Δ. These are quite general because:

1. Δ allows us to introduce a specific set of decisions for every variable x and every possible (sub)domain of x,
2. decisions are membership decisions (of the form $x \in D_x$ where D_x is a set of values taken from the initial domain of x) that generalize both variable assignments (of the form $x = a$) and value refutations (of the form $x \neq a$),
3. decisions may ignore some variables and/or values, and decisions may overlap each other,
4. ϕ is any well-behaved nogood-identifying consistency.

We study the relationships existing between them, including the case where Δ covers every variable and every value. We also show that SAC, partition-k-AC, BiSAC and DC are particular cases of S^{ϕ}_{Δ}, S^{ϕ}_{Δ}+E^{ϕ}_{Δ} (the two consistencies combined), B^{ϕ}_{Δ} and D^{ϕ}_{Δ}, respectively. BoundSAC, and many other forms of shaving, are also elements of the class S^{ϕ}_{Δ}. The general framework we depict provides a better insight into decision-based consistencies while allowing many new combinations and comparisons of such consistencies. For example, the class of consistencies S^{ϕ}_{Δ} induces a complete lattice where the partial order denotes the relative strength of every two consistencies.

2 Technical Background

This section provides technical background about constraint networks and consistencies, mainly taken from [1,11,3,13].

Constraint Networks. A *constraint network* (CN) P is composed of a finite set of n variables, denoted by $vars(P)$, and a finite set of e constraints, denoted by $cons(P)$. Each variable x has a domain which is the finite set of values that can be assigned to x. Each constraint c involves an ordered set of variables, called the *scope* of c and denoted by $scp(c)$, and is defined by a relation which is the set of tuples allowed for the variables involved in c. The initial domain of a variable x is denoted by $dom^{init}(x)$ whereas the current domain of x (in the context of P) is denoted by $dom^P(x)$, or more simply $dom(x)$. Assuming that the initial domain of each variable is totally ordered, $min(x)$ and $max(x)$ will denote the smallest and greatest values in $dom(x)$. The initial and current relations of a constraint c are denoted by $rel^{init}(c)$ and $rel(c)$, respectively.

A constraint is *universal* iff $rel^{init}(c) = \Pi_{x \in scp(c)} dom^{init}(x)$. For simplicity, a pair (x, a) with $x \in vars(P)$ and $a \in dom(x)$ is called a *value* of P, which is denoted by $(x, a) \in P$. A *unary* (resp., *binary*) constraint involves 1 (resp., 2) variable(s), and a *non-binary* one strictly more than 2 variables. Without any loss of generality, we only consider CNs that do not involve unary constraints, universal constraints and constraints of similar scope. The set of such CNs is denoted by $\mathscr{P}$. An *instantiation* I of a set $X = \{x_1, \ldots, x_k\}$ of variables is a set $\{(x_1, a_1), \ldots, (x_k, a_k)\}$ such that $\forall i \in 1..k, a_i \in dom^{init}(x_i)$; X is denoted by $vars(I)$ and each a_i is denoted by $I[x_i]$. An instantiation I *on* a CN P is an instantiation of a set $X \subseteq vars(P)$; it is *complete* if $vars(I) = vars(P)$. I is *valid* on P iff $\forall (x, a) \in I, a \in dom(x)$. I *covers* a constraint c iff $scp(c) \subseteq vars(I)$, and I *satisfies* a constraint c with $scp(c) = \{x_1, \ldots, x_r\}$ iff (i) I covers c and (ii) the tuple $(I[x_1], \ldots, I[x_r]) \in rel(c)$. An instantiation I on a CN P is *locally consistent* iff (i) I is valid on P and (ii) every constraint of P covered by I is satisfied by I. A *solution* of P is a complete locally consistent instantiation on P; $sols(P)$ denotes the set of solutions of P. An instantiation I on a CN P is *globally inconsistent*, or a *nogood*, iff it cannot be extended to a solution of P. Two CNs P and P' are *equivalent* iff $vars(P) = vars(P')$ and $sols(P) = sols(P')$.

The *nogood representation* of a CN is a set of nogoods, one for every value removed from the initial domain of a variable and one for every tuple forbidden by a constraint. More precisely, the nogood representation $\widetilde{x}$ of a variable x is the set $\{\{(x, a)\} \mid a \in \overline{dom}(x)\}$ with $\overline{dom}(x) = dom^{init}(x) \setminus dom(x)$. The nogood representation $\widetilde{c}$ of a constraint c is $\{\{(x_1, a_1), \ldots, (x_r, a_r)\} \mid (a_1, \ldots, a_r) \in \overline{rel}(c)\}$, with $scp(c) = \{x_1, \ldots, x_r\}$ and $\overline{rel}(c) = \Pi_{x \in scp(c)} dom^{init}(x) \setminus rel(c)$. The nogood representation $\widetilde{P}$ of a CN P is $(\cup_{x \in vars(P)} \widetilde{x}) \cup (\cup_{c \in cons(P)} \widetilde{c})$. Based on nogood representations, a general partial order can be introduced to relate CNs. Let P and P' be two CNs such that $vars(P) = vars(P')$, we have $P' \preceq P$ iff $\widetilde{P'} \supseteq \widetilde{P}$ and we have $P' \prec P$ iff $\widetilde{P'} \supsetneq \widetilde{P}$. $(\mathscr{P}, \preceq)$ is the partially ordered set (poset) considered in this paper. The search space of a CN can be reduced by a filtering process (called constraint propagation) based on some properties (called consistencies) that allow us to identify and record explicit nogoods in CNs; e.g., identified nogoods of size 1 correspond to inconsistent values that can be safely removed from variable domains. In $\mathscr{P}$, there is only one manner to discard an instantiation from a given CN, or equivalently to "record" a new explicit nogood. Given a CN P in $\mathscr{P}$, and an instantiation I on P, $P \setminus I$ denotes the CN P' in $\mathscr{P}$ such that $vars(P') = vars(P)$ and $\widetilde{P'} = \widetilde{P} \cup \{I\}$. $P \setminus I$ is an operation that retracts I from P and builds a new CN. If $I = \{(x, a)\}$, we remove a from $dom(x)$. If I corresponds to a tuple allowed by a constraint c of P, we remove this tuple from $rel(c)$. Otherwise, we introduce a new constraint allowing all possible tuples (from initial domains) except the one that corresponds to I.

Consistencies. A consistency is a property defined on CNs. When a consistency ϕ holds on a CN P, we say that P is *ϕ-consistent*; if ψ is another consistency, P is *ϕ+ψ-consistent* iff P is both ϕ-consistent and ψ-consistent. A consistency ϕ is *nogood-identifying* iff the reason why a CN P is not ϕ-consistent is that some instantiations, which are not in $\widetilde{P}$, are identified as globally inconsistent by ϕ; such instantiations are said to be *ϕ-inconsistent*. A *kth-order consistency* is a nogood-identifying consistency that allows the identification of nogoods of size k. A *domain-filtering consistency* [10,5] is a first-order consistency. A nogood-identifying consistency is *well-behaved* when for any CN P, the set $\{P' \in \mathscr{P} \mid P'$ is ϕ-consistent and $P' \preceq P\}$ admits a greatest element, denoted by $\phi(P)$, equivalent to P. Enforcing ϕ on a CN P means computing $\phi(P)$. Any well-behaved consistency ϕ is *monotonic*: for any two CNs P and P', we have: $P' \preceq P \Rightarrow \phi(P') \preceq \phi(P)$. To compare the pruning capability of consistencies, we use a preorder. A consistency ϕ is *stronger* than (or equal to) a consistency ψ, denoted by $\phi \trianglerighteq \psi$, iff whenever ϕ holds on a CN P, ψ also holds on P. ϕ is *strictly stronger* than ψ, denoted by $\phi \triangleright \psi$, iff $\phi \trianglerighteq \psi$ and there is at least a CN P such that ψ holds on P but not ϕ. ϕ and ψ are *equivalent*, denoted by $\phi \approx \psi$, iff both $\phi \trianglerighteq \psi$ and $\psi \trianglerighteq \phi$.

Now we introduce some concrete consistencies, starting with GAC (Generalized Arc Consistency). A value (x, a) of P is *GAC-consistent* iff for each constraint c of P involving x there exists a valid instantiation I of $scp(c)$ such that I satisfies c and $I[x] = a$. P is GAC-consistent iff every value of P is GAC-consistent. For binary constraints, GAC is often referred to as AC (Arc Consistency). Now, we introduce known consistencies based on decisions. When the domain of a variable of P is empty, P is unsatisfiable (i.e., $sols(P) = \emptyset$), which is denoted by $P = \bot$; to simplify, we consider that no value is present in a CN P such that $P = \bot$. The CN $P|_{x=a}$ is obtained from P by removing every value $b \neq a$ from $dom(x)$. A value (x, a) of P is *SAC-consistent* iff $GAC(P|_{x=a}) \neq \bot$ [9]. A value (x, a) of P is *1-AC-consistent* iff (x, a) is SAC-consistent and $\forall y \in vars(P) \setminus \{x\}, \exists b \in dom(y) \mid (x, a) \in GAC(P|_{y=b})$ [2]. A value (x, a) of P is *BiSAC-consistent* iff $GAC(P^{ia}|_{x=a}) \neq \bot$ where P^{ia} is the CN obtained after removing every value (y, b) of P such that $y \neq x$ and $(x, a) \notin GAC(P|_{y=b})$ [4]. P is SAC-consistent (resp., 1-AC-consistent, BiSAC-consistent) iff every value of P is SAC-consistent (resp., 1-AC-consistent, BiSAC-consistent). P is BoundSAC-consistent iff for every variable x, $min(x)$ and $max(x)$ are SAC-consistent [16]. A decision-based second-order consistency is dual consistency (DC) defined as follows. A locally consistent instantiation $\{(x, a), (y, b)\}$ on P, with $y \neq x$, is DC-consistent iff $(y, b) \in GAC(P|_{x=a})$ and $(x, a) \in GAC(P|_{y=b})$ [14]. P is *DC-consistent* iff every locally consistent instantiation $\{(x, a), (y, b)\}$ on P is DC-consistent. P is *sDC-consistent* (strong DC-consistent) iff P is GAC+DC-consistent, i.e. both GAC-consistent and DC-consistent. All consistencies mentioned above are well-behaved. Also, we know that sDC $\triangleright$ BiSAC $\triangleright$ 1-GAC $\triangleright$ SAC $\triangleright$ BoundSAC $\triangleright$ GAC.

3 Decision-Based Consistencies

In this section, we introduce decisions before presenting general classes of consistencies.

3.1 Decisions

A *positive decision* δ is a restriction on a variable x of the form $x = a$ whereas a *negative decision* is a restriction of the form $x \neq a$, with $a \in dom^{init}(x)$. A *membership decision* is a decision of the form $x \in D_x$, where x is a variable and $D_x \subseteq dom^{init}(x)$ is a non-empty set of values; note that D_x is not necessarily $dom(x)$, the current domain of x. Membership decisions generalize both positive and negative decisions as a positive (resp., negative) decision $x = a$ (resp., $x \neq a$) is equivalent to the membership decision $x \in \{a\}$ (resp., $x \in dom^{init}(x) \setminus \{a\}$). The variable involved in a decision δ is denoted by $var(\delta)$.

For a membership decision δ, we define $P|_\delta$ to be the CN obtained (derived) from P such that, if δ denotes $x \in D_x$ and if x is a variable of P then each value $b \in dom^P(x)$ with $b \notin D_x$ is removed from $dom^P(x)$. If Γ is a set of decisions, $P|_\Gamma$ is obtained by restricting P by means of all decisions in Γ, and $vars(\Gamma)$ denotes the set of variables occurring in Γ. Enforcing a given well-behaved consistency ϕ after taking a decision δ on a CN P may be quite informative. As seen later, analyzing the CN $\phi(P|_\delta)$ allows us to identify nogoods. Computing $\phi(P|_\delta)$ in order to make such inferences is called a decision-based ϕ-check on P from δ, or more simply a *decision-based check*. For SAC, a decision-based check from a pair (x, a), usually called a singleton check, aims at comparing $GAC(P|_{x=a})$ with $\perp$.

From now on, Δ will denote a mapping, called *decision mapping*, that associates with every variable x and every possible domain $dom_x \subseteq dom^{init}(x)$, a (possibly empty) set $\Delta(x, dom_x)$ of membership decisions on x such that for every decision $x \in D_x$ in $\Delta(x, dom_x)$, we have $D_x \subseteq dom_x$. For example, an illustrative decision mapping Δ^{ex} may be such that $\Delta^{ex}(x, \{a, b, c, d\}) = \{x \in \{a, b\}, x \in \{d\}\}$. For the current domain of x, i.e., the domain of x in the context of a current CN P, $\Delta(x, dom(x)) = \Delta(x, dom^P(x))$ will be simplified into $\Delta(x)$ when this is unambiguous. To simplify, we shall also refer to Δ as the set of all "current" decisions w.r.t. P, i.e., Δ will be considered as $\cup_{x \in vars(P)} \Delta(x)$. This quite general definition of decision mapping will be considered as our basis to perform decision-based checks. Sometimes, we need to restrict sets of decisions in order to have each value occurring at least once in a decision. A set of decisions Γ on a variable x is said to be a *cover* of $\cup_{(x \in D_x) \in \Gamma} D_x$. For example, $\Delta^{ex}(x, \{a, b, c, d\})$, as defined above, is a cover of $\{a, b, d\}$. Δ is a *cover* for (x, dom_x), where $dom_x \subseteq dom^{init}(x)$, iff $\Delta(x, dom_x)$ is a cover of dom_x. For example, Δ^{ex} is not a cover for $(x, \{a, b, c, d\})$. Δ is a *cover* for x iff for every $dom_x \subseteq dom^{init}(x)$, Δ is a *cover* for (x, dom_x). Δ is *covering* iff for every variable x, Δ is a *cover* for x.

As examples of decision mappings, we have for every variable x:

- $\Delta^{id}(x)$ containing only $x \in dom(x)$;
- $\Delta^{=}(x)$ containing $x = a$, $\forall a \in dom(x)$;
- $\Delta^{\neq}(x)$ containing $x \neq a$, $\forall a \in dom(x)$;
- $\Delta^{bnd}(x)$ containing $x = min(x)$ and $x = max(x)$;
- $\Delta^{P_2}(x)$ containing $x \in D^1_x$ and $x \in D^2_x$ where D^1_x and D^2_x resp. contain the first and last $|dom(x)|/2$ values of $dom(x)$.

For example, if P is a CN such that $vars(P) = \{x, y\}$ with $dom(x) = dom^P(x) = \{a, b, c\}$ and $dom(y) = dom^P(y) = \{a, b\}$ then:

- $\Delta^{id}(x) = \{x \in \{a, b, c\}\}$ and $\Delta^{id}(y) = \{y \in \{a, b\}\}$;
- $\Delta^{=}(x) = \{x = a, x = b, x = c\}$ and $\Delta^{=}(y) = \{y = a, y = b\}$;
- $\Delta^{\neq}(x) = \{x \neq a, x \neq b, x \neq c\}$ and $\Delta^{\neq}(y) = \{y \neq a, y \neq b\}$;
- $\Delta^{bnd}(x) = \{x = a, x = c\}$ and $\Delta^{bnd}(y) = \{y = a, y = b\}$;
- $\Delta^{P_2}(x) = \{x \in \{a, b\}, x = c\}$ and $\Delta^{P_2}(y) = \{y = a, y = b\}$.

Note that, except for Δ^{bnd}, all these decision mappings are covering. Also, the reader should be aware of the dynamic nature of decision mappings. For example, if P' is obtained from P after removing a from $dom^P(x)$ then we have $\Delta^{bnd}(x, dom^{P'}(x)) = \{x = b, x = c\}$.

3.2 Two Classes of First-Order Consistencies

Informally, a decision-based consistency is a property defined from the outcome of decision-based checks. From now on, we consider given a well-behaved nogood-identifying consistency ϕ and a decision mapping Δ. A first kind of inferences is made possible by considering the effect of a decision-based check on the domain initially reduced by the decision that has been taken.

Definition 1 (Consistency S_Δ^ϕ). *A value (x, a) of a CN P is S_Δ^ϕ-consistent iff for every membership decision $x \in D_x$ in $\Delta(x)$ such that $a \in D_x$, we have $(x, a) \in \phi(P|_{x \in D_x})$.*

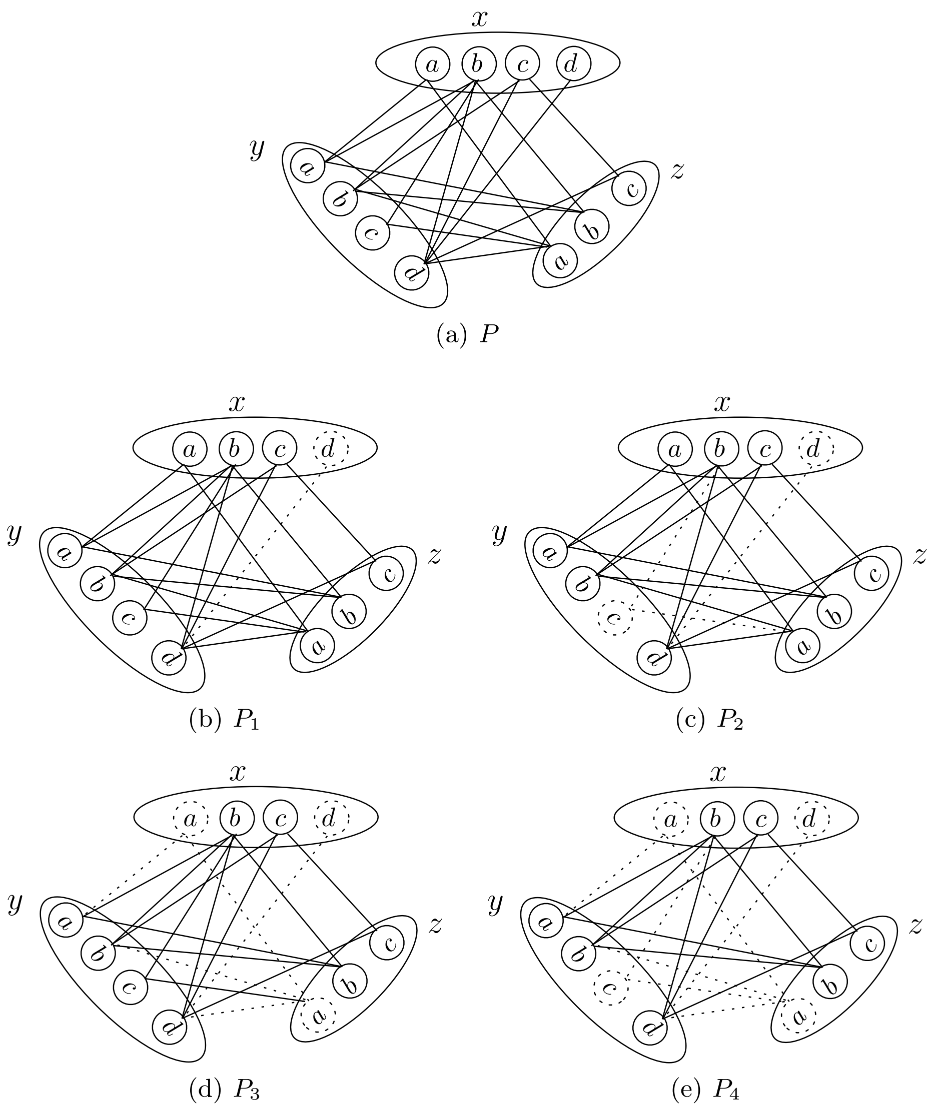
The following result can be seen as a generalization of Property 1 in [2].

Proposition 1. *Any S_Δ^ϕ-inconsistent value is globally inconsistent.*

Proof. If (x, a) is an S_Δ^ϕ-inconsistent value, then we know that there exists a decision $x \in D_x$ in $\Delta(x)$ such that $a \in D_x$ and $(x, a) \notin \phi(P|_{x \in D_x})$. We deduce that $x \in D_x \wedge x = a$ cannot lead to a solution because ϕ is nogood-identifying. This simplifies into $x = a$ being a nogood because $a \in D_x$. □

SAC is equivalent to $S_{\Delta^=}^{GAC}$ (because no value belongs to $\bot$), and BoundSAC[1] is equivalent to $S_{\Delta^{bnd}}^{GAC}$. Note also that GAC is equivalent to $S_{\Delta^{id}}^{GAC}$. As a simple illustration of S_Δ^ϕ, let us consider the five binary CNs depicted in Figure 1; each vertex denotes a value, each edge denotes an allowed tuple and each dotted vertex (resp., edge) means that the value (resp., tuple) is removed (resp., no more relevant). P_1, P_2, P_3 and P_4 are obtained from P by removing values that are S_Δ^{AC}-inconsistent when Δ is set to Δ^{id}, Δ^{P_2}, Δ^{bnd} and $\Delta^=$, respectively. For example, for Δ^{P_2}, we find that $(y, c) \notin AC(P|_{y \in \{c,d\}})$. Note that the CN P_4 is also obtained when setting Δ to $\Delta^{\neq}$.

[1] Another related consistency is Existential SAC [16], which guarantees that some value in the domain of each variable is SAC-consistent. However, there is no guarantee about the network obtained after checking Existential SAC due to the non-deterministic nature of this consistency. Existential SAC is not an element of S_Δ^ϕ.



(a) P (b) P_1 (c) P_2 (d) P_3 (e) P_4

Fig. 1. Illustration of S_{Δ}^{GAC}

In [2], it is also shown that inferences regarding values may be obtained by considering the result of several decision-based checks. This is generalized below. The idea is that a value (x, a) of P can be safely removed when there exist a variable y and a cover $\Gamma \subseteq \Delta(y)$ of $dom(y)$ such that every decision-based check, performed from a decision in Γ, eliminates (x, a).

Definition 2 (Consistency E_{Δ}^{ϕ}). *A value (x, a) of a CN P is E_{Δ}^{ϕ}-consistent w.r.t. a variable $y \neq x$ of P iff for every cover Γ of $dom(y)$ such that $\Gamma \subseteq \Delta(y)$, there exists a decision $y \in D_y$ in Γ such that $(x, a) \in \phi(P|_{y \in D_y})$. (x, a) is E_{Δ}^{ϕ}-consistent iff (x, a) is E_{Δ}^{ϕ}-consistent w.r.t. every variable $y \neq x$ of P.*

Proposition 2. *Any E^{ϕ}_{Δ}-inconsistent value is globally inconsistent.*

Proof. If (x, a) is an E^{ϕ}_{Δ}-inconsistent value, then we know that there exists a variable $y \neq x$ of P and a set $\Gamma \subseteq \Delta(y)$ such that (i) $dom^P(y) = \cup_{(y \in D_y) \in \Gamma} D_y$ and (ii) every decision $y \in D_y$ in Γ entails $(x, a) \notin \phi(P|_{y \in D_y})$. As Γ is a cover of $dom(y)$, we infer that $sols(P) = \cup_{(y \in D_y) \in \Gamma} sols(P|_{y \in D_y})$. Because ϕ preserves solutions, we have $sols(P) = \cup_{(y \in D_y) \in \Gamma} sols(\phi(P|_{y \in D_y}))$. For every $y \in D_y$ in Γ, we know that $(x, a) \notin \phi(P|_{y \in D_y})$. We deduce that (x, a) cannot be involved in any solution. □

As an illustration, let us consider the CN of Figure 1(a) and $\Delta(x) = \{x \in \{a, c\}, x \in \{b, d\}\}$. We can show that (z, a) is E^{GAC}_{Δ}-inconsistent because $(z, a) \notin AC(P|_{x \in \{a,c\}})$ and $(z, a) \notin AC(P|_{x \in \{b,d\}})$. The consistency P-k-AC, introduced in [2], corresponds to S^{ϕ}_{Δ}+E^{ϕ}_{Δ} where $\phi = AC$ and Δ necessarily corresponds to a partition of each domain into pieces of size at most k.

3.3 Classes Related to Nogoods of Size 2

Decision-based consistencies introduced above are clearly domain-filtering: they allow us to identify inconsistent values. However, decision-based consistencies are also naturally orientated towards identifying nogoods of size 2. $NG2(P)^{\phi}_{\Delta}$ denotes the set of nogoods of size 2 that can be directly derived from checks on P based on the consistency ϕ and the decision mapping Δ. From this set, together with a decision $x \in D_x$, we obtain a set $ND1(P, x \in D_x)^{\phi}_{\Delta}$ of negative decisions that can be used to make further inferences.

Definition 3. *Let P be a CN and $x \in D_x$ be a membership decision in $\Delta(x)$.*

- *$NG2(P)^{\phi}_{x \in D_x}$ denotes the set of locally consistent instantiations $\{(x, a), (y, b)\}$ on P such that $a \in D_x$ and $(y, b) \notin \phi(P|_{x \in D_x})$.*
- *$NG2(P)^{\phi}_{\Delta}$ denotes the set $\cup_{\delta \in \Delta} NG2(P)^{\phi}_{\delta}$.*
- *$ND1(P, x \in D_x)^{\phi}_{\Delta}$ denotes the set of negative decisions $y \neq b$ such that every value $a \in D_x$ is such that $\{(x, a), (y, b)\} \in \widetilde{P}$ or $\{(x, a), (y, b)\} \in NG2(P)^{\phi}_{\Delta \setminus \{x \in D_x\}}$.*

From *ND1* sets, we can define a new class B^{ϕ}_{Δ} of consistencies.

Definition 4 (Consistency B^{ϕ}_{Δ}). *A value (x, a) of a CN P is B^{ϕ}_{Δ}-consistent iff for every membership decision $x \in D_x$ in $\Delta(x)$ such that $a \in D_x$, we have $(x, a) \in \phi(P|_{\{x \in D_x\} \cup ND1(P, x \in D_x)^{\phi}_{\Delta}})$.*

Proposition 3. *Any B^{ϕ}_{Δ}-inconsistent value is globally inconsistent.*

Proof. The proof is similar to that of Proposition 1. The only difference is that the network P is made smaller by removing some additional values by means of negative decisions. However, in the context of a decision $x \in D_x$ taken on P, the inferred negative decisions correspond to inconsistent values because they are derived from nogoods of size 2 (showing that elements of $NG2(P)^{\phi}_{\Delta}$ are nogoods is immediate). □

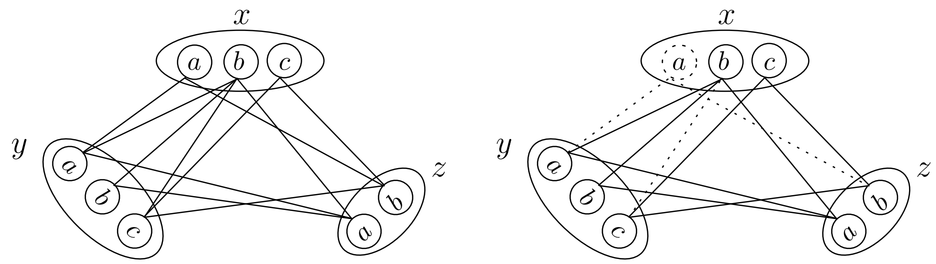


Fig. 2. Illustration of B^{GAC}_{Δ} and D^{GAC}_{Δ}

As an illustration of B^{ϕ}_{Δ}, let us consider the binary CN P in Figure 2(a). For $\phi = AC$ and $\Delta = \Delta^{P_2} = \{x \in \{a,b\}, x = c, y \in \{a,b\}, y = c, z = a, z = b\}$ we obtain $NG2(P)^{\phi}_{\Delta} = \{\{(x,a),(y,a)\}, \{(x,a),(z,b)\}, \{(x,b),(y,c)\}\}$ since for example $(x,a) \notin AC(P|_{y\in\{a,b\}})$. Because $\{(x,b),(z,b)\} \in \widetilde{P}$ and $\{(x,a),(z,b)\} \in NG2(P)^{\phi}_{\Delta}$, $ND1(P, x \in \{a,b\})^{\phi}_{\Delta} = \{z \neq b\}$, and (x,a) is B^{ϕ}_{Δ}-inconsistent as $(x,a) \notin AC(P|_{x\in\{a,b\}\cup\{z\neq b\}})$. Here, P is S^{ϕ}_{Δ}-consistent, but not B^{ϕ}_{Δ}-consistent.

Note that BiSAC [4] is equivalent to $B^{GAC}_{\Delta^{=}}$. On the other hand, there is a 2-order consistency that can be naturally defined as follows.

Definition 5 (Consistency D^{ϕ}_{Δ}). *A locally consistent instantiation $\{(x,a),(y,b)\}$ on a CN P is D^{ϕ}_{Δ}-consistent iff for every membership decision $x \in D_x$ in $\Delta(x)$ such that $a \in D_x$, $(y,b) \in \phi(P|_{x\in D_x})$ and for every membership decision $y \in D_y$ in $\Delta(y)$ such that $b \in D_y$, $(x,a) \in \phi(P|_{y\in D_y})$.*

Proposition 4. *Any D^{ϕ}_{Δ}-inconsistent instantiation is globally inconsistent.*

Proof. D^{ϕ}_{Δ}-inconsistent instantiations are exactly those in $NG2(P)^{\phi}_{\Delta}$, which are nogoods. □

Note that DC [15] is equivalent to $D^{GAC}_{\Delta^{=}}$, and recall that DC is equivalent to PC (Path Consistency) for binary CNs. D^{ϕ}_{Δ} (being 2-order) is obviously incomparable with previously introduced domain-filtering consistencies. However, a natural practical approach is to benefit from decision-based checks to record both S^{ϕ}_{Δ}-inconsistent values and D^{ϕ}_{Δ}-inconsistent instantiations. This corresponds to the combined consistency $S^{\phi}_{\Delta}+D^{\phi}_{\Delta}$.

As an illustration of D^{ϕ}_{Δ}, let us consider again Figure 2. For $\phi = AC$ and $\Delta = \Delta^{P_2} = \{x \in \{a,b\}, x = c, y \in \{a,b\}, y = c, z = a, z = b\}$, we have that P is S^{ϕ}_{Δ}-consistent, not B^{ϕ}_{Δ}-consistent and not D^{ϕ}_{Δ}-consistent. Enforcing $S^{\phi}_{\Delta} + D^{\phi}_{\Delta}$ on P yields the CN P', which is also the strong DC-closure (here, AC+PC-closure) of P.

4 Qualitative Study

In this section, we study the relationships between the different classes of consistencies (as well as some of their combinations), and discuss refinements and well-behavedness of consistencies.

4.1 Relationships between Consistencies

From Definitions 1 and 4, it is immediate that any S^{ϕ}_{Δ}-inconsistent value is necessarily B^{ϕ}_{Δ}-inconsistent.

Proposition 5. $B^{\phi}_{\Delta} \trianglerighteq S^{\phi}_{\Delta}$.

In order to relate B^{ϕ}_{Δ} with E^{ϕ}_{Δ}, we need to consider covering sets of decisions.

Proposition 6. *If Δ is covering, $B^{\phi}_{\Delta} \trianglerighteq E^{\phi}_{\Delta}$.*

Proof. We show that every E^{ϕ}_{Δ}-inconsistent value in a CN P is necessarily B^{ϕ}_{Δ}-inconsistent. Assume that (x, a) is a E^{ϕ}_{Δ}-inconsistent value. It means that there exists a variable $y \neq x$ of P and $\Gamma \subseteq \Delta(y)$ such that $dom^P(y) = \cup_{(y \in D_y) \in \Gamma} D_y$ and every decision $y \in D_y$ in Γ is such that $(x, a) \notin \phi(P|_{y \in D_y})$. We deduce that for every value $b \in dom^P(y)$, we have $\{(x, a), (y, b)\}$ in $NG2(P)^{\phi}_{\Delta}$. On the other hand, we know that there exists a decision $x \in D_x$ in Δ such that $a \in D_x$ (since Δ is covering). Hence, $ND1(P, x \in D_x)^{\phi}_{\Delta}$ contains a negative decision $y \neq b$ for each value in $dom^P(y)$. It follows that $\phi(P|_{\{x \in D_x\} \cup ND1(P, x \in D_x)^{\phi}_{\Delta}}) = \bot$, and (x, a) is B^{ϕ}_{Δ}-inconsistent. □

As a corollary, we have $B^{\phi}_{\Delta} \trianglerighteq S^{\phi}_{\Delta} + E^{\phi}_{\Delta}$ when Δ is covering. Note that there exist consistencies ϕ and decision mappings Δ such that B^{ϕ}_{Δ} is strictly stronger ($\triangleright$) than S^{ϕ}_{Δ} and E^{ϕ}_{Δ} (and also $S^{\phi}_{\Delta}+E^{\phi}_{\Delta}$). For example, when $\phi = AC$ and $\Delta = \Delta^{=}$, we have $B^{\phi}_{\Delta} = BiSAC$, $S^{\phi}_{\Delta} = SAC$ and $S^{\phi}_{\Delta} + E^{\phi}_{\Delta} = 1\text{-}AC$, and we know that $BiSAC \triangleright 1\text{-}AC$ [4], and $1\text{-}AC \triangleright SAC$ [2].

Because D^{ϕ}_{Δ} captures all 2-sized nogoods while S^{ϕ}_{Δ} can eliminate inconsistent values, it follows that the joint use of these two consistencies is stronger than B^{ϕ}_{Δ}.

Proposition 7. $S^{\phi}_{\Delta}+D^{\phi}_{\Delta} \trianglerighteq B^{\phi}_{\Delta}$.

Proof. Let P be a CN that is $S^{\phi}_{\Delta}+D^{\phi}_{\Delta}$-consistent. As P is S^{ϕ}_{Δ}-consistent, for every decision $x \in D_x$ in Δ and every $a \in D_x$, we have $(x, a) \in \phi(P|_{x \in D_x})$. But $\phi(P|_{x \in D_x}) = \phi(P|_{\{x \in D_x\} \cup ND1(P, x \in D_x)^{\phi}_{\Delta}})$ since P being D^{ϕ}_{Δ}-consistent entails $NG2(P)^{\phi}_{\Delta} = \emptyset$ and $ND1(P, x \in D_x)^{\phi}_{\Delta} = \emptyset$. We deduce that P is B^{ϕ}_{Δ}-consistent. □

One may expect that $S^{\phi}_{\Delta} \trianglerighteq \phi$. However, to guarantee this, we need both ϕ to be domain-filtering and Δ to be covering, For example, $S^{AC}_{\Delta} \trianglerighteq AC$ does not hold if for every $dom_x \subseteq dom^{init}(x)$, we have $\Delta(x, dom_x) = \emptyset$: it suffices to build a CN P with a value (x, a) being arc-inconsistent.

Proposition 8. *If ϕ is domain-filtering and Δ is covering, $S^{\phi}_{\Delta} \trianglerighteq \phi$.*

Proof. Assume that (x,a) is a ϕ-inconsistent value of a CN P. This means that $(x,a) \notin \phi(P)$. As Δ is covering, there exists a decision $x \in D_x$ in Δ with $a \in D_x$. We know that $P|_{x \in D_x} \preceq P$. By monotonicity of ϕ, $\phi(P|_{x\in D_x}) \preceq \phi(P)$. Since $(x,a) \notin \phi(P)$, we deduce that $(x,a) \notin \phi(P|_{x \in D_x})$. So, (x,a) is S^{ϕ}_{Δ}-inconsistent, and S^{ϕ}_{Δ} is stronger than ϕ. □

Figure 3 shows the relationships between the different classes of consistencies introduced so far. There are many ways to instantiate these classes because the choice of Δ and ϕ is left open. If we consider binary CNs, and choose $\phi = AC$ and $\Delta = \Delta^{=}$, we obtain known consistencies. We directly benefit from the relationships of Figure 3, and have just to prove strictness when it holds. Figure 4 shows this where an arrow denotes now $\triangleright$ (instead of $\trianglerighteq$). An extreme instantiation case is when $\Delta = \Delta^{id}$ and ϕ is domain-filtering. In this case, all consistencies collapse: we have $S^{\phi}_{\Delta^{id}} = E^{\phi}_{\Delta^{id}} = B^{\phi}_{\Delta^{id}} = D^{\phi}_{\Delta^{id}} = \phi$. This means that our framework of decision-based consistencies is general enough to encompass all classical local consistencies. Although this is appealing for theoretical reasons (e.g., see Proposition 11 later), the main objective of decision-based consistencies remains to learn relevant nogoods from nontrivial decision-based checks.

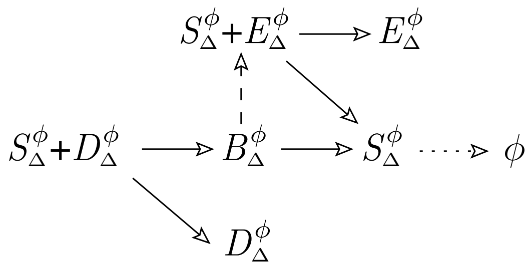


Fig. 3. Summary of the relationships between (classes of) consistencies. An arrow from φ to ψ means that $\varphi \trianglerighteq \psi$. A dashed (resp., dotted) arrow means that the relationship is guaranteed provided that Δ is covering (resp., Δ is covering and ϕ is domain-filtering).

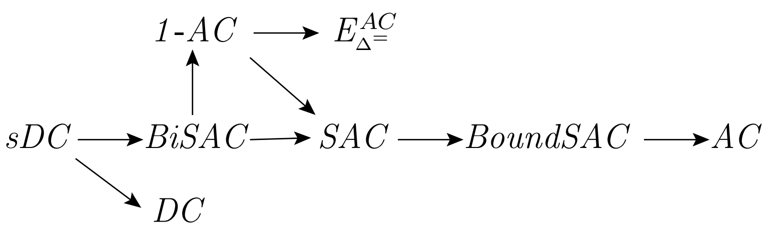


Fig. 4. Relationships between consistencies when $\phi = AC$ and $\Delta = \Delta^{=}$ (except for BoundSAC which is derived from Δ^{bnd}). An arrow from φ to ψ means that $\varphi \triangleright \psi$.

4.2 Refinements

Now, we show that two consistencies of the same class can be naturally compared when a refinement connection exists between their decision mappings.

Definition 6. *A decision mapping Δ' is a refinement of a decision mapping Δ iff for each decision $x \in D_x$ in Δ there exists a subset $\Gamma \subseteq \Delta'(x)$ that is a cover of D_x.*

For example, $\{x \in \{a,b\}, x = c\}$ is a refinement of $\{x \in \{a,b,c\}\}$, and $\{x \in \{a,b\}, x = c, y = a, y = b, y = c\}$ is a refinement of $\{x \in \{a,b,c\}, y \in \{a,b\}, y \in \{b,c\}\}$. Unsurprisingly, using refined sets of decisions improves inference capability as shown by the following proposition.

Proposition 9. *If Δ and Δ' are two decision mappings such that Δ' is a refinement of Δ, then $X^{\phi}_{\Delta'} \trianglerighteq X^{\phi}_{\Delta}$ where $X \in \{S,E,B,D\}$.*

Proof. Due to lack of space, we only show that $S^{\phi}_{\Delta'} \trianglerighteq S^{\phi}_{\Delta}$. Assume that (x,a) is an S^{ϕ}_{Δ}-inconsistent value of a CN P. This means that there exists a decision $x \in D_x$ in $\Delta(x)$ such that $a \in D_x$ and $(x,a) \notin \phi(P|_{x \in D_x})$. We know, by hypothesis, that there exists a subset $\Gamma \subseteq \Delta'(x)$ such that $D_x = \cup_{(x \in D'_x) \in \Gamma} D'_x$. Hence, there exists (at least) a decision $x \in D'_x$ in Γ such that $a \in D'_x$ and $D'_x \subseteq D_x$. As $D'_x \subseteq D_x$, we have $P|_{x \in D'_x} \preceq P|_{x \in D_x}$, and by monotonicity of ϕ, $\phi(P|_{x \in D'_x}) \preceq \phi(P|_{x \in D_x})$. Consequently, $(x,a) \notin \phi(P|_{x \in D_x})$ implies $(x,a) \notin \phi(P|_{x \in D'_x})$. We deduce that there exists a decision $x \in D'_x$ in $\Delta'(x)$ such that $a \in D'_x$ and $(x,a) \notin \phi(P|_{x \in D''_x})$. Then (x,a) is $S^{\phi}_{\Delta'}$-inconsistent. We conclude that $S^{\phi}_{\Delta'} \trianglerighteq S^{\phi}_{\Delta}$. □

As a corollary, for any decision mapping Δ, we have: $X^{\phi}_{\Delta^{=}} \trianglerighteq X^{\phi}_{\Delta} \trianglerighteq X^{\phi}_{\Delta^{id}}$ where $X \in \{S,E,B,D\}$. In particular, if $\phi = GAC$, we have $SAC = S^{GAC}_{\Delta^{=}} \trianglerighteq S^{GAC}_{\Delta} \trianglerighteq S^{GAC}_{\Delta^{id}} = GAC$.

Because, consistencies S^{ϕ}_{Δ} identify inconsistent values on the basis of a single decision, we obtain the two following results. In the spirit of our set view of decision mappings, for any two decision mappings Δ_1 and Δ_2, $\Delta_1 \cup \Delta_2$ is the decision mapping such that for every variable x and every $dom_x \subseteq dom^{init}(x)$, $(\Delta_1 \cup \Delta_2)(x, dom_x) = \Delta_1(x, dom_x) + \Delta_2(x, dom_x)$.

Proposition 10. *Let Δ_1 and Δ_2 be two decision mappings. We have $S^{\phi}_{\Delta_1} + S^{\phi}_{\Delta_2} = S^{\phi}_{\Delta_1 \cup \Delta_2}$.*

Proof. Let P be a CN and (x,a) be a value of P. (x,a) is $S^{\phi}_{\Delta_1 \cup \Delta_2}$-inconsistent $\Leftrightarrow$ there exists a decision $x \in D_x$ in $\Delta_1 \cup \Delta_2$ such that $(x,a) \notin \phi(P|_{x \in D_x}) \Leftrightarrow (x,a)$ is $S^{\phi}_{\Delta_1}$-inconsistent or (x,a) is $S^{\phi}_{\Delta_2}$-inconsistent $\Leftrightarrow (x,a)$ is $S^{\phi}_{\Delta_1} + S^{\phi}_{\Delta_2}$-inconsistent. □

$\mathscr{S}^{\phi}$ denotes the set of equivalence classes modulo $\approx$ of the consistencies S^{ϕ}_{Δ} that can be built from ϕ and all possible decision mappings Δ. It forms a complete lattice, in a similar way to what has been shown for qualitative constraint networks [7].

Proposition 11. $(\mathscr{S}^\phi, \trianglerighteq)$ *is a complete lattice with* $S^\phi_{\Delta^=}$ *as greatest element and* $S^\phi_{\Delta^{id}}$ *as least element.*

Proof. Let $S^\phi_{\Delta_1}$ and $S^\phi_{\Delta_2}$ be two consistencies in $\mathscr{S}^\phi$.
(Existence of binary joins) From Proposition 10, we can infer that $S^\phi_{\Delta_1 \cup \Delta_2}$ is the least upper bound of $S^\phi_{\Delta_1}$ and $S^\phi_{\Delta_2}$.

(Existence of binary meets) Let us define the set E as $E = \{S^\phi_\Delta \in \mathscr{S}^\phi : S^\phi_\Delta \trianglelefteq S^\phi_{\Delta_1}$ and $S^\phi_\Delta \trianglelefteq S^\phi_{\Delta_2}\}$. Note that $E \neq \emptyset$ since $S^\phi_{\Delta^{id}} \in E$. Next, let us define $S^\phi_{\Delta^E}$ such that $\Delta^E = \bigcup_{S^\phi_{\Delta_i} \in E} \Delta_i$. For every $S^\phi_{\Delta_i} \in E$, Δ^E is a refinement Δ_i, and so, from Proposition 9, we know that $S^\phi_{\Delta^E}$ is an upper bound of E. We now prove by contradiction that $S^\phi_{\Delta^E} \trianglelefteq S^\phi_{\Delta_1}$. Suppose that there is a value (x, a) of a CN P that is $S^\phi_{\Delta^E}$-inconsistent and $S^\phi_{\Delta_1}$-consistent. This means that there exists a decision $x \in D_x$ in $\Delta(x)$ such that $(x, a) \notin \phi(P|_{x \in D_x})$. From construction of Δ, we know that there exists a decision mapping Δ_i such that $S^\phi_{\Delta_i} \in E$ and $x \in D_x$ is in Δ_i. By definition of E, we know that $S^\phi_{\Delta_i} \trianglelefteq S^\phi_{\Delta_1}$. Consequently, (x, a) is $S^\phi_{\Delta_i}$-consistent and $(x, a) \in \phi(P|_{x \in D_x})$. This is a contradiction, so $S^\phi_\Delta \trianglelefteq S^\phi_{\Delta_1}$. Similarly, we have $S^\phi_\Delta \trianglelefteq S^\phi_{\Delta_2}$. Then S^ϕ_Δ is the greatest lower bound of $S^\phi_{\Delta_1}$ and $S^\phi_{\Delta_2}$. □

4.3 Well-Behavedness

Finally, we are interested in well-behavedness of consistencies. Actually, in the general case, the consistencies S^ϕ_Δ, E^ϕ_Δ, B^ϕ_Δ and D^ϕ_Δ are not necessarily well-behaved for $(\mathscr{P}, \preceq)$. Consider as an illustration three CNs P, P_1 and P_2 which differ only by the domain of the variable x: $dom^P(x) = \{a, b, c, d\}$, $dom^{P_1}(x) = \{a, b, c\}$ and $dom^{P_2}(x) = \{d\}$. Now, consider a decision mapping Δ defined for the variable x and the domains $\{a, b, c, d\}$, $\{a, b, c\}$ and $\{d\}$ by: $\Delta(x, \{a, b, c, d\}) = \{x \in \{a\}\}$, $\Delta(x, \{a, b, c\}) = \{x \in \{a, b, c\}\}$ and $\Delta(x, \{d\}) = \{x \in \{d\}\}$. Despite the fact that $dom^P(x) = dom^{P_1}(x) \cup dom^{P_2}(x)$, one can see that the value (x, a) could be S^ϕ_Δ-consistent in P_1 and P_2, whereas S^ϕ_Δ-inconsistent in P. With such a Δ, S^ϕ_Δ is not guaranteed to be well-behaved.

Nevertheless, there exist decision mappings for which consistencies are guaranteed to be well-behaved, at least those of the class S^ϕ_Δ. Informally, a relevant decision mapping is a decision mapping that keeps its precision (in terms of decisions) when domains are restricted.

Definition 7. *A decision mapping Δ is said to be* relevant *if and only if for any variable x, any two sets of values dom_x and dom'_x such that $dom'_x \subsetneq dom_x \subseteq dom^{init}(x)$ and any decision $x \in D_x$ in $\Delta(x, dom_x)$, we have:*

$$D_x \cap dom'_x \neq \emptyset \Rightarrow \exists \Gamma \subseteq \Delta(x, dom'_x) \mid D_x \cap dom'_x = \cup_{(x \in D'_x) \in \Gamma} D'_x.$$

We can notice that Δ^{id}, $\Delta^=$, $\Delta^{\neq}$, Δ^{bnd} are relevant decision mappings. For our proposition, we need some additional definitions. A CN P' is a sub-CN of a

CN P if P' can be obtained from P by simply removing certain values. If P_1 and P_2 are two CNs that only differ by the domains of their variables, then $P = P_1 \cup P_2$ is the CN such that P_1 and P_2 are sub-CNs of P and for every variable x, $dom^P(x) = dom^{P_1}(x) \cup dom^{P_2}(x)$.

Proposition 12. *Let Δ be a relevant decision mapping and let P, P_1, and P_2 be three CNs such that $P = P_1 \cup P_2$. If P_1 and P_2 are S^ϕ_Δ-consistent then P is S^ϕ_Δ-consistent.*

Proof. Let (x, a) be a value of $P = P_1 \cup P_2$. Let us show that this value is S^ϕ_Δ-consistent. Consider a membership decision $x \in D_x$ in $\Delta(x, dom^P(x))$ such that $a \in D_x$. We have to show that $(x, a) \in \phi(P|_{x \in D_x})$. We know that $dom^P(x) = dom^{P_1}(x) \cup dom^{P_2}(x)$. Hence, $a \in dom^{P_1}(x)$ or $x \in dom^{P_2}(x)$. Assume that $a \in dom^{P_1}(x)$ (the case $a \in dom^{P_2}(x)$ can be handled in a similar way). Since Δ is a relevant decision mapping, there exists $\Gamma \subseteq \Delta(x, dom^{P_1}(x))$ such that $D_x \cap dom^{P_1}(x) = \cup_{(x \in D'_x) \in \Gamma} D'_x$. It follows that there exists a decision $x \in D^1_x$ in $\Delta(x, dom^{P_1}(x))$ such that $a \in D^1_x$ and $D^1_x \subseteq D_x$. From the fact that P_1 is S^ϕ_Δ-consistent we know that $(x, a) \in \phi(P_1|_{x \in D^1_x})$. Since $a \in D^1_x$, $D^1_x \subseteq D_x$ and P_1 is a sub-CN of P we can assert that $(x, a) \in \phi(P|_{x \in D_x})$. We conclude that (x, a) is a S^ϕ_Δ-consistent value of P. □

Corollary 1. *If Δ is a relevant decision mapping then S^ϕ_Δ is well-behaved.*

Indeed, to obtain the closure of a CN P, it suffices to take the union of all sub-CNs of P which are S^ϕ_Δ-consistent. Hence, the consistency S^ϕ_Δ for which Δ is a relevant decision mapping is well-behaved for $(\mathscr{P}, \preceq)$.

5 Conclusion

In this paper, our aim was to give a precise picture of decision-based consistencies by developing a hierarchy of general classes. This general framework offers the user a vast range of new consistencies. Several issues have now to be addressed. First, me must determine the conditions under which overlapping between decisions may be beneficial. Overlapping allows us to cover domains while considering weak decisions (e.g., decisions in $\Delta^{\neq}$) that are quick to propagate, and might also be useful to tractability procedures (e.g., in situations where only some decisions lead to known tractable networks). Second, we must seek to elaborate dynamic procedures (heuristics) so as automatically select the right decision-based consistency (set of membership decisions) at each step of a backtrack search as in [19]; many new combinations are permitted. Finally, bound consistencies and especially singleton checks on bounds may be revisited by checking several values at once (using intervals at bounds with the mechanism of detecting X^ϕ_Δ-inconsistent values), so as to speed up the inference process in shaving procedures. These are some of the main perspectives.

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Consistency of qualitative constraint networks from tree decompositions

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Consistency of qualitative constraint networks from tree decompositions

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***Abstract*—A common way to decide the consistency problem of a qualitative constraint network (QCN) is to encode it as a boolean formula in order to benefit from the efficiency of SAT solvers. In recent works, a decomposition method of QCNs have been proposed to reduce the amount of boolean formulae. In this paper, we first show that the decompositions used can be expressed by particular tree decompositions. Furthermore, for some classes of relations, we prove that the consistency problem of a QCN can be decided by applying the method of the closure by weak composition on the clusters of a tree decomposition. This result allows us to extend the approach recently proposed to tree decompositions of QCNs.**

***Keywords*-Temporal qualitative constraints ; Consistency problem ; Tree decomposition**

I. Introduction

Reasoning about temporal or spatial knowledge is a major task in many domains of Artificial Intelligence, such as Geographic Information System (GIS), natural language processing, temporal/spatial scheduling, and so on. Qualitative reasoning is a way to express and process the qualitative aspect of knowledge about temporal or spatial entities. A qualitative calculus considers a domain from temporal or spatial entities and a finite set of base relations over these entities. Each base relation symbolizes a relative position between the entities, and is a factoring for configurations given as numeric information. These previous decades, many qualitative calculi have been studied. The Interval Algebra [1] represents temporal entities by intervals and considers thirteen base relations describing each possible relative position between two temporal entities (see Figure 1). Many qualitative calculi for temporal knowledge are derived from the Interval Algebra [2], [3], [4]. In spatial reasoning, the well-known qualitative calculus RCC [5], [6] is quite possibly the most studied. RCC is based on eight base relations between entities defined over all regions of a topological space. In the qualitative calculi frameworks, the set of temporal or spatial information may be represented by some specific constraint networks called qualitative constraint networks (QCNs). In a QCN, each variable stands for a temporal or spatial entity and each constraint restricts the possible configurations between entities by using a set of base relations. Given a QCN, the main decision problem is the consistency problem. In the general case, this problem is NP-complete. To solve it, we can cite two kinds of approaches can be used.

The first, introduced by Nebel [7], consists in a synchronous backtracking algorithm by maintaining a local consistency. This algorithm exploits certain tractable classes of relations with multi-valued assignments and the weak composition closure allowing to obtain a local consistency close to path-consistency. At each step during search, a constraint relation is split into sub-relations belonging to the tractable class. The constraint is iteratively replaced by each sub-relation. This cutting allows us to decrease the branching factor during the search. Most of the QCN solvers exploit this solving method, in particular GQR* [8] which is currently the most efficient solver.

The second approach [9], [10] consists in encoding the consistency problem of QCNs as boolean formulae in order to benefit from the efficiency of SAT solvers. However, the counter-part of this approach is the large amount of the boolean formulae obtained. Recently, in order to overcome this drawback for QCNs of the Interval Algebra, Li *et al.* [11] proposed a method of decomposition allowing to discard some constraints of the QCN. This method splits recursively the QCN considered in two equivalent sub-QCNs. Each constraint of the initial QCN which is absent from one of the two QCNs is considered as useless to decide the consistency, and is not considered in the encoding. The amount of the boolean formulae is greatly reduced compared to the full encoding, and the experimental results show an improvement in term of solving time.

In this paper, we define particular decompositions called RecPart decompositions. These specific decompositions formalize the decompositions of QCNs used by Li *et al.* in their process of SAT encoding. Then, we show that the RecPart decompositions can be equivalently defined as tree decompositions [12], widely studied in the framework of finite CSPs. Moreover, we study the consistency problem for the QCN through the tree decompositions. In particular, we show that, given a tree decomposition for a QCN defined over a tractable class of relations, the weak composition closure applied on each cluster is sufficient to decide the consistency of the QCN. Finally, by exploiting this result we propose to decide the consistency of QCNs by using a tractable class of relations and a tree decomposition in the framework of boolean formula encodings.

II. Preliminary notions on qualitative calculi

A. Qualitative calculi

A qualitative calculus considers a finite set of relations B, called *base relations*, over an infinite domain D representing the temporal or spatial entities. In our study, we focus on binary relations (a large part of qualitative calculi considers this kind of base relations). Each base relation of B represents a certain relative position between spatial or temporal entities. They are *Jointly Exhaustive and Pairwise Disjoint* (JEPD), *i.e.* each element $(x, y) \in \mathsf{D} \times \mathsf{D}$ belongs to one and only one $a \in \mathsf{B}$. The set B has some properties [13] : (1) B is a partition of $\mathsf{D} \times \mathsf{D}$, (2) B contains the identity relation id and, (3) B is closed by converse, *i.e.* the converse of an base relation in B is also in B. As an illustration, the *Interval Algebra* (IA), also known as *Allen's calculus* [1], considers intervals of the line to represent the temporal entities. The domain D is defined by $\mathsf{D} = \{(x^-, x^+) \in \mathbb{Q} \times \mathbb{Q} \mid x^- < x^+\}$. The base relations correspond to the set $\mathsf{B} = \{eq, p, pi, m, mi, o, oi, s, si, d, di, f, fi\}$. Each of these base relations symbolizes a relative position between two temporal intervals, which is illustrated in Figure 1.

A *complex relation*, also called *relation*, for a qualitative calculus is an union of base relations. It is customary to represent a relation by the set of the base relations which compose it. Hence, in the sequel we make no distinction between the set of relations and the set 2^{B} which will represent the set of relations of a qualitative calculus based on the set of base relations B. In the Interval Algebra, the relation $r = p \cup mi \cup eq$ will be represented by the set $\{p, mi, eq\}$. The usual set-theoretic operations *union* ($\cup$), *intersection* ($\cap$) and *converse* ($\cdot^{-1}$) are defined over 2^{B}. For a relation $r \in 2^{\mathsf{B}}$, the converse is defined as $r^{-1} = \bigcup\{a^{-1} | a \in r\}$. Among the relations of 2^{B}, Ψ denotes the relation that contains all the base relations of B. The set 2^B is also equipped with the *weak composition* operation, denoted by $\diamond$, defined as: $a \diamond b = \{c \in \mathsf{B} : \exists x, y, z \in \mathsf{D}$ with $x\ a\ z, z\ b\ y$ and $x\ c\ y\}$, with $a, b \in \mathsf{B}$, $r \diamond s = \bigcup_{a \in r, b \in s}\{a \diamond b\}$, with $r, s \in 2^{\mathsf{B}}$. The relation $r \diamond s$ is also defined as the strongest relation in 2^{B} which contains the usual composition $r \circ s = \{(x, y) \in \mathsf{D} \times \mathsf{D} : \exists x, y, z \in \mathsf{D}$ with $x\ a\ z, z\ b\ y$ and $x\ c\ y\}$. For some qualitative calculi, $r \circ s$ and $r \diamond s$ are equivalent.

A *class* of relations $\mathcal{C}$ is a subset of 2^{B} which contains the relation Ψ, all of the singleton relations of 2^{B}, and which is closed under converse, intersection and weak composition. Given $r \in 2^{\mathsf{B}}$ and a class $\mathcal{C}$, the smallest relation of $\mathcal{C}$ which contains r is denoted by $r^{\mathcal{C}}$ and is called the closure of r in $\mathcal{C}$. In IA, let us consider the relation $\{p, m\} \in 2^{\mathsf{B}}$ between two entities X and Y. $X\ \{p, m\}\ Y$ means that X *precedes* or *meets* Y. The converse relation $\{p, m\}^{-1} = \{p^{-1}, m^{-1}\} = \{pi, mi\}$ expresses the relation between Y and X, so Y *is preceded* or *is met by* X. Finally, let us introduce $Y\ \{m, s, eq\}\ Z$. We have some information about the relations between X, Y and Y, Z, so we can deduce information about the relation between X, Z by using the weak composition. Since $\{p, m\} \diamond \{m, s, eq\} = \{p, m\}$, we have $X\ \{p, m\}\ Z$.

B. Qualitative Constraint Networks

A Qualitative Constraint Network (QCN) consists of a finite set of m variables $V = \{v_1, \ldots, v_m\}$ which represent the spatial or temporal entities, and a map C from $V \times V$ to 2^{B} such that $C(v_i, v_i) \subseteq \{\mathsf{Id}\}$ for each $v_i \in V$, with Id the base relation corresponding to the identity relation over D, and $C(v_i, v_j) = C(v_j, v_i)^{-1}$ for all $v_i, v_j \in V$. In the rest of this paper, we will also denote $C(v_i, v_j)$ by $\mathcal{N}[v_i, v_j]$. The figure 2 illustrates a QCN $\mathcal{N}$ of the Interval Algebra. In this figure, a variable is represented by a node, and a constraint by an arc labelled with the associated relation. Note that, for simplicity, there is no arc going from v_i to v_j when either there is already an arc going from v_j to v_i or $i = j$.

Given a QCN $\mathcal{N} = (V, C)$, a *partial instantiation* of $\mathcal{N}$ on $V' \subseteq V$ is a map s from V' to D. A *partial solution* on $\mathcal{N}$ on $V' \subseteq V$ is a partial instantiation on V' such that $(s(v_i), s(v_j))$ satisfies $C(v_i, v_j)$ for all $v_i, v_j \in V'$, *i.e.* there exists a base relation $b \in C(v_i, v_j)$ such that $(s(v_i), s(v_j)) \in b$ for all $v_i, v_j \in V'$. A *solution* of $\mathcal{N}$ is a partial solution of $\mathcal{N}$ on V. $\mathcal{N}$ is *consistent* if, and only if,

Relation	Symbol	Converse	Illustration
precedes	p	pi	
meets	m	mi	
overlaps	o	oi	
starts	s	si	
during	d	di	
finishes	f	fi	
equals	eq	eq	

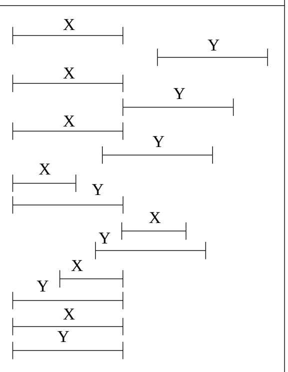

Figure 1. The base relations of the Interval Algebra.

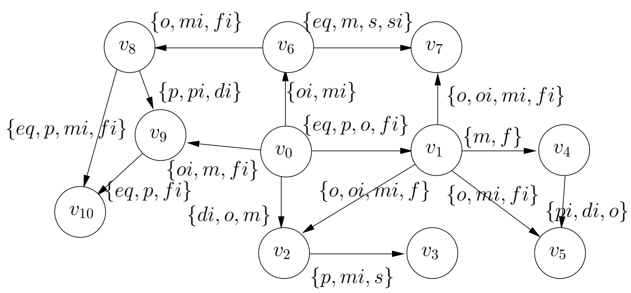

Figure 2. A QCN $\mathcal{N} = (V, C)$ of the Interval Algebra.

there exists a solution of $\mathcal{N}$. $\mathcal{N}$ is *trivially inconsistent* when there exist two variables $v, v' \in V$ such that $\mathcal{N}[v, v'] = \emptyset$. $\mathcal{N}$ is *globally consistent* if, and only if, each partial solution of $\mathcal{N}$ can be extended to a solution of $\mathcal{N}$. The *projection* of the QCN $\mathcal{N}$ to V' with $V' \subseteq V$, denoted by $\mathcal{N}_{V'}$, is the QCN (V', C_{proj}) with C_{proj} the restriction of C to the set V'. A sub-QCN $\mathcal{N}'$ of $\mathcal{N}$ is a QCN (V, C') such that $C'(v_i, v_j) \subseteq C(v_i, v_j)$, for all $v_i, v_j \in V$. Let $\mathcal{N}^1$ and $\mathcal{N}^2$ be two QCNs defined respectively on the sets of variables V^1 and V^2, with for each pair of variables $v, v' \in V^1 \cap V^2$, $\mathcal{N}^1[v, v'] = \mathcal{N}^2[v, v']$. We denote by $\mathcal{N}^1 \cup \mathcal{N}^2$ the unique QCN $\mathcal{N}$ defined on $V^1 \cup V^2$ such that $\mathcal{N}[v, v'] = \mathcal{N}^1[v, v']$ for all $v, v' \in V^1$, $\mathcal{N}[v, v'] = \mathcal{N}^2[v, v']$ for all $v, v' \in V^2$, $\mathcal{N}[v, v'] = \Psi$ for all $v \in V^2 \setminus V^1$ and $v' \in V^1 \setminus V^2$.
A QCN $\mathcal{N} = (V, C)$ is $\diamond$-consistent or closed by weak composition if, and only if, $C(v_i, v_j) \subseteq C(v_i, v_k) \diamond C(v_k, v_j)$ for all $v_i, v_j, v_k \in V$. The weak composition closure of the QCN $\mathcal{N}$, denoted by $\diamond(\mathcal{N})$ is the largest (for $\subseteq$) $\diamond$-consistent sub-QCN of $\mathcal{N}$. This closure may be obtained by iterating the operation $C(v_i, v_j) \leftarrow C(v_i, v_j) \cap (C(v_i, v_k) \diamond C(v_k, v_j))$ for all $v_i, v_j, v_k \in V$ until a fixpoint is reached. The worst-case time-complexity of this method is $O(m^3)$, with m the number of variables. For some classes of relations, such as the set of the ORD-Horn relations or the set of the convex relations [14], [15] of IA, the consistency problem of a QCN can be decided by enforcing $\diamond$-consistency. Hence, a $\diamond$-consistent ORD-Horn QCN with no empty constraint is a consistent QCN. The class of convex relations admits a stronger property : each $\diamond$-consistent convex QCN non trivially inconsistent is globally consistent [16]. We conclude this section with some definitions about trees. Given a rooted tree (a connected acyclic graph with a root) $T = (X, F)$ and a node $X_i \in X$, we denote by $\mathsf{desc}(X_i)$ (resp. $\mathsf{asc}(X_i)$) the set of the descendant nodes (resp. the ancestor nodes) of X_i (note that X_i belongs to $\mathsf{desc}(X_i)$ and $\mathsf{asc}(X_i)$). Given $X' \subseteq X$ a non-empty subset of nodes, $\mathsf{lca}(X')$ denote the node of T which is the lowest common ancestor of the nodes belonging to X'. The set $\mathsf{leaves}(T)$ corresponds the set of the leaf nodes of T. Finally, given a $X_i \in X$, T_{X_i} denotes the sub-tree of T rooted in X_i.

III. Decompositions of QCNs

A. Tree decompositions

The relation Ψ consists of all the possible base relations of B and is satisfied by any pair of elements of the domain D. A constraint between two variables of a QCN defined by the relation Ψ specifies that locally there is no constraint concerning the relative position of both entities represented. So, in a natural way, we define the graph of constraints of a QCN $\mathcal{N} = (V, C)$, by the undirected graph $\mathsf{G}(\mathcal{N}) = (V, E)$ with $(v, v') \in E$ if, and only if, $\mathcal{N}[v, v'] \neq \Psi$ and $v \neq v'$. In the sequel we suppose that given a QCN $\mathcal{N}$, $\mathsf{G}(\mathcal{N})$ is connected. In the contrary case, $\mathcal{N}$ may be trivially split in two independent QCNs without common variables. As within the framework of the discrete CSPs we define a tree decomposition of a QCN as a decomposition of its constraint graph:

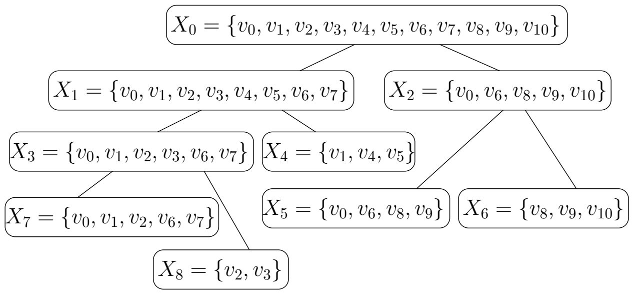

Figure 3(a). A RecPart $T = (X, F)$ decomposition of $\mathcal{N}$.

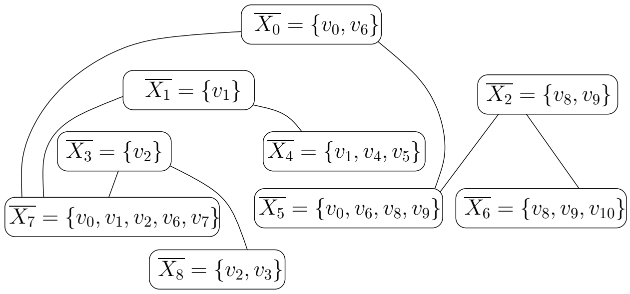

Figure 3(b). A tree decomposition of $\mathcal{N}$ corresponding to $\overline{T}$.

Definition 1: Let $\mathcal{N} = (V, C)$ be a QCN and $\mathsf{G}(\mathcal{N}) = (V, E)$ be its constraint graph. A tree decomposition of $\mathcal{N}$ is a tree $T = (X = \{X_0, \ldots, X_n\}, F)$ with n a positive integer, where X is a family of subsets of variables of V ($X_i \subseteq V$), such that :

(1) $\bigcup \{X_i \in X\} = V$;
(2) $\forall (v, v') \in E$, there exists $X_i \in X$ with $v, v' \in X_i$;
(3) for all $X_i, X_j, X_k \in X$, if X_j is on the unique path between X_i and X_k then $X_i \cap X_k \subseteq X_j$.

Given a tree decomposition $T = (X = \{X_0, \ldots, X_n\}, F)$ of a QCN, the treewidth of T is equal to $max\{|X_i| - 1 : X_i \in X\}$. Furthermore, every set of variables X_i is called a cluster. In Figure 3(b), a tree decomposition of the QCN $\mathcal{N}$ of Figure 2 is represented.

B. *The* RecPart *decompositions*

Recently, Li *et al.* [11] proposed a method allowing to translate QCNs of the Interval Algebra into boolean satisfiability problem (SAT instances). This method recursively decomposes at each step a QCN $\mathcal{N} = (V, C)$ into two QCNs $\mathcal{N}^1$ and $\mathcal{N}^2$ such that $\mathcal{N} = \mathcal{N}^1 \cup \mathcal{N}^2$. The constraints defined by the relation Ψ in the QCN $\mathcal{N}$ not belonging to $\mathcal{N}^1$ and $\mathcal{N}^2$ are characterized as not necessary in the search of a solution of the initial QCN and thus not translated. The advantage of such a translation is that the obtained SAT instance is of smaller size than an instance stemming from a complete translation. Taking inspiration from this method, we define particular decompositions of QCNs, called RecPart decompositions (for recursive partitioning), in the following way:

Definition 2: Let $\mathcal{N}$ be a QCN $= (V, C)$. A RecPart decomposition of $\mathcal{N}$ is a rooted tree $T = (X = \{X_0, \ldots, X_n\}, F)$, with n a positive integer, where X is a family of subsets of V ($X_i \subseteq V$) and $X_r = V$ with X_r the root of T. Moreover, each node of T has two or no child nodes and, given $X_i, X_j, X_k \in X$ such that X_j and X_k are child nodes of X_i, T must satisfy the following properties :

(1) $\mathcal{N}_{X_i} = \mathcal{N}_{X_j} \cup \mathcal{N}_{X_k}$, $X_j \setminus X_k \neq \emptyset$ and $X_k \setminus X_j \neq \emptyset$.

(2) For every $X_m \in \mathsf{desc}(X_i)$, if $X_m \cap (X_j \cap X_k) \neq \emptyset$ then $(X_j \cap X_k) \subseteq X_m$.

In Figure 3(a) is represented a RecPart decomposition of the QCN $\mathcal{N}$ illustrated in Figure 2. We define the treewidth of a RecPart decomposition $T = (X, F)$ by $max\{|X_i| - 1 : X_i \in \mathsf{leaves}(T)\}$. Note that from the property (1) of the previous definition, $X_i \neq \emptyset$ for every $X_i \in X$. A RecPart decomposition satisfies the following properties :

Proposition 1: Let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$. For each $X_i \in X$, we have :

(1) For each $X_j \in X$ such that $i \neq j$, $X_i \neq X_j$ and moreover, if $X_j \in \mathsf{desc}(X_i)$ then $X_j \subset X_i$;

(2) T_{X_i} is a RecPart decomposition of $\mathcal{N}_{X_i}$;

(3) $\mathcal{N}_{X_i} = \bigcup\{\mathcal{N}_{X_j} : X_j \in \mathsf{leaves}(T_{X_i})\}$;

(4) $X_i = \bigcup\{X_j \in \mathsf{leaves}(T_{X_i})\}$.

The proofs of these properties are omitted, they can be directly established from Definition 2. Note that from the previous property (1), we have for all $X_i, X_j \in X$, the sets X_i and X_j which are distinct sets if, and only if, $i \neq j$. Hence, for each $X_i \in X$, the set of variables belonging to X_i characterises one and only one node of T. Let $T = (X, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$. Given $X_i \in X$, $\overline{X_i}$ will denote the set X_i in the case where $X_i \in \mathsf{leaves}(T)$. In the case where $X_i \notin \mathsf{leaves}(T)$, $\overline{X_i}$ is defined by the set $X_j \cap X_k$ with X_j and X_k the two child nodes of X_i. Note that $\overline{X_i} \subseteq X_i$ since $X_j \subseteq X_i$ and $X_k \subseteq X_i$. Moreover, we have the following properties :

Proposition 2: Let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$.

(1) For each $X_i \in X$, $\overline{X_i} \neq \emptyset$.

(2) Given $X_i, X_j, X_k \in X$ with $X_k = \mathsf{lca}(\{X_i, X_j\})$, if $X_k \neq X_i$ and $X_k \neq X_j$ then $X_i \cap X_j \subseteq \overline{X_k}$.

(3) For each $X_i \in X$, there exists $X_j \in \mathsf{leaves}(X)$ such that $\overline{X_i} \subseteq X_j$.

Proof.

(1) Consider the case where $X_i \notin \mathsf{leaves}(T)$ (for the case where $X_i \in \mathsf{leaves}(T)$ the property is trivially satisfied since $\overline{X_i} = X_i$ and $X_i \neq \emptyset$). Let $X_i, X_j, X_k \in X$ such that X_i is the parent node of X_j and X_k. Suppose by contradiction that $X_j \cap X_k = \emptyset$. From Definition 2, we have $\mathcal{N}_{X_i} = \mathcal{N}_{X_j} \cup \mathcal{N}_{X_k}$. Hence, for all $v \in X_j$ and $v' \in X_k$, we can assert that $\mathcal{N}[v, v'] = \Psi$ and that (v, v') is not an edge of $\mathsf{G}(\mathcal{N})$. Moreover, we have previously assumed that for every QCN $\mathcal{N}$ considered, $\mathsf{G}(\mathcal{N})$ is connected. Consequently, for all $v \in X_j$ and $v' \in X_k$, there exists a path of $\mathsf{G}(\mathcal{N})$ between v and v'. Moreover this path passes necessarily through two edges (v, v'') and (v, v''') with $v'' \in V \setminus \{X_j \cup X_k\}$ and $v''' \in V \setminus \{X_j \cup X_k\}$. As $X_j \cup X_k = X_i$, we have $v'' \notin X_i$. Let X_l be the nearest common ancestor of X_i such that $v'' \in X_l$. Let X_m and X_o be the two child nodes of X_l with $X_m \in \mathsf{asc}(X_i)$. We have $v'' \notin X_m$ and $v \in X_m$. Moreover, $v'' \in X_o$ since $X_o \cup X_m = X_l$. We also know $\mathcal{N}_{X_l} = \mathcal{N}_{X_m} \cup \mathcal{N}_{X_o}$. Consequently, we have $\mathcal{N}[v, v''] = \Psi$. We know that (v, v'') is an edge of $\mathsf{G}(\mathcal{N})$. There is a contradiction. We conclude that $X_j \cap X_k \neq \emptyset$.

(2) Let $X_i, X_j, X_k \in X$ with $X_k = \mathsf{lca}(\{X_i, X_j\})$. Assume that $X_k \neq X_i$ and $X_k \neq X_j$. We have necessarily $X_k \notin \mathsf{leaves}(T)$. Let us denote by X_l and X_m the two child nodes of X_k. We have $X_i \in \mathsf{desc}(X_l)$ and $X_j \in \mathsf{desc}(X_m)$ or, $X_i \in \mathsf{desc}(X_m)$ and $X_j \in \mathsf{desc}(X_l)$. Hence, from Proposition 1 (1), we have $X_i \subseteq X_l$ and $X_j \subseteq X_m$ or, $X_i \subseteq X_m$ and $X_j \subseteq X_l$. Consequently, we have $X_i \cap X_j \subseteq X_m \cap X_l$. We conclude that $X_i \cap X_j \subseteq \overline{X_k}$.

(3) Let $X_i \in X$. First, consider the case where $X_i \in \mathsf{leaves}(X)$. By definition, $\overline{X_i} = X_i$, hence the property is satisfied. Now, assume that $X_i \notin \mathsf{leaves}(X)$ and let X_k and X_l be the two child nodes of X_i. We have $X_k \cap X_l = \overline{X_i} \neq \emptyset$. Let $v \in \overline{X_i}$, we have $v \in X_i$. From Proposition 1 (4), there exists $X_j \in \mathsf{leaves}(T) \cap \mathsf{desc}(X_i)$ such that $v \in X_j$. From the property (2) of Definition 2, we have $X_k \cap X_l \subseteq X_j$ since $(X_k \cap X_l) \cap X_j \neq \emptyset$. $\dashv$

IV. From RecPart decompositions to tree decompositions

In this section, we are going to show that from a RecPart decomposition T of a QCN we can define a tree decomposition T' with same treewidth and considering the same clusters of variables (more exactly the same binary constraints). Before this, we introduce a new notation : given

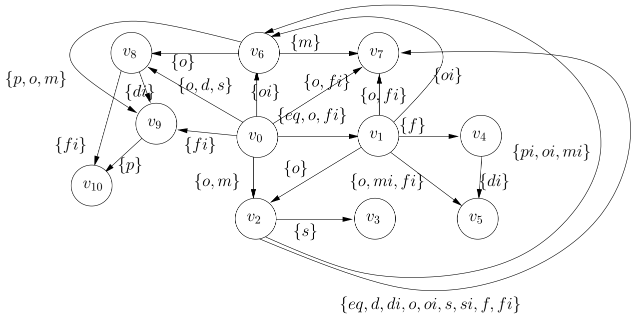

Figure 4. $\mathcal{N}'$ a ORD-Horn $\diamond_X$–consistent sub-QCN of $\mathcal{N}$.

$X_i, X_j \in X$ with X_j child node of X_i, $\widetilde{X_iX_j}$ will denote the set $\{X_k : X_k \in \mathsf{desc}(X_j)$ and $\overline{X_i} \subseteq \overline{X_k}\}$. Concerning $\widetilde{X_iX_j}$, we have the following properties :

Proposition 3: Let $T = (X, F)$ be a RecPart decomposition of a QCN $\mathcal{N} = (V, C)$ and let $X_i, X_j \in X$ with X_j a child node of X_i. We have : (1) $\widetilde{X_iX_j} \neq \emptyset$, (2) $\mathsf{lca}(\widetilde{X_iX_j}) \in \widetilde{X_iX_j}$.

Proof.
(1) We know that $\overline{X_i} \neq \emptyset$ (Proposition 2 (1)), hence there exists $v \in \overline{X_i}$. By definition of $\overline{X_i}$, we know that $v \in X_j$. From Proposition 1 (4), there exists $X_k \in \mathsf{leaves}(T) \cap \mathsf{desc}(X_j)$ such that $v \in X_k$. From the property (3) of Definition 2, we can assert that $\overline{X_i} \subseteq X_k$. Hence, $X_k \in \widetilde{X_iX_j}$ and we can conclude that $\widetilde{X_iX_j} \neq \emptyset$.
(2) Let $X_l = \mathsf{lca}(\widetilde{X_iX_j})$. There exists $X_k, X_m \in \widetilde{X_iX_j}$ such that $X_l = \mathsf{lca}(\{X_k, X_m\})$. In the case where $X_l = X_k$ or $X_l = X_m$ the property is trivially satisfied. In the case where $X_l \neq X_k$ and $X_l \neq X_m$, from Proposition 2 (2) we have $X_k \cap X_m \subseteq \overline{X_l}$. As $\overline{X_i} \subseteq \overline{X_k}$ and $\overline{X_i} \subseteq \overline{X_m}$ we can assert that $\overline{X_i} \subseteq \overline{X_l}$. Moreover, since $X_l = \mathsf{lca}(\{X_k, X_m\})$, $X_k \in \mathsf{desc}(X_j)$ and $X_m \in \mathsf{desc}(X_j)$, we have $X_l \in \mathsf{desc}(X_j)$. We conclude that $X_l \in \widetilde{X_iX_j}$. $\dashv$

Now, we define from a RecPart decomposition T of a QCN $\mathcal{N}$ a tree denoted by $\overline{T}$. We will prove in the sequel that this tree is a tree decomposition of the QCN $\mathcal{N}$ satisfying some particular properties.

Definition 3: Let $\mathcal{N}$ be a QCN and $T = (X = \{X_0, \dots, X_n\}, F)$ a tree decomposition of $\mathcal{N}$. From T and an element $X_i \in X$, we inductively define a rooted tree $\overline{T}_{X_i} = (X_{X_i}, F_{X_i})$ with root $\overline{X_i}$ in the following way :

- Base case : $X_i \in \mathsf{leaves}(T)$, $\overline{T}_{X_i} = (\{\overline{X_i}\}, \emptyset)$.
- Inductive case : $X_i \notin \mathsf{leaves}(T)$. By considering T, let X_j and X_k the child nodes of X_i, $X_l = \mathsf{lca}(\widetilde{X_iX_j})$ and $X_m = \mathsf{lca}(\widetilde{X_iX_k})$. X_{X_i} and F_{X_i} are defined by $X_{X_i} = X_{X_j} \cup X_{X_k} \cup \{\overline{X_i}\}$, $F_{X_i} = F_{X_j} \cup F_{X_k} \cup \{(\overline{X_i}, \overline{X_l}), (\overline{X_i}, \overline{X_m})\}$.

$\overline{T}$ is defined by the rooted tree $\overline{T}_{X_r}$ with X_r the root of T. Let us show that the tree $\overline{T}$ is a tree decomposition of the QCN $\mathcal{N}$ for which T is a RecPart decomposition.

Proposition 4: Let $\mathcal{N} = (V, C)$ be a QCN and a RecPart decomposition $T = (X = \{X_0, \dots, X_n\}, F)$. We have $\overline{T} = (\overline{X} = \{\overline{X_0}, \dots, \overline{X_n}\}, \overline{F})$ which is a tree decomposition of $\mathcal{N}$ such that for each $\overline{X_i} \in \overline{X}$ there exists $X_j \in \mathsf{leaves}(T)$ with $\overline{X_i} \subseteq X_j$.

Proof. Properties (1) and (2) of the definition 1 arise from the fact that for each $X_i \in \mathsf{leaves}(X)$, $\overline{X_i} = X_i$ and from Proposition 1. Now, let us prove that the property (3) of Definition 1 is satisfied by $\overline{T}$. Let $\overline{X_i}, \overline{X_j}, \overline{X_k} \in \overline{X}$ with $\overline{X_j}$ on the unique path between $\overline{X_i}$ and $\overline{X_k}$ *w.r.t.* $\overline{T}$. We are going to show that if $v \in \overline{X_i}$ and $v \in \overline{X_k}$ then $v \in \overline{X_j}$. If the length of the path between $\overline{X_i}$ and $\overline{X_k}$ is 0 the property is trivially satisfied. Now, assume that the property is satisfied for each path with a length $l \geq 0$ and let us show in an inductive way that the property holds for each path between $\overline{X_i}$ and $\overline{X_k}$ of length $l + 1$. First, assume that $X_i \notin \mathsf{desc}(X_k)$ and $X_k \notin \mathsf{desc}(X_i)$ *w.r.t.* T. By examining the definition 3 we notice that for $\overline{T}$ a path between $\overline{X_i}$ and $\overline{X_k}$ necessarily passes through $\overline{X_o}$ with $X_o = \mathsf{lca}(\{X_i, X_k\})$. From Proposition 2 (2) we have $X_i \cap X_k \subseteq \overline{X_o}$. Consequently, $v \in \overline{X_o}$. By considering the two paths $\overline{X_i}, \dots, \overline{X_o}$ and $\overline{X_o}, \dots, \overline{X_k}$ which have a length lower or equal to l, and by using the induction hypothesis, we can assert that the nodes on the path $\overline{X_i}, \dots, \overline{X_o}$ and the nodes on the path $\overline{X_o}, \dots, \overline{X_k}$ contain v since $\overline{X_i}, \overline{X_o}$ and $\overline{X_k}$ contain v. Now, assume that $X_i \in \mathsf{desc}(X_k)$ in T (the case $X_k \in \mathsf{desc}(X_i)$ can be handled in a similar way). The case $X_i = X_k$ is a trivial case, assume that $X_i \neq X_k$. We have $X_i \in \mathsf{desc}(X_o)$ with X_o one of the child nodes of X_k in T. By examining Definition 3 we remark that for $\overline{T}$, a path between $\overline{X_k}$ and $\overline{X_i}$ passes necessarily through $\overline{X_m}$ with $X_m = \mathsf{lca}(\widetilde{X_kX_o})$. From Proposition 3 (2), we know that $\overline{X_k} \subseteq \overline{X_m}$. It results that $v \in \overline{X_m}$. By considering the paths $\overline{X_i}, \dots, \overline{X_m}$ and $\overline{X_m}, \dots, \overline{X_k}$ which have a length lower or equal to l, and by using the induction hypothesis, we can assert that the nodes of the path $\overline{X_i}, \dots, \overline{X_m}$ and the nodes of the path $\overline{X_m}, \dots, \overline{X_k}$ contain v since $\overline{X_i}, \overline{X_m}$ and $\overline{X_k}$ contain v.
The fact that for each $\overline{X_i} \in \overline{X}$, there exists $X_j \in \mathsf{leaves}(T)$ such that $\overline{X_i} \subseteq X_j$ results from Proposition 2 (3). $\dashv$

V. Tree decompositions and consistency of QCNs

In this section we are going to show that to decide the consistency of a QCN from one of its tree decomposition we can leave aside some of its constraints. In particular, we show that for some classes of relations, the closure by weak composition restricted to constraints of the clusters stemming from a tree decomposition is complete for the consistency problem. First of all, we introduce a new local consistency corresponding to the property of $\diamond$-consistency restricted to some subsets of variables of a QCN:

Definition 4: Let $\mathcal{N} = (V, C)$ be a QCN and $X = \{X_0, \dots, X_n\}$ a family of subsets of V. $\mathcal{N}$ is $\overset{\diamond}{X}$-consistent if, and only if, for each $X_i \in X$, the QCN $\mathcal{N}_{X_i}$ is a $\diamond$-consistent QCN.
Given a QCN $\mathcal{N} = (V, C)$ and $X = \{X_0, \dots, X_n\}$ a family of subsets of V, we will denote by $\overset{\diamond}{X}(\mathcal{N})$ the larger (for $\subseteq$) $\overset{\diamond}{X}$-consistent sub-QCN of $\mathcal{N}$.
The following result extends the one of Li *et al.* on atomic networks. It concerns QCNs whose constraints are defined by relations stemming from a class $\mathcal{C}$ for which any QCN closed by weak composition is globally consistent. As illustration, we can consider the QCNs defined by relations belonging to the class of the convex relations of the Interval Algebra which admits this property.

Theorem 1: Let $\mathcal{N} = (V, C)$ be a QCN defined on a class of relations $\mathcal{C}$ for which each QCN $\diamond$-consistent is globally consistent, and let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a tree decomposition of $\mathcal{N}$. If $\mathcal{N}$ is a non trivially inconsistent and $\overset{\diamond}{X}$-consistent QCN then $\mathcal{N}$ is a consistent QCN.

Proof. We suppose without loss of generality that T has a root. Let $X_i \in X$ and $T_{X_i} = (X_{X_i}, F_{X_i})$ the sub-tree of T. Given a partial instantiation s on V' with $V' \cap X'_i \subseteq X_i$ for each $X'_i \in X_{X_i}$ and such that for each $X_j \in X$ with $X_j \subseteq V'$, s_{X_j} is a solution of $\mathcal{N}_{X_j}$. We are going to prove the following property : s can be extended to a partial instantiation s' on $V'' = V' \cup \bigcup\{X'_i \in X_{X_i}\}$ such that for each $X_j \in X$ with $X_j \subseteq V''$, s'_{X_j} is a solution of $\mathcal{N}_{X_j}$. We are going to prove this property in an inductive way on the size of X_{X_i}.

• Base case: $|X_{X_i}| = 1$. We have $X_{X_i} = \{X_i\}$. Since $\mathcal{N}$ is a $\overset{\diamond}{X}$-consistent QCN, we have the QCN $\mathcal{N}_{X_i}$ which is a QCN $\diamond$-consistent and hence globally consistent. $s_{V' \cap X_i}$ is a partial solution of $\mathcal{N}_{X_i}$ which can be extended to a solution s'' of $\mathcal{N}_{X_i}$. We define by s' the partial instantiation on $V' \cup X_i$ in the following way : if $v \in V'$ then $s'(v) = s(v)$ else $s'(v) = s''(v)$. We have s'_{X_i} which is a solution of $\mathcal{N}_{X_i}$ and more generally s'_{X_k} is a solution of $\mathcal{N}_{X_k}$ for each $X_k \in X$ and $X_k \subseteq V' \cup X_i$.

• Inductive step: $|X_{X_i}| > 1$. We assume that the property holds for each sub-tree $T_{X_j} = (X_{X_j}, F_{X_j})$ with $|X_{X_j}| < |X_{X_i}|$. As in the previous case, we extend s to a partial instantiation s' on $V' \cup X_i$ such that s'_{X_k} is a solution of $\mathcal{N}_{X_k}$ for each $X_k \in X$ and $X_k \subseteq V' \cup X_i$. By the induction hypothesis, this partial instantiation s' can be extended to the set of variables belonging to the descendant nodes of X_i. Indeed, consider X_l a child node of X_i. First, we remark that by denoting by $T_{X_l} = (X_{X_l}, F_{X_l})$ the sub-tree of T, we have $|X_{X_l}| < |X_{X_i}|$. Moreover, as T is a tree decomposition, we have for each $X_m \in X_{X_l}$, $X_m \cap (V' \cup X_i) \subseteq X_l$ (from the property (3) of the definition 1). Hence, the induction hypothesis can be applied on $T_{X_l} = (X_{X_l}, F_{X_l})$.

By applying the previous property on X_r the root of T, we know that there exists an instantiation s on the set of variables $\bigcup\{X_i : X_i \in X\}$ such that s_{X_i} is a solution of the QCN $\mathcal{N}_{X_i}$ for each $X_i \in X$. First, from the property (1) of Definition 1 we can assert that $V = \bigcup\{X_i : X_i \in X\}$. Hence, s is an instantiation on V. Moreover we can show that s is a solution of $\mathcal{N}$. Indeed, let $v, v' \in V$, if $\mathcal{N}[v, v'] = \Psi$ we have $s(v)$ and $s(v')$ which satisfy the constraint $\mathcal{N}[v, v']$. Now, assume that $\mathcal{N}[v, v'] \neq \Psi$. From the property (2) of Definition 1, there exists $X_i \in X$ such that $v \in X_i$ and $v' \in X_i$. We know that $s(v)$ and $s(v')$ satisfy $\mathcal{N}_{X_i}[v, v']$. As $\mathcal{N}_{X_i}[v, v'] = \mathcal{N}[v, v']$ we can assert that $s(v)$ and $s(v')$ satisfy $\mathcal{N}[v, v']$. We can conclude that s is a solution of $\mathcal{N}$ $\dashv$

We are going to characterize a similar result for the particular class of the ORD-Horn relations of the Interval Algebra. In [15], Ligozat attributes a dimension (an integer included between 0 and 2) to each base relations of the Interval Algebra : the dimension of the base relations p, pi, o, oi, d, di is 2, this one of the base relations m, mi, s, si, f, fi is 1, and the dimension of eq is 0. A partial solution of maximal dimension is a solution satisfying for every pair of variables a base relation of maximal dimension with regard to the dimensions of the base relations belonging to the constraint. For illustration, consider the QCN $\mathcal{N}'$ in Figure 4, a maximal instantiation s of $\mathcal{N}'_{\{v_0,v_1,v_2,v_6,v_7\}}$ is represented in Figure 5(b), the atomic QCN corresponding to this solution is given in Figure 5(a). For example, the base relation satisfied between $s(v_0)$ and $s(v_1)$ is the base relation o of dimension 2, it is a maximal dimension *w.r.t.* to the dimensions of the base relations of the relation $\mathcal{N}'[v_0, v_1] = \{eq, o, fi\}$. Given a QCN $\mathcal{N}$ closed by weak composition defined by ORD-Horn relations, in the general case $\mathcal{N}$ is not a globally consistent QCN. Nevertheless we have a nearest property which is satisfied : each partial solution of maximal dimension of $\mathcal{N}$ can be extended to a maximal solution of $\mathcal{N}$ (see Proposition 6 in [15]). From this property we can establish the following result :

Theorem 2: Let $\mathcal{N} = (V, C)$ be a QCN defined by relations of the ORD-Horn class of the Interval Algebra and let $T = (X = \{X_0, \ldots, X_n\}, F)$ be a tree decomposition of $\mathcal{N}$. If $\mathcal{N}$ is a $\overset{\diamond}{X}$-consistent QCN non trivially inconsistent then $\mathcal{N}$ is consistent.

Proof. The proof is similar to the proof of Theorem 1 except that the manipulated partial instantiations are maximal partial instantiations. $\dashv$

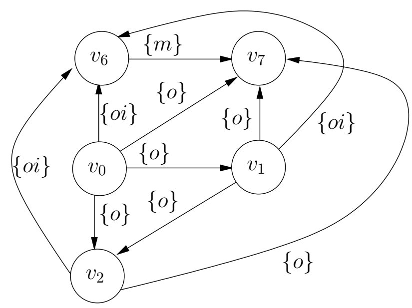

Figure 5(a). An atomic sub-QCN of $\mathcal{N}'_{\{v_0,v_1,v_2,v_6,v_7\}}$

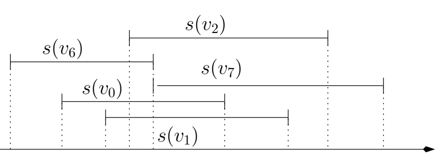

Figure 5(b). A maximal solution of $\mathcal{N}'_{\{v_0,v_1,v_2,v_6,v_7\}}$

We proved in the previous section that given a QCN $\mathcal{N}$ and a RecPart decomposition $T = (X = \{X_0, \ldots, X_n\}, F)$ of this QCN we can define a tree decomposition $T' = (X' = \{X'_0, \ldots, X'_n\}, F) = \overline{T}$ such that for each $X'_i \in X'$ there exists $X_j \in \mathsf{leaves}(T)$ such that $X'_i \subseteq X_j$. From this property and the previous theorems we can establish the following properties :

Corollary 1: Let $\mathcal{N} = (V, C)$ be a QCN defined on a class of relations $\mathcal{C}$ and let $T = (X = \{X_0, \ldots, X_n\}, F)$ a RecPart decomposition of $\mathcal{N}$.

- If $\mathcal{C}$ is such that each $\diamond$-consistent QCN defined on $\mathcal{C}$ is globally consistent and if $\mathcal{N}$ is a $\diamond_{\mathsf{leaves}(T)}$-consistent QCN then $\mathcal{N}$ is consistent.
- If $\mathcal{C}$ is the ORD-Horn class of the Interval Algebra and if $\mathcal{N}$ is a $\diamond_{\mathsf{leaves}(T)}$-consistent QCN then $\mathcal{N}$ is consistent.

VI. From QCNs to boolean formulae

To decide the consistency problem of QCNs, recent studies [9], [10] propose to exploit the theoretical and practical framework of the propositional logic, by using SAT encodings. Given a QCN $\mathcal{N} = (V, C)$, a first part of these encodings allows to represent the possible base relations of $C(v_i, v_j)$ for each pair of variables $v_i, v_j \in V$. A second part is defined by a set of clauses allowing to the SAT solver to enforce the property of $\diamond$-consistency during search. Intuitively, these clauses represent the possible configurations for each triple of variables $v_i, v_j, v_k \in V$ with regard to the weak composition operation. Hence, a SAT instance resulting of these encodings will be consistent if, and only if, there exists a $\diamond$-consistent sub-QCN of $\mathcal{N}$. The encoding proposed in [9] leads to a sub-QCN defined by singleton relations whereas the approach proposed in [10] leads to a convex sub-QCN.

From Theorem 1, we can restrict these encodings to the constraints belonging to clusters of a tree decomposition of $\mathcal{N}$. For illustration, we consider two kinds of decompositions : RecPart decompositions obtained by a method similar as in [11], and tree decompositions obtained from triangulation of the constraint graphs of the QCNs by using the lexBFS algorithm [17]. We have focused on QCNs of the Interval Algebra, randomly generated by following the model $A(n, d, s)$ [7]. This model involves the generation of QCNs according to three parameters: n the number of variables, d the density of constraints not defined by the relation Ψ (the average degree of the nodes in the constraint graph) and s the average number of base relations in each constraint. The results presented concern QCN instances from series $A(100, d, 6.5)$ for d varying from 4 to 24 with a step of 0.25, for each point We generated 100 networks for each serie.

In Figure 6(c) are given the percentages of pairs and triples of variables belonging to clusters for each kind of tree decompositions. Clearly, we can observe that the lexBFS-based tree decompositions discard much more constraints than the RecPart decompositions. Remark that the less is the density of non-trivial constraints, the more is the number of discarded constraints. In Figure 6(d) are given the ratios of the size of the SAT instances using tree decompositions to the size of the SAT instances using the complete encoding. The SAT encodings used are based on the SAT encoding defined in [10]. Unsurprisingly, the using of the LexBFS-based tree decompositions always performs the using of RecPart decompositions.

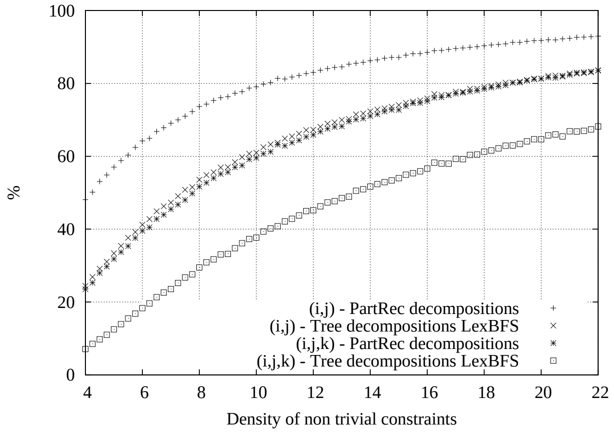

Figure 6(c). Percentages of pairs and triples of variables belonging to the clusters

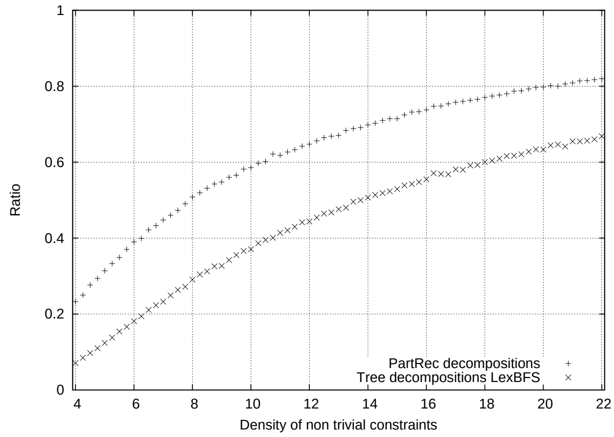

Figure 6(d). Ratios of the size of the SAT instances by using tree decompositions to the size of the SAT instances by using the complete encoding.

Figure 6 shows the number of solved instances against CPU time. The results are given for QCN instances with the parameter d varying from 8 to 12, and *Minisat 2.2* [18] was used to solve generated SAT instances. CPU time is restricted to solving time, and QCN instances are not preprocessed before encoding into SAT instances. As we can see, the lexBFS-based tree decompositions allows to improve the performance for solving SAT instances.

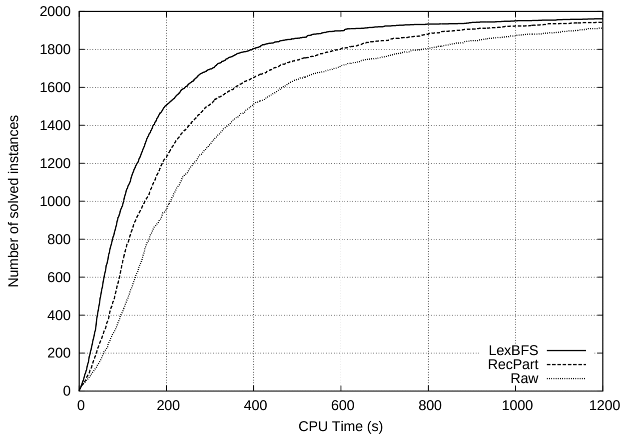

Figure 6. Number of solved instances against CPU time.

VII. Conclusion and future works

In this paper, we have introduced and studied the RecPart decompositions. We proved that these decompositions are equivalent to particular tree decompositions. Moreover, we have studied the consistency problem of QCNs with regard to tree decompositions. We proved that, for some tractable classes of relations such as ORD-Horn class, we can decide the consistency problem of a QCN by enforcing the $\diamond$-consistency restricted to the constraints belonging to clusters of a tree decomposition. In order to illustrate these results, we have compared two kinds of decompositions : RecPart decompositions and tree decompositions obtained from triangulation of the constraint graphs of QCNs by using the lexBFS algorithm. A future work is to conduct extensive experiments in order to compare more tree decompositions into the framework of SAT encodings of QCNs.

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