Stabilisation phenomena in moduli spaces of curves

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Phénomènes de stabilisation dans des espaces de modules de courbes

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Résumé. Dans cette thèse, on étudie l'espace de modules des morphismes d'une courbe projective lisse géométriquement irréductible vers une variété de Fano, ou plus généralement, l'espace de modules des sections d'une famille dont la fibre générique est Fano, ou proche de l'être.

En se plaçant dans un anneau d'intégration motivique, on décrit le comportement attendu de la classe des espaces de modules des sections très libres de grand degré par rapport à tout fibré en droites effectif. Une telle prédiction peut être vue comme des analogues motiviques du principe de Batyrev-Manin-Peyre en géométrique arithmétique. Dans cette analogie, le rôle de la hauteur d'un point rationnel est joué par le degré d'une section. Notamment, on définit les versions motiviques de la constante de Peyre et du principe d'équidistribution.

On démontre ensuite la validité de ces prédictions pour les exemples suivants : compactifications équivariantes d'espaces vectoriels en caractéristique nulle, variétés toriques déployées projectives, ainsi que certains produits tordus en variétés toriques. Dans le premier cas, on démontre de plus une variante pour les courbes dites de Campana.

Abstract. In this thesis, we study the moduli space of morphisms from a smooth, projective and geometrically irreducible curve to a smooth Fano variety, or more generally, the moduli space of sections of a family whose generic fibre is Fano (or close to being Fano).

Working in a ring of motivic integration, we describe the expected behavior of the class of the moduli space of very free morphisms having high degree with respect to every effective line bundle. One can understand this prediction as being a motivic analogue of the Batyrev-Manin-Peyre principle in arithmetic geometry. In our setting, the degree of a morphism plays the role of the height of a rational point. In particular, we define the motivic versions of Peyre's constant and of the equidistribution principle.

We show that this principle holds for equivariant compactifications of vector spaces in characteristic zero, for smooth split projective toric varieties, as well as for a certain kind of twisted toric products. Additionally, for the first example we also consider Campana curves.

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Chapitre 1

Introduction

Les variétés de Fano et les courbes rationnelles font partie des briques élémentaires du programme du modèle minimal en géométrie birationnelle, cet algorithme reposant sur la contraction des courbes extrémales [**Mor79**, **Mor82**]. Dans ce cadre, les fibrations en variétés de Fano font partie d'un ensemble de modèles dits minimaux.

Dans cette thèse on formule un certain nombre de prédictions concernant le comportement des composantes irréductibles de l'espace de modules des sections d'un morphisme propre fidèlement plat $\mathscr{V} \to \mathscr{C}$ vers une courbe projective lisse et géométriquement irréductible, et dont la fibre générique est une variété de Fano (ou proche de l'être, voir plus loin). Notons que le cas particulier de l'espace de modules des morphismes de la droite projective $\mathbf{P}^1_k$ vers une variété projective V définie sur le corps de base k est obtenue en considérant les sections du modèle isotrivial $V \times_k \mathbf{P}^1_k$. Par abus de langage, de tels morphismes sont appelés *courbes rationnelles*. On vérifie ensuite ces prédictions sur certaines classes de variétés : variétés toriques lisses et déployées, compactifications équivariantes d'espaces vectoriels, ainsi que certains produits tordus de variétés toriques.

Le point de vue adopté dans ce travail est grandement inspiré d'un problème de comptage en géométrie arithmétique connu sous le nom de conjecture de Batyrev-Manin-Peyre. Cette conjecture prédit le comportement asymptotique du nombre de points rationnels de hauteur bornée sur une variété de Fano, lorsque cette borne devient arbitrairement grande. L'histoire de cette question débute à la fin des années quatre-vingt avec les travaux précurseurs de Franke-Manin-Tschinkel [**FMT89**] et Batyrev-Manin [**BM90**]. La conjecture sous sa forme actuelle résulte de l'agrégation de ceux-ci et de nombreux autres travaux qui ont suivi ; on citera ceux de Peyre [**Pey95**, **Pey17**], Batyrev-Tschinkel [**BT98b**], Salberger [**Sal98**], Lehmann-Tanimoto [**LT17**] et Lehmann-Sengupta-Tanimoto [**LST22**], entre beaucoup d'autres.

L'analogie entre notre étude et la conjecture précédente repose notamment sur le dictionnaire classique entre le corps $\mathbf{Q}$ des rationnels et le corps $\mathbf{C}(t)$ des fonctions rationnelles à une indéterminée, ou plus généralement entre les corps de nombres et les corps de fonctions d'une courbe projective lisse. Or, les morphismes de la droite projective vers une variété projective complexe sont donnés par ses $\mathbf{C}(t)$-points. Plus généralement, les sections d'un morphisme propre $\mathscr{V} \to \mathscr{C}$ corespondent exactement aux points rationnels de sa fibre générique.

Néanmoins, les corps $\mathbf{Q}$ et $\mathbf{C}(t)$ présentent une différence de taille : les complétés du premier sont localement compacts tandis que ceux du second ne le sont pas. Dès lors, il n'est pas possible de faire usage des outils de l'intégration p-adique pour “compter” des $\mathbf{C}(t)$-points. Ce sont ceux de l'intégration motivique qui prennent le relais.

Dans ce nouveau cadre, on peut considérer la famille des classes, dans un certain anneau d'intégration motivique, des composantes irréductibles de l'espace de modules de sections. La question de leur stabilisation rejoint celle de la stabilisation homologique

[**EVW16**]. La philosophie sous-jacente est la suivante : étant donnée une suite (X_n) de variétés algébriques définie sur un corps fini $\mathbf{F}_q$ et dont la dimension croît avec n, alors la quantité $|X_n(\mathbf{F}_q)|q^{-\dim(X_n)}$ devrait admettre une limite lorsque n tend vers l'infini précisément lorsque les groupes de cohomologie de X_n "se stabilisent" [**EVW16**, §1.8]. Lorsque le corps de base de la suite (X_n) n'est plus $\mathbf{F}_q$ mais le corps des complexes $\mathbf{C}$, travailler avec les outils de l'intégration motivique permet encore de mimer des questions de comptage, tout en respectant les réalisations cohomologiques.

Les résultats de cette thèse s'inscrivent dans le domaine relativement récent des *statistiques motiviques*. Ils peuvent être mis en comparaison avec un ensemble de résultats de stabilisation asymptotique dans des anneaux d'intégration motivique. Citons notamment les travaux de Vakil-Wood concernant les hypersurfaces [**VW15**], ceux de Bilu-Howe sur les fibrés vectoriels [**BH21**], de Bilu-Das-Howe sur la convergence dite de Hadamard [**BDH22**], ou encore ceux de Landesman-Vakil-Wood à propos des espaces de Hurwitz de petits degrés [**LVW22**].

Cette introduction est organisée de la façon suivante. Dans la Section 1.1, on introduit les espaces de modules de sections $\mathrm{Hom}_{\mathscr{C}}(\mathscr{C},\mathscr{V})$ et on rappelle quelques propriétés géométriques connues à leur sujet. On donne ensuite dans la Section 1.2 quelques définitions de base d'intégration motivique, afin de formuler une ébauche d'un principe de Batyrev-Manin-Peyre motivique, sous la forme d'une première question directrice. Puis l'on introduit la filtration par le poids sur l'anneau d'intégration motivique ainsi que les produits eulériens motiviques, ceci afin de permettre plus loin une formulation complète. La Section 1.3 est un pas-de-côté permettant de faire le lien avec les travaux récents de Lehmann, Tanimoto et Riedl concernant la classification des composantes irréductibles de $\mathrm{Hom}_{\mathscr{C}}(\mathscr{C},\mathscr{V})$. Ceci permet alors en Section 1.4 de formuler un certain nombre de prédictions. Les résultats de cette thèse venant appuyer de telles prédictions sont présentés dans la Section 1.5. Enfin, on conclut cette introduction par un résumé rapide des différents chapitres en Section 1.6.

1.1. Espaces de modules de morphismes depuis une courbe

1.1.1. Le cadre isotrivial rationel. Donnons-nous une variété V projective et lisse définie sur un corps k. Son fibré tangent sera noté $\mathcal{T}_V$ et son fibré canonique ω_V. Typiquement, on considère $k = \mathbf{F}_q$ où q est une puissance d'un nombre premier, ou bien $k = \mathbf{C}$, la seconde situation justifiant bien davantage l'usage des techniques développées plus bas. Il est bien connu [**Gro60**, **Deb01**] que les courbes rationnelles $\mathbf{P}^1_k \to V$ sont paramétrées par un k-schéma noté

$$\mathrm{Hom}(\mathbf{P}^1_k, V)$$

localement noethérien. Tout morphisme $f : \mathbf{P}^1_k \to V$ définit un *multidegré*

$$\mathbf{deg}\, f : [L] \in \mathrm{Pic}(V) \mapsto \deg(f^*L)$$

et $\mathrm{Hom}(\mathbf{P}^1_k, V)$ admet une décomposition selon celui-ci, c'est-à-dire en tant que l'union dénombrable des sous-espaces de modules

$$\mathrm{Hom}^{\delta}(\mathbf{P}^1_k, V)$$

paramétrant les courbes de multidegré $\delta \in \mathrm{Pic}(V)^{\vee}$. Ces sous-schémas sont quasi-projectifs [**Deb01**, Chap. 2]. Par un argument de déformation [**Deb01**, Theorem 2.6], sa dimension en un point $[f]$ correspondant à un morphisme $f : \mathbf{P}^1_k \to V$ est bornée inférieurement par une quantité de l'ordre du degré anticanonique de la courbe :

$$\dim_{[f]} \operatorname{Hom}(\mathbf{P}^1_k, V) \geqslant \deg(f^*\omega_V^{-1}) + \dim(V). \tag{1.1.1.1}$$

Cette borne inférieure est communément appelée *dimension attendue*. Si V est une variété de Fano, c'est-à-dire que son fibré anticanonique ω_V^{-1} est ample, alors cette quantité est toujours strictement positive.

Deux questions générales peuvent d'ores-et-déjà être posées :

(1) Concernant la classification des composantes irréductibles de $\operatorname{Hom}^\delta(\mathbf{P}^1_k, V)$:
   (a) Quelles sont les composantes irréductibles qui sont de la dimension attendue, c'est-à-dire vérifient l'égalité dans (1.1.1.1) ?
   (b) Combien y a-t-il de telles composantes et comment les distinguer entre elles ?
(2) Comment se comportent par déformation les courbes d'une "bonne" composante pour des degrés grands relativement à un ou plusieurs fibrés ? La composante associée se stabilise-t-elle, dans un sens à préciser ?

Les résultats de cette thèse concernent essentiellement la seconde question. On présentera néanmoins les liens que ceux-ci entretiennent avec la première question dans la Section 1.3 de cette introduction.

1.1.2. Variétés définies sur un corps de fonction. Par analogie avec le cadre dit arithmétique, c'est-à-dire celui de l'étude des points rationnels sur un corps de nombres, un cadre plus général est le suivant. On se donne une fois pour toutes

— une courbe $\mathscr{C}$, définie sur un corps de base absolu k, supposée projective, lisse et géométriquement irréductible, et dont le corps des fonctions sera noté F,
— ainsi qu'une variété projective et lisse V définie sur F, également supposée géométriquement irréductible.

Soit $\mathscr{V}$ un modèle de V au-dessus de $\mathscr{C}$, c'est-à-dire un schéma de type fini sur k muni d'un morphisme $\mathscr{V} \to \mathscr{C}$ fidèlement plat. On peut supposer que ce morphisme est propre, que son lieu lisse est un modèle de Néron faible, et pour simplifier que $\mathscr{V}$ est projectif sur k. Un modèle vérifiant ces hypothèses sera appelé un *bon modèle*.

Dans cette situation, les points sur F de V sont en bijection avec l'ensemble des sections de $\mathscr{V} \to \mathscr{C}$. Celles-ci sont paramétrées par un schéma

$$\operatorname{Hom}_{\mathscr{C}}(\mathscr{C}, \mathscr{V}).$$

On suppose le groupe de Picard de V libre de rang r et l'on se donne un ensemble de fibrés $\mathscr{L}_1, ..., \mathscr{L}_r$ sur $\mathscr{V}$ dont les restrictions $L_1, ..., L_r$ à V fournissent une base de $\operatorname{Pic}(V)$. Toute section $\sigma : \mathscr{C} \to \mathscr{V}$ fournit alors une forme linéaire sur $\operatorname{Pic}(V)$ donnée par

$$\otimes_{i=1}^r L_i^{\otimes \lambda_i} \longmapsto \deg\left(\sigma^*(\otimes_{i=1}^r \mathscr{L}_i^{\otimes \lambda_i})\right)$$

et généralisant le *multidegré* de la section précédente. L'espace de modules

$$\operatorname{Hom}^\delta_{\mathscr{C}}(\mathscr{V}, \mathscr{C})$$

des sections de multidegré δ est alors bien défini en tant que schéma noetherien quasi-projectif sur k (voir 2.3.6 page 50).

Étant donné que l'on a supposé que le lieu lisse $\mathscr{V}^\circ$ de $\mathscr{V} \to \mathscr{C}$ est un modèle de Néron faible de V, l'image de toute section $\mathscr{C} \to \mathscr{V}$ est incluse dans celui-ci. Il fait donc sens de parler du degré d'intersection d'une section avec le fibré anticanonique relatif $(\Omega^{\dim(V)}_{\mathscr{V}^\circ/\mathscr{C}})^\vee$. La dimension de $\mathrm{Hom}_{\mathscr{C}}(\mathscr{C},\mathscr{V})$ au point paramétrant $\sigma : \mathscr{C} \to \mathscr{V}$ admet la minoration

$$\dim_{[\sigma]} \mathrm{Hom}_{\mathscr{C}}(\mathscr{C},\mathscr{V}) \geqslant \deg\left(\sigma^*(\Omega^{\dim(V)}_{\mathscr{V}^\circ/\mathscr{C}})^\vee\right) + (1-g(\mathscr{C}))\dim(V) \tag{1.1.2.2}$$

où g désigne le genre, voir [**Deb01**, page 47]. Une borne supérieure est donnée par

$$\deg\left(\sigma^*(\Omega^{\dim(V)}_{\mathscr{V}^\circ/\mathscr{C}})^\vee\right) + (1-g(\mathscr{C}))\dim(V) + \dim(H^1(\mathscr{C},\sigma^*\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}})) \tag{1.1.2.3}$$

où $\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}}$ est le fibré tangent relatif $(\Omega^1_{\mathscr{V}^\circ/\mathscr{C}})^\vee$. En particulier, si $H^1(\mathscr{C},\sigma^*\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}}) = 0$ alors $\mathrm{Hom}_{\mathscr{C}}(\mathscr{C},\mathscr{V})$ est de la dimension attendue en $[\sigma]$.

Les questions du paragraphe précédent s'étendent aux composantes de $\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C},\mathscr{V})$.

1.1.3. Ajout de contraintes. Dans ce paragraphe on présente un premier type de contraintes que l'on peut imposer aux sections et on introduit les espaces de modules correspondants.

Soit $\mathscr{S} \subset \mathscr{C}$ un sous-schéma fermé de dimension nulle, ainsi qu'un morphisme $\varphi : \mathscr{S} \to \mathscr{V}$ au-dessus de $\mathscr{C}$. On dispose alors pour tout multidegré $\delta \in \mathrm{Pic}(V)^\vee$ d'un morphisme de restriction

$$\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C},\mathscr{V}) \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})$$

dont la fibre au-dessus de φ est notée

$$\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C},\mathscr{V} \mid \varphi).$$

On a alors

$$\dim_{[\sigma]} \mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C},\mathscr{V} \mid \varphi) \geqslant -\sigma_*\mathscr{C} \cdot \Omega^{\dim(V)}_{\mathscr{V}^\circ/\mathscr{C}} + (1-g(\mathscr{C})-\ell(\mathscr{S}))\dim(V)$$

où ℓ désigne la longeur [**Deb01**, §2.3], tandis qu'une borne supérieure est donnée par

$$\deg\left(\sigma^*(\Omega^{\dim(V)}_{\mathscr{V}^\circ/\mathscr{C}})^\vee\right) + (1-g(\mathscr{C}))\dim(V) + \dim(H^1(\mathscr{C},(\sigma^*\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}})\otimes I_{\mathscr{S}}))$$

où $I_{\mathscr{S}}$ est le faisceau d'idéaux de $\mathscr{O}_{\mathscr{C}}$ définissant $\mathscr{S}$. Plus généralement, si $W \subset \mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})$ est un sous-ensemble localement fermé, la fibre au-dessus de W de la restriction sera notée

$$\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C},\mathscr{V} \mid W).$$

Sa dimension au point paramétrant $\sigma : \mathscr{C} \to \mathscr{V}$ est minorée par

$$\deg\left(\sigma^*(\Omega^{\dim(V)}_{\mathscr{V}^\circ/\mathscr{C}})^\vee\right) + (1-g(\mathscr{C})-\ell(\mathscr{S}))\dim(V) + \dim(W) \tag{1.1.3.4}$$

et majorée par

$$\deg\left(\sigma^*(\Omega^{\dim(V)}_{\mathscr{V}^\circ/\mathscr{C}})^\vee\right)+(1-g(\mathscr{C}))\dim(V)+\dim(H^1(\mathscr{C},\sigma^*\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}}\otimes I_{\mathscr{S}}))+\dim(W). \tag{1.1.3.5}$$

On en déduit que la dimension en $[\sigma]$ est celle attendue dès lors que $H^1(\mathscr{C},\sigma^*\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}}\otimes I_{\mathscr{S}})$ est trivial.

1.2. Le point de vue motivique

1.2.1. Anneaux de variétés. On donne brièvement quelques définitions essentielles [**CLNS18**, Chap. 2].

1.2.1.1. *Anneaux d'intégration motivique.* Soit k un corps. Le groupe de Grothendieck des variétés sur k

$$K_0(\mathbf{Var}_k)$$

est le $\mathbf{Z}$-module engendré par les classes d'isomorphismes $[X]$ de variétés algébriques (non nécessairement irréductibles) sur k et muni des relations de la forme

$$[X] - [U] - [X \setminus U]$$

où U est un ouvert d'une variété algébrique X. Ce groupe admet une structure d'anneau dont la loi de multiplication est donnée par

$$[X] \times [Y] = [X \times_k Y].$$

La classe de la droite affine est notée $\mathbf{L}_k$. Notons que par un théorème de Borisov [**Bor14**] la classe $\mathbf{L}_k$ est un diviseur de zéro. Néanmoins, on peut toujours inverser formellement $\mathbf{L}_k$ en localisant, pour obtenir l'anneau d'intégration motivique $\mathscr{M}_k$ introduit par Kontsevich.

1.2.1.2. *La filtration dimensionnelle.* L'anneau $\mathscr{M}_k$ admet une filtration par la dimension virtuelle : pour tout $m \in \mathbf{Z}$, soit $\mathscr{F}^m\mathscr{M}_k$ le sous-groupe de $\mathscr{M}_k$ engendré par les éléments de la forme

$$[X]\mathbf{L}_k^{-i}$$

où X est une variété algébrique sur k et i un entier relatif, tels que

$$\dim(X) - i \leqslant -m.$$

Ces sous-groupes fournissent une filtration décroissante. Le complété de $\mathscr{M}_k$ par rapport à celle-ci est alors la limite projective

$$\widehat{\mathscr{M}_k}^{\dim} = \varprojlim \mathscr{M}_k/\mathscr{F}^m\mathscr{M}_k$$

laquelle est munie d'un morphisme $\mathscr{M}_k \to \widehat{\mathscr{M}_k}^{\dim}$ [**CLNS18**, Chap.2, §4].

1.2.1.3. *Versions relatives.* Les définitions données ci-dessus se généralisent toutes au cadre relatif au-dessus d'un schéma S fixé, en remplaçant les variétés sur k par des morphismes $X \to S$ de présentation finie. On obtient *verbatim* des anneaux $K_0(\mathbf{Var}_S)$ et $\mathscr{M}_S$ filtrés par la dimension relative, ainsi qu'un complété $\widehat{\mathscr{M}_S}^{\dim}$.

1.2.1.4. *Puissances rationnelles de* $\mathbf{L}$. Dans notre étude de l'espace de module des sections de Campana sur les compactifiés équivariants d'espaces vectoriels, on sera naturellement conduit à introduire des puissances rationnelles de la droite affine au-dessus d'un schéma S. En effet, les degrés considérés pourront prendre des valeurs rationnelles non-entières. On travaillera alors dans l'anneau quotient

$$K_0(\mathbf{Var}_{S,r}) = K_0(\mathbf{Var}_S)[x]/(x^r - \mathbf{L}_S)$$

pour r un entier naturel non nul (en pratique, choisi suffisamment grand). Sa localisation en $\mathbf{L}_S$ sera notée $\mathscr{M}_{S,r}$. On dispose à nouveau d'une filtration dimensionnelle décroissante $\mathscr{F}^m\mathscr{M}_{S,r}$, cette fois-ci indexée par $m \in \frac{1}{r}\mathbf{Z}$, et d'un complété associé

$$\widehat{\mathscr{M}_{S,r}}^{\dim} = \varprojlim \mathscr{M}_{S,r}/\mathscr{F}^m\mathscr{M}_{S,r}.$$

1.2.2. Quelques rudiments d'intégration motivique. L'étude locale des sections nous amènera à nous placer dans le cadre suivant, voir [CLNS18, Chap. 3-5].

Soit R un anneau de valuation discrète, supposé complet et d'égale caractéristique, d'idéal maximal $\mathfrak{m}$, de corps résiduel κ, de corps de fractions K. Soit $\mathscr{X}$ une variété sur R. Pour tout entier naturel m, le m-ième schéma de Greenberg de $\mathscr{X}$

$$\mathrm{Gr}_m(\mathscr{X})$$

aussi appelé l'espace des m-jets de $\mathscr{X}$, est le κ-schéma réprésentant le foncteur

$$A \rightsquigarrow \mathrm{Hom}_R(\mathrm{Spec}((R/\mathfrak{m}^{m+1}) \otimes_\kappa A), \mathscr{X})$$

de la catégorie des algèbres sur κ [CLNS18, Chap. 4, §2.1]. Ses κ-points peuvent être compris comme des développements de Taylor à l'ordre m de sections de $\mathscr{X}$. Cette famille de schémas est munie de morphismes canoniques de troncation

$$\theta^{m+1}_m : \mathrm{Gr}_{m+1}(\mathscr{X}) \to \mathrm{Gr}_m(\mathscr{X})$$

et la limite projective

$$\mathrm{Gr}_\infty(\mathscr{X}) = \varprojlim \mathrm{Gr}_m(\mathscr{X})$$

est le pro-schéma sur κ représentant le foncteur

$$A \mapsto \mathrm{Hom}_\kappa(\mathrm{Spec}(A \otimes_\kappa R), X).$$

Celui-ci est muni pour tout entier naturel m d'un morphisme de troncation

$$\theta^\infty_m : \mathrm{Gr}_\infty(\mathscr{X}) \to \mathrm{Gr}_m(\mathscr{X}).$$

Lorsque $\mathscr{X}$ est de dimension relative pure égale à d, l'intégration motivique, dans l'une de ses versions les plus simples, consiste à associer à tout cylindre

$$C = (\theta^\infty_m)^{-1}(C_m),$$

où $C_m \subset \mathrm{Gr}_m(\mathscr{X})$ est un sous-ensemble constructible dont on suppose qu'il évite le lieu singulier de $\mathscr{X}$, une classe

$$\mu_{\mathscr{X}}(C) = [\theta^m_0(C_m)]\mathbf{L}^{-(m+1)d} \in \mathscr{M}_{\mathscr{X}_k}$$

où $\mathscr{X}_\kappa$ désigne ici la fibre spéciale de $\mathscr{X}$. Étant donnée une fonction $f : C \to \mathbf{Z} \cup \{\infty\}$ telle que $f^{-1}(\{n\})$ est constructible pour tout $n \in \mathbf{Z}$, son intégrale

$$\int_A \mathbf{L}^{-f} \mathrm{d}\mu_{\mathscr{X}} = \sum_{n \in \mathbf{Z}} \mu_{\mathscr{X}}(f^{-1}(\{n\}))\mathbf{L}^{-n}$$

est alors bien définie dans $\mathscr{M}_{\mathscr{X}_k}$.

1.2.3. Une première question. L'objet d'étude de cette thèse est le comportement de la classe

$$\left[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})\right] \in K_0(\mathbf{Var}_k)$$

lorsque δ *devient grand*. On tâche à présent de préciser ce que l'on entend par là.

1.2.3.1. *Variétés quasi de Fano.* Tout d'abord, afin que cette étude quantitative ait du sens, on doit supposer que $\mathscr{V}$ a beaucoup de sections, autrement dit que V a potentiellement suffisamment de points rationnels. Par exemple, il suffit de supposer que V est une variété de Fano, c'est-à-dire que son faisceau anticanonique ω_V^{-1} est ample. Mori a démontré que de telles variétés sont uniréglées [**Mor79**]. On a en fait mieux : deux points distincts de V peuvent être reliés par une courbe rationnelle. On dit que les variétés de Fano sont *rationnellement connexes*.

Dans ce travail on considèrera une classe de variétés un peu plus grande que celles des variétés de Fano.

Définition A. Si V est une F-variété géométriquement irréductible, lisse et projective, on dira que V est une variété *quasi de Fano* [1] si

(i) $V(F)$ est dense dans V pour la topologie de Zariski ;

(ii) les groupes de cohomologie $H^1(V, \mathscr{O}_V)$ et $H^2(V, \mathscr{O}_V)$ sont nuls ;

(iii) le groupe de Picard de V coïncide avec son groupe géométrique, lequel est sans torsion ;

(iv) son groupe de Brauer géométrique est trivial ;

(v) la classe du faisceau anticanonique de V appartient à l'intérieur du cône effectif $\mathrm{Eff}(V)$, et celui-ci est polyèdral rationnel : il est engendré par un nombre fini de classes de faisceaux inversibles effectifs.

L'hypothèse $H^1(V, \mathscr{O}_V) = 0$ dans (ii) assure que le groupe de Picard géométrique de V coïncide avec le groupe de Néron-Severi.

Par le théorème d'annulation de Kodaira, les variétés de Fano vérifient les hypothèses (ii) et (iv) ci-dessus. Étant rationnellement connexes, elles vérifient également (i). Quant à l'hypothèse (v), pour les variétés de Fano complexes, c'est une conséquence de [**BCHM10**, Cor. 1.3.2].

Les variétés toriques, les variétés en drapeaux, les compactifications équivariantes, vérifient ces hypothèses.

1.2.3.2. *Première heuristique.* Lorsque V est une variété algébrique complexe, par le théorème de [**BDPP12**], les courbes dites *mobiles* correspondent exactement aux courbes dont la classe est duale au cône effectif de V. Dès lors, les courbes dont la classe s'éloigne du bord de ce cône dual sont "de plus en plus mobiles" tandis que leur degré par rapport à chacun des générateurs du cône polyèdral $\mathrm{Eff}(V)$ croît infiniment. On s'attend alors à ce que leur "multidegré de liberté infiniment grand" leur permette de se déformer en tout point tout en recouvrant V.

Notons $\mathrm{Eff}(V)^\vee_{\mathbf{Z}}$ l'intersection de $\mathrm{Pic}(V)^\vee$ avec le dual du cône effectif de V. Pour toute classe $\delta \in \mathrm{Eff}(V)^\vee_{\mathbf{Z}}$,

$$d(\delta, \partial\,\mathrm{Eff}(V)^\vee)$$

désigne la distance de δ au bord de celui-ci.

Question 1. *La classe normalisée*

$$\left[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})\right] \mathbf{L}_k^{-\delta\cdot\omega_V^{-1}} \in \mathscr{M}_k$$

se stabilise-t-elle lorsque $d(\delta, \partial\,\mathrm{Eff}(V)) \to \infty$ *?*

1. On dira *"Fano-like"* en anglais.

Afin de fixer les idées, donnons nous un diviseur anticanonique K_V sur la variété quasi de Fano V ainsi qu'un modèle $\mathscr{K}_{\mathscr{V}}$ de celui-ci obtenu en choisissant une section rationnelle de $(\Omega^n_{\mathscr{V}/\mathscr{C}})^\vee$. Considérons alors les sections de degré d par rapport à $-\mathscr{K}_{\mathscr{V}}$ (on rappelle que l'on peut supposer que leur image tombe dans le lieu lisse de $\mathscr{V} \to \mathscr{C}$). En tout point fermé $p \in \mathscr{C}$, il existe une fonction constructible

$$\mathrm{ord}_{-\mathscr{K}_{\mathscr{V}}} : \mathrm{Gr}_\infty(\mathscr{V}^\circ_{R_p}) \longrightarrow \mathbf{Z} \cup \{\infty\}$$

bornée inférieurement, correspondant au degré local en p (voir le §4.4 de [**CLNS18**, Chap. 4]). Les densités motiviques des *germes de courbes* (c'est-à-dire de l'espace d'arcs en p) de tous ordres par rapport à $-\mathscr{K}_{\mathscr{V}}$ sont encodées par la série

$$\int_{\mathrm{Gr}_\infty(\mathscr{V}^\circ_{R_p})} t^{\mathrm{ord}_{-\mathscr{K}_{\mathscr{V}}}} \mathrm{d}\mu_{\mathscr{V}_{R_p}}.$$

Très grossièrement, on pourrait espérer qu'une section générale de multidegré infiniment grand puisse être déformée arbitrairement en tout point dans toutes les directions, donc être de degré anticanonique infiniment grand en tout point de $\mathscr{C}$. D'où l'heuristique suivante : il s'agirait de prendre la limite de la suite des coefficients – normalisés par la puissance de $\mathbf{L}$ adéquate – du produit

$$\text{“}\prod_{p\in\mathscr{C}} \int_{\mathrm{Gr}_\infty(\mathscr{V}^\circ_{R_p})} t^{\mathrm{ord}_{-\mathscr{K}_{\mathscr{V}}}} \mathrm{d}\mu_{\mathscr{V}_{R_p}}\text{”}$$

lequel restant encore largement à définir. Pour le moment, on peut garder en tête que ce produit admet un pôle en $t = \mathbf{L}_k^{-1}$ d'ordre $\mathrm{rg}(\mathrm{Pic}(V))$, et que son résidu devrait alors être (à une puissance de $\mathbf{L}_k$ près) la limite attendue du rapport

$$\left[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C},\mathscr{V})\right] \mathbf{L}_k^{-\delta\cdot\omega_V^{-1}}$$

lorsque $d(\delta, \partial\,\mathrm{Eff}(V)) \to \infty$.

En réalité, on considèrera plutôt une fonction constructible bornée encodant la différence de degrés locaux entre

- — celui relatif au modèle de ω_V^{-1} donné par le choix que l'on a fait de modèles d'une base de $\mathrm{Pic}(V)$,
- — et celui relatif à $(\Omega^n_{\mathscr{V}^\circ/\mathscr{C}})^\vee$.

Notons d'ores-et-déjà que cette différence est nulle au-dessus de presque tous les points fermés de $\mathscr{C}$.

À travers un exemple pathologique, on verra en Section 1.3 que cela n'est pas parce qu'une section est de multidegré arbitrairement grand qu'elle peut être déformée arbitrairement dans toutes les directions. Il faut en fait raisonner dans l'autre sens, en ne considérant que des sections dites *très libres* se déformant beaucoup dans toutes les directions, et donc de degré anticanonique grand – mais aussi et toujours de multidegré grand.

1.2.4. Réalisations mixtes et filtration par le poids. En général il s'avère que la filtration dimensionnelle ne fournit pas une notion de convergence suffisamment fine pour définir ne serait-ce que la limite attendue du rapport $\left[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C},\mathscr{V})\right] \mathbf{L}^{-\delta\cdot\omega_V^{-1}}$. À la place, on utilisera une filtration par le poids sur les réalisations cohomologiques. On présente ici succinctement les constructions afférentes, en se référant principalement aux sections 3.3 à 3.5 de [**CLNS18**, Chapitre 2].

1.2.4.1. *Anneaux de catégories abéliennes ou triangulées.* Si $\mathbf{A}$ est une catégorie abélienne essentiellement petite [2], le groupe de Grothendieck

$$K_0(\mathbf{A})$$

est le quotient du $\mathbf{Z}$-module libre engendré par les classes d'isomorphismes d'objets de $\mathbf{A}$ par les relations

$$[X] - [Y] - [Z]$$

lorsqu'il existe une suite exacte

$$0 \to X \to Y \to Z \to 0.$$

Une variante $K_0^{\text{split}}(\mathbf{A})$ est obtenue en se restreignant aux suites exactes *scindées*. Si $\mathbf{A}$ est munie d'un bifoncteur additif et exact $(X, Y) \mapsto X \otimes Y$, celui-ci fournit une structure d'anneau sur $K_0(\mathbf{A})$ et $K_0^{\text{split}}(\mathbf{A})$.

Si $\mathbf{A}$ est une catégorie triangulée essentiellement petite, son groupe de Grothendieck

$$K_0^{\text{tri}}(\mathbf{A})$$

est défini de la même façon en remplaçant les suites exactes ci-dessus par les triangles distingués

$$X \to Y \to Z \to X[1].$$

Dans ce cas, une structure d'anneau sur $K_0^{\text{tri}}(\mathbf{A})$ est fournie par tout bifoncteur additif et triangulé $(X, Y) \mapsto X \otimes Y$.

Étant donnée une catégorie abélienne $\mathbf{A}$ essentiellement petite, on note $\mathbf{D}^{\mathrm{b}}(\mathbf{A})$ sa catégorie dérivée des complexes bornés. Il est alors équivalent de travailler dans $K_0(\mathbf{A})$ ou dans $K_0^{\text{tri}}(\mathbf{D}^{\mathrm{b}}(\mathbf{A}))$. En effet, le foncteur envoyant un objet de $\mathbf{A}$ sur le complexe concentré en degré zéro correspondant induit un isomorphisme de groupe $K_0(\mathbf{A}) \to K_0^{\text{tri}}(\mathbf{D}^{\mathrm{b}}(\mathbf{A}))$, l'inverse étant donné par le morphisme envoyant la classe d'un complexe borné $C_\bullet$ sur la somme

$$\sum_{i \in \mathbf{Z}} [\mathcal{H}^i(C_\bullet)] \in K_0(\mathbf{A}).$$

1.2.4.2. *Réalisation de Hodge mixte dans le cas complexe.* Si S est une variété définie sur le corps des complexes, on désigne par $\mathbf{MHM}_S$ la catégorie abélienne des modules de Hodge mixtes [**Sai88**, **Sai90**, **Sai16**]. La réalisation de Hodge mixte

$$\chi_S^{\text{Hdg}} : K_0(\mathbf{Var}_S) \to K_0(\mathbf{MHM}_S)$$

est le morphisme d'anneaux envoyant la classe de $X \xrightarrow{f} S$ sur

$$\left[p_! \mathbf{Q}_X^{\text{Hdg}}\right] = \sum_{i \in \mathbf{Z}} (-1)^i \left[\mathcal{H}^i \left(p_! \mathbf{Q}_X^{\text{Hdg}}\right)\right].$$

En particulier, $\mathbf{L}_S$ est envoyée sur $\mathbf{Q}_S^{\text{Hdg}}(-1)$, qui est inversible dans $K_0(\mathbf{MHM}_S)$, et induit un morphisme d'anneaux

$$\chi_S^{\text{Hdg}} : \mathscr{M}_S \to K_0(\mathbf{MHM}_S).$$

Les modules de Hodge mixtes sont munis d'une filtration par le poids, induisant par conséquent une filtration sur $K_0(\mathbf{MHM}_S)$ et le tiré-en-arrière de celle-ci à $\mathscr{M}_S$ via χ_S^{Hdg}

2. On entend par là que les flèches entre deux objets quelconques de $\mathbf{A}$ forment un ensemble et qu'il en va de même pour les classes d'isomorphismes d'objets de $\mathbf{A}$.

fournit une nouvelle filtration. La composée du poids sur $K_0(\mathbf{MHM}_S)$ avec χ_S^{Hdg} sera notée w_X, et le complété associé

$$\widehat{\mathscr{M}_S}^w.$$

1.2.4.3. *Réalisation étale mixte.* Si S est une variété définie sur un corps k finiment engendré, et si ℓ est un nombre premier inversible dans k, alors les faisceaux étales ℓ-adiques fournissent, par passage à leur catégorie dérivée, deux catégories triangulées $\mathbf{D}^{\mathrm{b}}(S,\mathbf{Q}_\ell)$ et $\mathbf{D}^{\mathrm{b}}_{\mathrm{cons}}(S,\mathbf{Q}_\ell)$ correspondant respectivement aux complexes de faisceaux étales ℓ-adiques et à ceux dont les faisceaux de cohomologie sont des faisceaux ℓ-adiques constructibles [**Del80**].

Tout comme celle des modules de Hodge mixtes introduite ci-avant, ces catégories sont munies d'un formalisme des six foncteurs à la Grothendieck. Notamment, si $f : X \to S$ est une S-variété séparée, alors $f_!(\mathbf{Q}_\ell)_X^{\text{ét}}$ est un complexe de faisceaux ℓ-adiques constructibles sur S. La réalisation étale sur S est alors le morphisme d'anneaux

$$\chi_S^{\text{ét}} : K_0(\mathbf{Var}_S) \longrightarrow K_0^{\mathrm{tri}}(\mathbf{D}^{\mathrm{b}}_{\mathrm{cons}}(S,\mathbf{Q}_\ell))$$

envoyant la classe de $X \xrightarrow{f} S$ sur

$$\left[f_!(\mathbf{Q}_\ell)_X^{\text{ét}}\right] = \sum_{i\in\mathbf{Z}}(-1)^i\left[\mathcal{H}^i\left(f_!(\mathbf{Q}_\ell)_X^{\text{ét}}\right)\right].$$

La catégorie triangulée $\mathbf{D}^{\mathrm{b}}_{\mathrm{cons}}(S,\mathbf{Q}_\ell)$ admet une sous-catégorie triangulée $\mathbf{D}^{\mathrm{b}}_{\mathrm{mix}}(S,\mathbf{Q}_\ell)$ stable par les six opérateurs et dont les objets sont les complexes de faisceaux ℓ-adiques dont les faisceaux de cohomologies sont constructibles et mixtes, dans le sens qu'ils admettent une filtration par le poids [**Del80**, §6]. Étant donnée une S-variété séparée $f : X \to S$, le complexe $(\mathbf{Q}_\ell)_X^{\mathrm{mix}}$ est le tiré en arrière $f^*(\mathbf{Q}_\ell)_S^{\mathrm{mix}}$.

Ceci fournit une nouvelle réalisation

$$\begin{array}{rcl}\chi_S^{\mathrm{mix}} : K_0(\mathbf{Var}_S) & \longrightarrow & K_0^{\mathrm{tri}}(\mathbf{D}^{\mathrm{b}}_{\mathrm{mix}}(S,\mathbf{Q}_\ell)) \\ \left[X \xrightarrow{f} S\right] & \longmapsto & [f_!(\mathbf{Q}_\ell)_X^{\mathrm{mix}}]\end{array}$$

qui factorise $\chi_S^{\text{ét}}$

$$\begin{tikzcd} K_0(\mathbf{Var}_S) \arrow[r, "\chi_S^{\mathrm{mix}}"] \arrow[dr, "\chi_S^{\text{ét}}"'] & K_0^{\mathrm{tri}}(\mathbf{D}^{\mathrm{b}}_{\mathrm{mix}}(S,\mathbf{Q}_\ell)) \arrow[d] \\ & K_0^{\mathrm{tri}}(\mathbf{D}^{\mathrm{b}}_{\mathrm{cons}}(S,\mathbf{Q}_\ell)) \end{tikzcd}$$

ainsi qu'une filtration par le poids sur $\mathscr{M}_S$ et un complété $\widehat{\mathscr{M}_S}^w$ associé.

1.2.5. Produits eulériens motiviques. On présente maintenant les constructions permettant de donner sens au produit définissant la limite attendue. La notion de produit eulérien motivique que nous utilisons est celle introduite et largement développée dans le chapitre 3 de la thèse de Bilu [**Bil23**]. On pourra également se rapporter à la présentation faite dans l'introduction [**Bil23**, §1.5.1].

1.2.5.1. *Produits symétriques.* Soit toujours S une variété sur un corps k quelconque et X une variété sur S. La première brique du produit eulérien motivique est la *m-ième puissance symétrique relative*

$$\mathrm{Sym}^m_{/S}(X) = (\underbrace{X \times_S \ldots \times_S X}_{m \text{ fois}})/\mathfrak{S}_m.$$

Ensuite, pour toute famille $\mathscr{X} = (X_i)_{i\in I}$ de variétés quasi-projectives sur X, indexée par un ensemble I, et toute partition $\mu = (m_i)_{i\in I} \in \mathbf{N}^{(I)}$, on définit la *$\mu$-ième puissance symétrique de X* par

$$\mathrm{Sym}^\mu_{/S}(X) = \prod_{i\in I} \mathrm{Sym}^{m_i}_{/S}(X)$$

ainsi que celle de $\mathscr{X}$ par

$$\mathrm{Sym}^\mu_{X/S}(\mathscr{X}) = \prod_{i\in I} \mathrm{Sym}^{m_i}_{X/S}(X_i)$$

laquelle est une variété au-dessus de $\mathrm{Sym}^\mu_{/S}(X)$.

Le produit symétrique restreint

$$\mathrm{Sym}^\mu_{/S}(X)_*$$

est alors l'image dans $\mathrm{Sym}^\mu_{/S}(X)$ du complémentaire dans $\prod_{i\in I} X^{m_i}$ de la grande diagonale, c'est-à-dire l'image de l'ouvert des points de coordonnées deux-à-deux distinctes. De la même façon, le produit symétrique restreint

$$\mathrm{Sym}^\mu_{X/S}(\mathscr{X})_*$$

est l'image dans $\mathrm{Sym}^\mu_{X/S}(\mathscr{X})$ du complémentaire dans $\prod_{i\in I} X_i^{m_i}$ de la grande diagonale.

1.2.5.2. *Produits eulériens motiviques.* En plus des hypothèses précédentes, on se donne une famille d'indéterminées $(t_i)_{i\in I}$ et on note $\mathbf{t}^\mu = \prod_{i\in I} t_i^{m_i}$ pour toute partition $\mu = (m_i)_{i\in I} \in \mathbf{N}^{(I)}$. Alors le produit

$$\prod_{x\in X/S}\left(1 + \sum_{i\in I} X_{i,x} t_i\right)$$

est une *notation* désignant la série formelle

$$\sum_{\mu\in\mathbf{N}^{(I)}} [\mathrm{Sym}^\mu_{X/S}(\mathscr{X})_*]\mathbf{t}^\mu \tag{1.2.5.6}$$

à coefficients dans $K_0(\mathbf{Var}_k)$, $K_0^{\mathrm{uh}}(\mathbf{Var}_k)$ ou $\mathscr{M}_k$ selon le contexte (ou bien encore dans un anneau de variétés avec exponentielles comme dans le Chapitre 4).

Cette série formelle a les propriétés attendues d'un produit. Notamment, le découpage de X/S en un ouvert U/S et son complémentaire Y/S se traduit en l'identité

$$\prod_{x\in X/S}\left(1 + \sum_{i\in I} X_{i,x} t_i\right) = \left(\prod_{x\in U/S}\left(1 + \sum_{i\in I} X_{i,x} t_i\right)\right)\left(\prod_{x\in Y/S}\left(1 + \sum_{i\in I} X_{i,x} t_i\right)\right)$$

(où l'on s'est permis de conserver les mêmes notations pour les restrictions de $\mathscr{X}$ à U et Y). Elle est aussi compatible avec les changements de variables de la forme $\mathbf{t} \mapsto \mathbf{L}_k^{\mathbf{n}}\mathbf{t}$ pour tout $\mathbf{n} \in \mathbf{N}^I$.

Une propriété plus délicate à démontrer est qu'elle est compatible avec le produit extérieur. Pour cela, on suppose que $I_0 = I \cup \{\mathbf{0}\}$ est un monoïde vérifiant une certaine condition de finitude permettant de se placer dans l'algèbre large de I_0 sur $K_0(\mathbf{Var}_X)$

(voir la condition (D) page 27 dans [Bou07, Chap. III]). Étant données deux familles $\mathscr{A} = (A_i)_{i\in I}$ et $\mathscr{B} = (B_i)_{i\in I}$ de variétés sur X, ceci permet de donner sens à la famille $\mathscr{A} *_X \mathscr{B} = (\sum_{i+j=\ell} A_i \boxtimes_X B_j)_{\ell\in I}$ sur X. Alors, dans l'algèbre large de $\mathbf{N}^{(I)}$ sur $K_0(\mathbf{Var}_k)$, on a

$$\begin{aligned}\prod_{x\in X/S}&\left(\left(1+\sum_{i\in I} A_{i,x}t_i\right)\left(1+\sum_{i\in I} B_{i,x}t_i\right)\right)\\ &= \prod_{x\in X/S}\left(1+\sum_{i\in I} A_{i,x}t_i\right)\prod_{x\in X/S}\left(1+\sum_{i\in I} B_{i,x}t_i\right).\end{aligned}$$

De cette propriété on pourra déduire plus loin que le nombre de Tamagawa motivique d'un produit fini de variétés est donné par le produit des nombres de Tamagawa. On invite le lecteur intéressé par les détails de cette construction à se rapporter à [Bil23, Chap. 3].

1.3. Conjecture de Manin géométrique

Dans cette section on présente les liens qu'entretient cette thèse avec les travaux récents de Brian Lehmann, Sho Tanimoto et leurs collaborateurs. Ceux-ci concernent la partie de la conjecture de Manin dite *géométrique* visant à classifier les composantes irréductibles de $\mathrm{Hom}_{\mathscr{C}}(\mathscr{C},\mathscr{V})$.

Supposons que V est une variété de Fano et que le corps de base absolu est $\mathbf{C}$. Suivant [LRT23] on dira qu'une section $\sigma : \mathscr{C} \to \mathscr{V}$ est relativement libre si $\sigma^*\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}}$ est engendré par ses sections globales et $H^1(\mathscr{C},\sigma^*\mathcal{T}_{\mathscr{V}^\circ/\mathscr{C}}) = 0$. Notons que les bornes (1.1.2.2) et (1.1.2.3) assurent qu'une composante irréductible paramétrant une telle section est de la dimension attendue. Étant donnée M une composante irréductible quelconque de $\mathrm{Hom}_{\mathscr{C}}(\mathscr{C},\mathscr{V})$, les auteurs de [LRT23] distinguent deux possibilités.

(1) Le morphisme d'évaluation $\mathscr{C}\times M \to \mathscr{V}$ n'est pas dominant, c'est-à-dire que les déformations des sections paramétrées par M recouvrent une sous-variété stricte de $\mathscr{V}$.

(2) Le morphisme d'évaluation est dominant. Alors,

   (a) M paramètre une famille de sections dites *relativement libres*, auquel cas M est génériquement lisse et de la dimension attendue,

   (b) ou bien une section générale paramétrée par M n'est pas relativement libre et se factorise alors par un morphisme génériquement fini au-dessus de $\mathscr{C}$.

D'après [LRT23, Theorem 1.3] ces différents cas peuvent être distingués à l'aide d'un invariant birationnel (invariant a de Fujita).

1.3.1. Courbes rationelles libres. On suppose que V est définie sur le corps de base k. Étant donnée une courbe rationnelle $f : \mathbf{P}^1_k \to V$, le théorème de Birkhoff-Grothendieck [Har77, V, Ex. 2.6] fournit des entiers $a_1 \leqslant a_2 \leqslant \dots \leqslant a_n$ tels que

$$f^*\mathcal{T}_V \simeq \mathscr{O}_{\mathbf{P}^1_k}(a_1)\oplus\dots\oplus\mathscr{O}_{\mathbf{P}^1_k}(a_n).$$

On dira que f est *libre* si $a_1 \geqslant 0$ et *très libre* si $a_1 > 0$. Plus généralement, pour tout entier positif r, on dira qu'une courbe rationnelle $f : \mathbf{P}^1_k \to V$ est r-libre si $f^*\mathcal{T}_V \otimes \mathscr{O}_{\mathbf{P}^1_k}(-r)$ est engendré par ses sections globales, ce qui équivaut à $a_1 \geqslant r$ avec les notations précédentes.

Les déformations des courbes rationnelles libres recouvrent la variété. En fait, V possède une courbe libre si et seulement si elle est uniréglée [Deb01, Corollary 4.11], c'est-à-dire qu'il existe une variété M de dimension $n-1$ et une application rationnelle dominante $\mathbf{P}^1_{\mathbf{C}} \times M \dashrightarrow V$.

1.3.2. Des exemples pathologiques. On se place encore dans le cas où $k = \mathbf{C}$ et on décrit des exemples de variétés complexes dont l'espace de module de courbes rationnelles admet des composantes additionnelles qu'il s'agira donc de ne pas prendre en compte.

1.3.2.1. *Volumes cubiques de Fano dans $\mathbf{P}^4_{\mathbf{C}}$.* Supposons que V soit une hypersurface lisse dans $\mathbf{P}^4_{\mathbf{C}}$. Le fibré anticanonique de V est alors

$$\omega_V^{-1} \simeq \mathscr{O}_V(2)$$

et la borne inférieure (1.1.1.1) devient pour les morphismes $\mathbf{P}^1_{\mathbf{C}} \to V$ de H-degré $d \geqslant 1$

$$\dim \operatorname{Hom}^d_k(\mathbf{P}^1_{\mathbf{C}}, V) \geqslant 2d+3.$$

Par [**LT19**, Theorem 7.9], pour tout $d \geqslant 2$ l'espace de module $\operatorname{Hom}^d_{\mathbf{C}}(\mathbf{P}^1_{\mathbf{C}}, V)$ est de dimension $2d+3$ et possède deux composantes irréductibles distinctes R_d et N_d, la première paramétrant génériquement des courbes libres, la seconde paramétrant des revêtements de degré d d'une droite de V. Comme le groupe de Picard de V est de rang 1, le H-degré est le seul degré à notre disposition, et il faut trouver un autre moyen de distinguer ces deux composantes.

Il est connu que la variété $\Sigma(V)$ des droites sur V est une surface lisse de type général [**Isk80**]. On note également $\widetilde{\Sigma(V)} \subset V \times \Sigma(V)$ sa famille universelle, qui est une fibration en $\mathbf{P}^1_k$ au-dessus de $\Sigma(V)$.

Donnons-nous une telle droite ℓ contenue dans V. Les morphismes

$$\mathbf{P}^1_{\mathbf{C}} \longrightarrow \ell$$

de degré d fournissent par composition des courbes rationnelles dans V

$$\mathbf{P}^1_{\mathbf{C}} \to \ell \hookrightarrow V$$

de degré d, paramétrées par $\operatorname{Hom}^d_k(\mathbf{P}^1_k, \ell) \simeq \operatorname{Hom}^d_k(\mathbf{P}^1_k, \mathbf{P}^1_k)$. La dimension de cet espace de modules est $2d+1$. L'espace de module N_d des morphismes $\mathbf{P}^1_{\mathbf{C}} \to V$ de degré d *se factorisant par une fibre de* $\widetilde{\Sigma(V)} \to \Sigma(V)$ est ainsi de dimension $2d+3$ et muni d'une fibration $N_d \to \Sigma(V)$ de fibre $\operatorname{Hom}^d_k(\mathbf{P}^1_k, \ell)$ en $[\ell]$.

$$\begin{tikzcd}
\mathbf{P}^1_{\mathbf{C}} \times N_d \arrow[r] \arrow[dr, "\mathrm{ev}"'] & \widetilde{\Sigma(V)} \arrow[r] \arrow[d] & \Sigma(V) \\
& V &
\end{tikzcd}$$

La classe de N_d dans $K_0(\mathbf{Var}_{\mathbf{C}})$ est donc donnée par

$$[N_d] = \left[\operatorname{Hom}^d_k(\mathbf{P}^1_k, \mathbf{P}^1_k)\right][\Sigma(V)]$$

et dès que $d \geqslant 1$ on a donc plus précisément dans $\widehat{\mathscr{M}_{\mathbf{C}}}^{\dim}$

$$[N_d]\mathbf{L}^{-2d} = \frac{\mathbf{L}^2-1}{\mathbf{L}-1}(1-\mathbf{L}^{-1})[\Sigma(V)] = (\mathbf{L}^2-1)[\Sigma(V)]\mathbf{L}^{-1}.$$

D'après la Proposition III.1.3(ii) de [**Isk80**] les seules possibilités pour $N_{\ell/V}$ sont
— $N_{\ell/V} \simeq \mathscr{O}_\ell \oplus \mathscr{O}_\ell$
— $N_{\ell/V} \simeq \mathscr{O}_\ell(-1) \oplus \mathscr{O}_\ell(1)$

or par la suite exacte

$$0 \longrightarrow \mathscr{O}_\ell(2) \longrightarrow (\mathcal{T}_V)_{|\ell} \longrightarrow N_{\ell/V} \longrightarrow 0$$

de faisceaux localement libres, ces deux cas correspondent respectivement à

$$(\mathcal{T}_V)_{|\ell} \simeq \mathscr{O}_\ell \oplus \mathscr{O}_\ell \oplus \mathscr{O}_\ell(2)$$

et

$$(\mathcal{T}_V)_{|\ell} \simeq \mathscr{O}_\ell(-1) \oplus \mathscr{O}_\ell(1) \oplus \mathscr{O}_\ell(2).$$

Dans le premier cas la droite ℓ est libre sans être très libre, dans le second cas ℓ n'est pas libre. Ceci démontre au passage que N_d paramètre génériquement des morphismes $f : \mathbf{P}^1_{\mathbf{C}} \to V$ de H-degré d qui sont libres sans néanmoins être très libres.

Par ailleurs, le morphisme $\widetilde{\Sigma(V)} \to V$ est génériquement fini : par un point général de V passent un nombre fini de droites contenues dans V. En effet, les seules possibilités pour $N_{\ell/V}(-1)$ sont

— $N_{\ell/V}(-1) \simeq \mathscr{O}_\ell(-1) \oplus \mathscr{O}_\ell(-1)$ auquel cas $\dim(H^0(\ell, N_{\ell/X}(-1)) = 0$,

— $N_{\ell/V}(-1) \simeq \mathscr{O}_\ell(-2) \oplus \mathscr{O}_\ell$ auquel cas $\dim(H^0(\ell, N_{\ell/X}(-1)) = 1$,

et donc par un point fixé de V passe soit un nombre fini de droites, soit une famille unidimensionnelle de droites. Or dans le second cas ℓ n'étant pas libre, elle est donc contenue dans un fermé strict de V. On en déduit qu'il n'existe qu'un nombre fini de points appartenant à une famille unidimensionnelle de droites, et donc que passent par un point général de V un nombre fini de droites.

On a donné là un exemple d'ensemble *mince* accumulateur de courbes rationnelles.

1.3.2.2. *Schéma de Hilbert des paires de points dans* $\mathbf{P}^n_k$. Du côté des points rationnels, le cas du schéma de Hilbert $\mathrm{Hilb}^2(\mathbf{P}^n_k)$ des paires de points dans $\mathbf{P}^n_k$, traité notamment par C. Le Rudulier dans sa thèse [**LR14**] et repris dans une note par W. Sawin [**Saw20**] suggère que la notion de liberté des courbes rationnelles ne soit pas suffisante pour éviter certains ensembles accumulateurs, et qu'il faille aussi considérer tous les degrés à la fois. Ceci justifie l'introduction du multidegré faite plus haut. Dans le cadre arithmétique, A. Demirhan a déjà testé cette approche en considérant “toutes les hauteurs” dans le cadre torique [**Dem21**].

1.4. Des prédictions

1.4.1. Un résultat préliminaire de convergence lorsque $k = \mathbf{C}$. On reprend les notations de la Section 1.2.2. Soit $\mathscr{X}$ un R-schéma de dimension pure égale à n dont on suppose que sa fibre générique X est lisse. On démontre que la différence des valuations relatives à deux modèles $\mathscr{L}$ et $\mathscr{L}'$ d'un même faisceau inversible L sur X est constante sur chaque fibre de L. Celle-ci définit donc une fonction constructible $\varepsilon_{\mathscr{L}-\mathscr{L}'}$ sur l'espace des arcs de $\mathscr{X}$.

Soit $\mathscr{L}_{\mathscr{X}}$ un modèle de ω_X^{-1}. Pour tout ensemble constructible A de $\mathrm{Gr}_\infty(\mathscr{X})$ évitant le lieu singulier de $\mathscr{X}$, la densité motivique de A par rapport à $\mathscr{L}_{\mathscr{X}}$ est définie par

$$\mu_{\mathscr{L}_{\mathscr{X}}}(A) = \int_A \mathbf{L}^{-\varepsilon_{\mathscr{L}_{\mathscr{X}}-\omega^\vee_{\mathscr{X}/R}}} \mathrm{d}\mu_{\mathscr{X}}$$

où l'on a noté $\omega_{\mathscr{X}/R}$ le faisceau des formes volumes relatives de $\mathscr{X}/R$.

La convergence de la limite attendue mentionnée plus haut est assurée par la proposition suivante, première étape dans la formulation d'un principe de Batyrev-Manin-Peyre motivique.

Proposition A. *Soit $\mathscr{V}$ un bon modèle d'une variété V quasi de Fano définie sur le corps de fonctions d'une courbe complexe projective et lisse.*

Alors pour tout ouvert dense $\mathscr{C}' \subset \mathscr{C}$ le produit eulérien motivique

$$\prod_{p \in \mathscr{C}'} \left(1 - \mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))} \mu_{\mathscr{L}_{\mathscr{X}}}(\mathrm{Gr}_\infty(\mathscr{V}_{R_p}^\circ))$$

est bien défini et converge dans $\widehat{\mathscr{M}_{\mathbf{C}}}^w$.

À une constante multiplicative près, ce produit est le résidu associé au pôle d'ordre $\mathrm{rg}(\mathrm{Pic}(V))$ en $t = 1$ du produit eulérien motivique

$$\prod_{p \in \mathscr{C}} \left(1 + t\left(\int_{\mathrm{Gr}_\infty(\mathscr{V}_{R_p}^\circ)} \mathbf{L}_p^{-\varepsilon_{\mathscr{L}_{\mathscr{X}}} - \omega^\vee_{\mathscr{X}/R}} \mathrm{d}\mu_{\mathscr{X}} - 1\right)\right).$$

1.4.2. Formulation d'un principe de Batyrev-Manin-Peyre motivique. On suppose à nouveau k quelconque et on reprend le cadre des sections 1.1.2 et 1.2.3.

Soit $\mathscr{L}$ le modèle de ω_V^{-1} donné par

$$\mathscr{L} = \otimes_{i=1}^r \mathscr{L}_i^{\otimes \lambda_i}$$

où les λ_i sont les uniques entiers tels que

$$\omega_V^{-1} \simeq \otimes_{i=1}^r L_i^{\otimes \lambda_i}.$$

On suppose que le nombre de Tamagawa motivique $\tau_{\mathscr{L}}(\mathscr{V})$ est bien défini dans $\widehat{\mathscr{M}_k}^w$.[3]

Pour tout multidegré $\delta \in \mathrm{Pic}(V)^\vee$ et tout ouvert dense de V, soit

$$\widetilde{\mathrm{Hom}}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$$

la réunion des composantes irréductibles de $\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$ paramétrant génériquement des sections relativement très libres.

Principe A. *Il existe un ouvert dense $U \subset V$ tel que la classe normalisée*

$$\left[\widetilde{\mathrm{Hom}}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U\right] \mathbf{L}_k^{-\delta \cdot \omega_V^{-1}} \in \mathscr{M}_k$$

tend vers le nombre de Tamagawa motivique

$$\tau_{\mathscr{L}}(\mathscr{V}) = \frac{\mathbf{L}_k^{(1-g(\mathscr{C}))\dim(V)} \left[\mathrm{Pic}^0(\mathscr{C})\right]^{\mathrm{rg}(\mathrm{Pic}(V))}}{\left(1 - \mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))}} \prod_{p \in \mathscr{C}} \left(1 - \mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))} \mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}(\mathrm{Gr}(\mathscr{V}_{R_p}^\circ))$$

dans le complété $\widehat{\mathscr{M}_k}^w$, lorsque $\delta \in \mathrm{Pic}(V)^\vee$ appartient au dual du cône effectif de V et s'éloigne arbitrairement loin du bord de celui-ci.

Le symbole $\tau_{\mathscr{L}}(\mathscr{V})$ est l'analogue motivique de la constante prédite par Peyre dans le cadre arithmétique original de la conjecture de Manin [**Pey95**].

3. Ceci est démontré plus loin pour $k = \mathbf{C}$ mais n'a pas été vérifié pour $k = \mathbf{F}_q$.

1.4.3. Formulation d'un principe plus général d'équidistribution de courbes. Les conditions de restriction à un sous-schéma $\mathscr{S} \subset \mathscr{C}$ se reformulent naturellement en termes d'espaces de jets et d'arcs, ce qui permettra notamment de donner une formule commode pour la limite attendue. En effet, $\mathrm{Hom}_{\mathscr{C}}(\mathscr{S}, \mathscr{V})$ peut être vu comme le produit

$$\prod_{p \in |\mathscr{S}|} \mathrm{Gr}_{\ell_p}(\mathscr{V}_{R_p})$$

sur les points fermés de $\mathscr{S}$, où les entiers ℓ_p sont tels que

$$\mathscr{S} = \coprod_{p \in |\mathscr{S}|} \mathscr{S}_p$$

$$\mathscr{S}_p \simeq \mathrm{Spec}(\kappa(p)[[t]]/t^{\ell_p+1}).$$

Tout ensemble constructible $W \subset \mathrm{Hom}_{\mathscr{C}}(\mathscr{S}, \mathscr{V})$ fournit un cylindre

$$(\prod_{p \in |\mathscr{S}|} \theta^{\infty}_{\ell_p})^{-1}(W)$$

où l'on rappelle que les $\theta^{\infty}_{\ell_p}$ sont les morphismes de troncation

$$\theta^{\infty}_{\ell_p} : \mathrm{Gr}_{\infty}(\mathscr{V}_{R_p}) \longrightarrow \mathrm{Gr}_{\ell_p}(\mathscr{V}_{R_p}).$$

Dans cette thèse on se restreindra aux ensembles W qui sont des produits sur $|\mathscr{S}|$ d'ensembles constructibles

$$W = \prod_{p \in |\mathscr{S}|} W_p$$

avec $W_p \subset \mathrm{Hom}_{\mathscr{C}}(\mathscr{S}_p, \mathscr{V})$. Le cylindre précédent est alors un produit de cylindres au sens usuel, auquel on peut associer le produit de mesures motiviques

$$\prod_{p \in |\mathscr{S}|} \frac{[W_p]}{\mathbf{L}^{(\ell_p+1)\dim(V)}}.$$

On adopte donc aussi ce point de vue dans la suite. En particulier, pour toute famille $(W_p)_{p \in |\mathscr{C}|}$ d'ensembles constructibles de $\mathrm{Gr}_{\infty}(\mathscr{V}_{R_p})$, indexée par les points fermés de $\mathscr{C}$ et telle que $W_p = \mathrm{Gr}_{\infty}(\mathscr{V}_{R_p})$ pour tout point fermé $p \in |\mathscr{C}|$ en dehors d'un ensemble fini, il fait sens de considérer pour tout multidegré $\delta \in \mathrm{Eff}(V)^{\vee}_{\mathbf{Z}}$ l'espace de module

$$\widetilde{\mathrm{Hom}}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V} \mid W)_U$$

donné par les sections $\sigma \in \widetilde{\mathrm{Hom}}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$ appartenant à la pré-image de W par le morphisme

$$\widetilde{\mathrm{Hom}}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V} \mid W)_U \to \prod_{p \in |\mathscr{C}|} \mathrm{Gr}_{\infty}(\mathscr{V}_{R_p}).$$

Notons que celui-ci induit sur les k-points la restriction du plongement diagonal

$$V(F) \longrightarrow \prod_{p \in |\mathscr{C}|} V(F_p)$$

aux F-points induisant en tout p des F_p-points tombant dans W_p.

Principe B (Équidistribution de courbes). *Il existe un ouvert dense $U \subset V$ tel que pour toute famille $(W_p)_{p\in|\mathscr{C}|}$ d'ensembles constructibles de $\mathrm{Gr}_\infty(\mathscr{V}_{R_p})$, indexée par les points fermés de $\mathscr{C}$ et telle que $W_p = \mathrm{Gr}_\infty(\mathscr{V}_{R_p})$ pour tout point fermé $p \in |\mathscr{C}|$ en dehors d'un ensemble fini, la classe normalisée*

$$\left[\widetilde{\mathrm{Hom}}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V} \mid W)_U\right] \mathbf{L}_k^{-\delta\cdot\omega_V^{-1}} \in \mathscr{M}_k$$

tend vers le nombre de Tamagawa motivique

$$\tau_{\mathscr{L}}(\mathscr{V} \mid W) = \frac{\mathbf{L}_k^{(1-g(\mathscr{C}))\dim(V)}\left[\mathrm{Pic}^0(\mathscr{C})\right]^{\mathrm{rg}(\mathrm{Pic}(V))}}{\left(1-\mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))}} \prod_{p\in\mathscr{C}} \left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))} \mu_{\mathscr{L}_{|\mathscr{V}_{R_p}}}(W_p)$$

dans le complété $\widehat{\mathscr{M}_k}^{w}$ lorsque $\delta \in \mathrm{Pic}(V)^\vee$ appartient au dual du cône effectif de V et s'éloigne arbitrairement loin du bord de celui-ci.

Étant donnée $(W_p)_{p\in|\mathscr{C}|}$ une famille d'ensembles constructibles de $\mathrm{Gr}_\infty(\mathscr{V}_{R_p})$ indexée par les points fermés de $\mathscr{C}$, telle que $W_p = \mathrm{Gr}_\infty(\mathscr{V}_{R_p})$ pour tout point fermé p en dehors d'un ensemble fini, on appellera niveau de W et on notera $\ell(W)$ la somme sur les points fermés de $\mathscr{C}$ des niveaux des W_p, c'est-à-dire la somme des entiers positifs ℓ_p tels que pour tout p il existe un constructible de $\mathrm{Gr}_{\ell_p}(\mathscr{V}_{R_p})$ dont W_p est la préimage par la troncation $\theta^\infty_{\ell_p}$ et ℓ_p est minimal pour cette propriété. Par hypothèse, $\ell_p = 0$ sauf pour un nombre fini de point p fermé, donc $\ell(W)$ est bien défini.

On peut alors être optimiste et raffiner la conjecture précédente en espérant un contrôle uniforme du terme d'erreur de la forme

$$\begin{aligned} &w_k\left(\tau_{\mathscr{L}}(\mathscr{V} \mid W) - \left[\widetilde{\mathrm{Hom}}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{V} \mid W)_U\right] \mathbf{L}_k^{-\delta\cdot\omega_V^{-1}}\right) \\ &\leqslant -ad(\delta, \partial\,\mathrm{Eff}(V)^\vee) + f(\ell(W)) + 2(\dim(V) - g(\mathscr{C})) - 2\ell(W)(\dim(V) - \dim(W_\ell)) \end{aligned}$$

où a est un réel strictement positif, f est une fonction linéaire (indépendante de W) et W_ℓ est l'image de W par $\prod_{p, W_p\neq \mathrm{Gr}_\infty(\mathscr{V}_{R_p})} \theta^\infty_{\ell_p}$.

1.4.4. Vers une extension aux courbes de Campana ? Un autre ensemble de conditions, en un certain sens orthogonales aux précédentes, consiste à

— fixer un ensemble fini de $\mathbf{Q}$-diviseurs de Weil de V, supposés linéairement indépendants,

— et à imposer un degré minimal d'intersection locale vis-à-vis de ceux-ci en presque tout point fermé de $\mathscr{C}$.

Les sections vérifiant ce type de conditions seront appelées *courbes de Campana*. On donne ici quelques définitions à ce sujet, suivant [**Cam05**, **PSTVA21**].

Dans ce travail, un orbifold de Campana sur F est une paire (V, D) où V est une variété projective lisse sur F comme auparavant, et D est un $\mathbf{Q}$-diviseur de Weil effectif sur V de la forme

$$D = \sum_{\alpha\in\mathscr{A}} \epsilon_\alpha D_\alpha$$

où les D_α sont des diviseurs irréductibles sur V indexés par un ensemble fini $\mathscr{A}$, et les nombres rationnels $\epsilon_\alpha \in [0,1] \cap \mathbf{Q}$ sont de la forme

$$\epsilon_\alpha = 1 - \frac{1}{m_\alpha} \qquad m_\alpha \in \mathbf{N}_{>0} \cup \{\infty\}.$$

De plus, on se donne un diviseur anticanonique K_V et on requiert que $K_V + D$ soit $\mathbf{Q}$-Cartier. On suppose également que le support de D est un diviseur à croisements normaux stricts.

On se donne comme ci-avant un bon modèle $\mathscr{V}$ de V au-dessus de la courbe $\mathscr{C}$. Pour tout $\alpha \in \mathscr{A}$, soit $\mathscr{D}_\alpha$ l'adhérence de D_α dans $\mathscr{V}$ et $\mathscr{D}$ la somme des $\mathscr{D}_\alpha$. Pour tout point fermé $p \in \mathscr{C}$, on note $\widehat{\mathscr{O}_{\mathscr{C},p}}$ le complété de l'anneau local de $\mathscr{C}$ en p. Le degré d'intersection local d'une section $\sigma : \mathscr{C} \to \mathscr{V}$ en un point fermé p est l'entier $(\sigma, \mathscr{D}_\alpha)_p$ donné par la multiplicité du tiré-en-arrière de $\mathscr{D}_\alpha$ sur le germe $\mathrm{Spec}(\widehat{\mathscr{O}_{\mathscr{C},p}})$ de $\mathscr{C}$ en p, où $\widehat{\mathscr{O}_{\mathscr{C},p}}$ désigne le complété de l'anneau local de $\mathscr{C}$ en p.

Soit S un ensemble fini de points fermé de $\mathscr{C}$. Une section $\sigma : \mathscr{C} \to \mathscr{V}$ est dite *de Campana* si pour tout point fermé p en-dehors de S,

$$(\sigma, \mathscr{D}_\alpha)_p = 0 \quad \text{ou} \quad (\sigma, \mathscr{D}_\alpha)_p \geqslant m_\alpha = \frac{1}{1-\epsilon_\alpha}$$

pour tout $\alpha \in \mathscr{A}$ [**PSTVA21**, Definition 3.4]. Pour tout multidegré δ, ces conditions définissent un sous-ensemble constructible de $\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})$.

Le multidegré s'étend naturellement à $\mathrm{Pic}(V) \otimes_{\mathbf{Z}} \mathbf{Q}$. Si l'on reprend *ad hoc* la borne (1.1.2.2) en remplaçant naïvement K_V par $K_V + D$ et $\Omega_{\mathscr{V}^\circ/\mathscr{C}}$ par $\Omega_{\mathscr{V}^\circ/\mathscr{C}}(\mathscr{D})$, la “dimension attendue” de la composante irréductible paramétrant une section $\sigma : \mathscr{C} \to \mathscr{V}$ vérifiant les conditions précédentes a la particularité d'être rationnelle :

$$\deg(\sigma^*\Omega_{\mathscr{V}^\circ/\mathscr{C}}(\mathscr{D})^\vee) + (1 - g(\mathscr{C}))\dim(V). \tag{1.4.4.7}$$

Notons qu'au sens de la géométrie birationnelle et des singularités du programme du modèle minimale, la paire (V, D) est *divisoriellement log-terminale*. En particulier, si l'on n'avait pas supposé V lisse, alors dans le pire des cas ses singularités auraient été rationnelles [**KM98**, Theorem 5.22]. Si de plus $\epsilon_\alpha < 1$ pour tout $\alpha \in \mathscr{A}$, alors (V, D) est *Kawamata log-terminale*. On renvoit à [**KM98**, Definition 2.37] ou bien [**BCHM10**, §3.1] pour les définitions correspondantes.

Soit U le complémentaire dans V de $\cup_{\epsilon_\alpha=1} D_\alpha$. On suppose d'une part pour simplifier que S est vide et d'autre part qu'il n'existe pas d'obstruction locale à l'existence de sections de Campana. Par soucis de prudence, on ne formulera pas de principe pour les sections de Campana dans cette introduction. Néanmoins, les résultats de cette thèse prédisent que le nombre de Tamagawa modifié à considérer soit de la forme

$$\tau^\epsilon_{\mathscr{L}}(\mathscr{V}) = \frac{\mathbf{L}_k^{(1-g)\dim(V)} \left[\mathrm{Pic}^0(\mathscr{C})\right]^{\mathrm{rg}(\mathrm{Pic}(U))}}{\left(1 - \mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(U))}} \prod_{p \in \mathscr{C}} \left(1 - \mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(U))} \int_{\mathrm{Gr}_\infty(\mathscr{V}_{R_p})^\epsilon} \mathbf{L}_p^{-\mathrm{ord}_{\mathscr{L}-\mathscr{D}_\epsilon} + \mathrm{ord}_{K_{\mathscr{V}/\mathscr{C}}}}$$

où $\mathrm{Gr}_\infty(\mathscr{V}_{R_p})^\epsilon$ est l'ensemble des arcs vérifiant les conditions de Campana au point p. On reporte à la Section 1.5.4 et au Théorème E ci-dessous pour plus de détails.

1.5. Des résultats

1.5.1. Le principe d'équidistribution de courbes ne dépend pas du choix de multidegré.

1.5.1.1. *L'énoncé.* Le principe d'équidistribution de courbes est une meilleure propriété qu'une simple stabilisation de la classe de l'espace de module tout entier, en ce qu'elle ne dépend pas du choix de multidegré que l'on fait.

Tandis que le Principe A concerne la convergence hypothétique de la mesure motivique de l'espace de modules de morphismes tout entier, le Principe B peut être vu comme décrivant la convergence de la mesure elle-même pour une classe d'ensembles mesurables donnée.

Théorème A (Theorem 3.2.6 page 82). *Soient $\mathscr{C}$ une courbe projective lisse définie sur le corps de base k et V une variété quasi de Fano définie sur le corps de fonctions de $\mathscr{C}$.*

Soient $\mathscr{V}$ et $\mathscr{V}'$ deux modèles propres de V au-dessus de $\mathscr{C}$, ainsi que $\underline{\mathscr{L}}$ et $\underline{\mathscr{L}}'$ deux r-uplets de faisceaux inversibles sur $\mathscr{V}$ et $\mathscr{V}'$ dont les restrictions à V forment une base de $\operatorname{Pic}(V)$. Soient respectivement δ et δ', les multidegrés associés à $\underline{\mathscr{L}}$ et $\underline{\mathscr{L}}'$.

Alors le Principe B est vérifié pour $\mathscr{V}$ muni de δ si et seulement s'il l'est pour $\mathscr{V}'$ muni de δ'.

1.5.1.2. *Principe et outils de la preuve.* En appliquant la théorie des modèles de Néron faibles, on démontre facilement qu'il existe sur $\mathscr{C}$ un modèle propre $\widetilde{\mathscr{V}}$ de V, dont le lieu lisse est un modèle de Néron faible, dominant à la fois $\mathscr{V}$ et $\mathscr{V}'$. Par ailleurs, tous les modèles (y compris de faisceaux inversibles) sont isomorphes au-dessus d'un certain ouvert dense de $\mathscr{C}$.

On ramène donc le problème à celui du changement de degré au-dessus d'un nombre fini de points de $\mathscr{C}$. D'abord, on applique la formule de changement de variable localement autant de fois qu'il y a de tels points et on démontre que l'equidistribution de courbes pour $\mathscr{V}$ et $\underline{\mathscr{L}}$ équivaut à l'equidistribution de courbes pour $\widetilde{\mathscr{V}}$ muni du tiré-en-arrière de $\underline{\mathscr{L}}$. La différence des degrés étant une fonction constructible, l'equidistribution sur $\widetilde{\mathscr{V}}$ permet de passer du multidegré relatif au tiré-en-arrière de $\underline{\mathscr{L}}$ à celui donné par le tiré-en-arrière de $\underline{\mathscr{L}}'$. On redescend alors à $\mathscr{V}'$ par l'argument précédent.

1.5.2. Le principe d'équidistribution dans le cas torique. Dans le chapitre 5, on démontre que les courbes rationnelles se comportent comme attendu sur certaines variétés toriques.

1.5.2.1. *Cadre et énoncés.* On rappelle qu'une variété torique V d'orbite ouverte isomorphe à un tore T, définie sur le corps de base k, est donnée par un éventail Σ dans $\operatorname{Hom}(\mathcal{X}^*(T), \mathbf{G}_{m,k})$ où $\mathcal{X}^*(T)$ est le groupe des caractères de T. Elle est projective si et seulement si Σ est complet. La variété torique V est dite *déployée* si son tore T est isomorphe sur k à un $\mathbf{G}_{m,k}^n$, ou de façon équivalente, si l'action du groupe de Galois absolu de k sur l'éventail Σ est triviale.

Théorème B (Theorem 5.1.7). *Les variétés toriques projectives lisses et déployées définies sur un corps de base k quelconque vérifient le principe d'équidistribution de courbes rationnelles pour la filtration dimensionnelle sur $\mathscr{M}_k$.*

Plus précisément, si V est une telle variété, de dimension n et de rang de Picard r, et si $(W_p)_{p\in|\mathbf{P}^1_k|}$ est une famille d'ensembles constructibles de $\operatorname{Gr}_\infty(V_{R_p})$ indexée par les points fermés de $\mathbf{P}^1_k$, telle que $W_p = \operatorname{Gr}_\infty(V_{R_p})$ pour tout point fermé p en dehors d'un *ensemble fini, alors*

$$\begin{aligned}&\left[\mathrm{Hom}_k^\delta(\mathbf{P}^1_k, V \mid W)\right]\mathbf{L}_k^{-\delta\cdot\omega_V^{-1}}\\ &\longrightarrow \frac{\mathbf{L}_k^n}{\left(1-\mathbf{L}_k^{-1}\right)^r}\prod_{p\in\mathbf{P}^1_k}\left(1-\mathbf{L}_{\kappa(p)}^{-1}\right)^r\mu_{V_{R_p}}(W_p)=\tau(V\mid W)\end{aligned}$$

dans $\widehat{\mathscr{M}_k}^{\dim}$ *lorsque* δ *appartient au dual du cône effectif de* V *et s'éloigne arbitrairement du bord de celui-ci.*

De plus, pour tout multidegré δ*, le terme d'erreur*

$$\mathrm{Hom}_k^\delta(\mathbf{P}^1_k, V \mid W) - \tau(V\mid W)$$

est de dimension virtuelle bornée par

$$-\frac{1}{4}d(\delta,\partial\,\mathrm{Eff}(V)^\vee)+2\ell(W)-1+\dim(V)(1-\ell(W))+\dim(\widetilde{W})$$

où

$$\widetilde{W}=\prod_{\substack{p\in|\mathbf{P}^1_k|\\ W_p\subsetneq\mathrm{Gr}_\infty(V_{R_p})}}\theta^\infty_{\ell_p}(W_p).$$

En particulier,

$$\mathrm{Hom}_k^\delta(\mathbf{P}^1_k, V \mid W)$$

est de la dimension attendue dès que

$$d(\delta,\partial\,\mathrm{Eff}(V)^\vee)\geqslant 4(2\ell(W)-1).$$

On démontre en fait le résultat pour des ensembles constructibles d'une forme un peu plus générale, voir Theorem 5.1.7 page 136.

On peut poser la question de l'extension du Théorème B aux morphismes $\mathscr{C}\to V$ depuis une courbe projective lisse géométriquement irréductible quelconque.

1.5.2.2. *Outils et principes de la preuve.* La preuve du résultat précédent repose sur une montée au torseur universel couplée à une inversion de Möbius motivique. Pour les variétés toriques déployées et lisses, le torseur universel admet une description totalement explicite en tant que complémentaire d'intersection d'hyperplans dans un espace affine [**Sal98**] :

$$\mathscr{T}_V=\mathbf{A}_k^{\Sigma(1)}\setminus\bigcup_{\substack{I\subset\Sigma(1)\\ \bigcap_{\alpha\in I}D_\alpha=\varnothing}}\bigcap_{i\in I}\{x_i=0\}$$

où $\Sigma(1)$ désigne l'ensemble des rayons de l'éventail Σ définissant V et les D_α, $\alpha\in\Sigma(1)$, sont les diviseurs de V correspondant. La variété V est alors obtenue comme quotient de $\mathscr{T}_V$ sous l'action du tore de Néron-Severi T_{NS} de V.

À multidegré δ donné, les morphismes de $\mathbf{P}^1_k$ dans la variété torique V correspondent exactement aux orbites sous $T_{\mathrm{NS}}(k)$ des morphismes équivariants de $\mathbf{A}^2_k\setminus\{0\}$ dans le torseur universel de V. Un tel morphisme est donné par un $\Sigma(1)$-uplet de polynômes homogènes à deux variables, de degrés donnés par δ, vérifiant certaines conditions de coprimalité – celles-là-mêmes qui assurent que l'image dudit morphisme $\mathbf{A}^2_k\setminus\{0\}\to\mathbf{A}_k^{\Sigma(1)}$ tombe bien dans l'ouvert $\mathscr{T}_V\subset\mathbf{A}_k^{\Sigma(1)}$.

Les conditions de réduction que l'on impose remontent également au niveau du torseur. Cependant, elles sont peu "compatibles" avec les conditions de coprimalité, comme l'illustre la remarque élémentaire suivante : l'ajout de facteurs communs à un ensemble fini de polynômes modifie potentiellement la valeur de leur réduction en un point donné, puisqu'on multiplie celle-ci par celle du facteur commun que l'on rajoute. Or l'inversion de Möbius repose sur le principe d'inclusion-exclusion : pour obtenir l'ensemble des $\Sigma(1)$-uplets de polynômes vérifiant les conditions de coprimalité voulues, on considère l'ensemble de tous les $\Sigma(1)$-uplets de polynômes sans condition de coprimalité, puis on retire les $\Sigma(1)$-uplets de polynômes ayant un certain facteur commun, appelée dégénérescence, et ainsi de suite en compensant les intersections éventuelles, *etc.*

La difficulté technique essentielle réside donc dans le fait qu'il s'agit de "détordre" morceau par morceau les conditions de réduction, et ceci de façon uniforme en la dégénérescence. Par ailleurs, il n'est pas possible de détordre les morphismes de trop petit degré par rapport à l'un des D_α, ce qui introduit un terme d'erreur additionnel qu'il s'agit alors de contrôler. L'augmentation, avec le degré, du nombre de termes d'erreur, ne pose pas problème tant que ceux-ci demeurent tous dans un cran de la filtration dimensionnelle bien identifié.

1.5.3. Application à des fibrations. Soit T un tore déployé sur un corps k, de dimension n, orbite ouverte d'une variété torique projective et lisse X, et $\mathscr{B}$ une variété projective lisse. Soit $\mathscr{T}$ un T-torseur sur $\mathscr{B}$. Dans [**CLT01b**, **CLT01a**], Chambert-Loir et Tschinckel ont étudié la conjecture de Batyrev-Manin-Peyre pour les points rationnels (c'est-à-dire lorsque k est corps de nombre) sur le produit tordus

$$\mathscr{X} = X \times^{\mathscr{T}} \mathscr{B}$$

obtenu par recollement de X sur $\mathscr{T}$ via l'action de T, si l'on suppose que les points rationnels sur $\mathscr{B}$ se répartissent comme attendu (la validité de la conjecture étant alors déjà bien connue pour les $\mathbf{Q}$-points sur X par [**BT98a**, **Sal98**]). Sous de bonnes hypothèses [**CLT01a**, Hypothèses 5.2.2], les deux auteurs démontrent que la fonction zeta des hauteurs de $\mathscr{X}$ admet un prolongement méromorphe dans un voisinage de ses pôles, et en déduisent via des théorèmes taubériens [**CLT01a**, Appendice] un développement asymptotique pour le nombre de points de hauteur bornée. Dans cette thèse on démontre l'analogue suivant.

THÉORÈME C. *Soit $\mathscr{B}$ une variété quasi de Fano sur k vérifiant le principe de Batyrev-Manin-Peyre pour les courbes rationnelles. Soit X une variété torique déployée, projective et lisse. Soit enfin $\mathscr{T}$ un torseur sous T au-dessus de $\mathscr{B}$.*

Alors le produit tordu

$$\mathscr{X} = X \times^{\mathscr{T}} \mathscr{B}$$

vérifie également le principe de Batyrev-Manin-Peyre pour les courbes rationnelles.

La preuve de ce résultat repose sur le théorème d'équidistribution des courbes sur les variétés toriques ainsi que sur l'argument suivant de montée à la fibration. Étant donné un morphisme $f : \mathbf{P}^1_k \to \mathscr{B}$, on étudie l'espace de modules des morphismes $g : \mathbf{P}^1_k \to \mathscr{X}$ dont la composée égale f. En tirant en arrière par f, un tel morphisme g induit une section d'un modèle de X au-dessus de $\mathbf{P}^1_k$.

$$\begin{tikzcd}
\mathscr{X}_f \arrow[r] \arrow[d, shift right] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d, "\pi"] \\
\mathbf{P}^1_k \arrow[u, dashed, shift right, "\sigma"'] \arrow[ur, "g"] \arrow[r, "f"] & \mathscr{B}
\end{tikzcd}$$

Ceci permet de démontrer que le morphisme de composition

$$\pi_* : \mathrm{Hom}_k(\mathbf{P}^1_k, \mathscr{X}) \to \mathrm{Hom}_k(\mathbf{P}^1_k, \mathscr{B})$$

induit à multidegrés fixés des fibrations triviales par morceaux. Une étude fine des multidegrés en jeu permet d'écrire la classe $\mathrm{Hom}^{\delta}_k(\mathbf{P}^1_k, \mathscr{X})$ comme le produit de celles de $\mathrm{Hom}^{\delta_{\mathscr{B}}}_k(\mathbf{P}^1_k, \mathscr{B})$ et $\mathrm{Hom}^{\delta_X^T}_{\mathbf{P}^1_k}(\mathbf{P}^1_k, \mathscr{X}_f)$ pour des multidegrés $\delta_{\mathscr{B}}$ et δ_X^T bien identifiés. Par le théorème d'indépendance vis-à-vis du modèle, et sachant que l'on a équidistribution de courbes rationnelles sur X, on peut alors passer de $\mathrm{Hom}^{\delta_X}_{\mathbf{P}^1_k}(\mathbf{P}^1_k, \mathscr{X}_f)$ à $\mathrm{Hom}^{\delta_X}_k(\mathbf{P}^1_k, X)$.

Il est probable que supposer de plus l'équidistribution de courbes sur $\mathscr{B}$ permette d'en déduire l'équidistribution de courbes sur le produit tordu $\mathscr{X}$.

1.5.4. Les compactifications équivariantes d'espaces vectoriels. Dans ce paragraphe on suppose que k est un corps algébriquement clos de caractéristique nulle.

En s'appuyant sur les travaux de thèse de M. Bilu [**Bil23**], eux-même étant notamment une généralisation importante de travaux d'A. Chambert-Loir et F. Loeser [**CLL16**], on démontre dans [**Fai22**], repris ici dans le chapitre 4, que l'espace de module des sections d'un modèle d'une compactification équivariante d'un espace vectoriel se comporte comme attendu. Dans cette thèse, on étend l'étude de [**Fai22**] au cas des courbes dites de Campana, obtenant un analogue motivique des résultats de M. Pieropan, A. Smeets, S. Tanimoto et A. Várilly-Alvarado obtenus dans le cadre arithmétique [**PSTVA21**].

1.5.4.1. *Cadre géométrique.* On précise d'abord le cadre géométrique et les notations. Par soucis de cohérence entre [**CLL16**, **Bil23**, **Fai22**] et le présent manuscrit, dans la suite de cette introduction ainsi que dans le chapitre dédié, la fibre générique sera notée X en place de V et le modèle $\mathscr{X}$ en place de $\mathscr{V}$. Soit G le groupe additif $\mathbf{G}_a^n$. On suppose que X est une compactification équivariante de $\mathbf{G}_F$, c'est-à-dire que X est un schéma projectif contenant un ouvert dense isomorphe à G_F de telle sorte que l'action de G_F s'étend à X tout entier.

Par [**HT99**, Theorem 2.7], il est connu que le complémentaire de G_F dans X est un diviseur dont les composantes irréductibles $(D_\alpha)_{\alpha\in\mathscr{A}}$ engendrent librement le groupe de Picard de X ainsi que son cône effectif. De plus, un diviseur anticanonique de X est donné par

$$-K_X = \sum_{\alpha\in\mathscr{A}} \rho_\alpha D_\alpha$$

où les ρ_α sont des entiers au moins égaux à 2. On se donne

$$\underline{\mathscr{L}} = (\mathscr{L}_\alpha)_{\alpha\in\mathscr{A}}$$

une famille de diviseurs de Cartier sur $\mathscr{X}$ qui étendent les D_α et on considère le multidegré associé, c'est-à-dire pour toute section $\sigma : \mathscr{C} \to \mathscr{X}$

$$\mathbf{deg}_{\underline{\mathscr{L}}}(\sigma) : \sum_{\alpha\in\mathscr{A}} \lambda_\alpha D_\alpha \longmapsto \sum_{\alpha\in\mathscr{A}} \lambda_\alpha \deg(\sigma^*\mathscr{L}_\alpha).$$

On considère alors pour tout $\delta \in \mathrm{Pic}(X)^\vee$ l'espace de module

$$\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{X})_G$$

des sections $\sigma : \mathscr{C} \to \mathscr{X}$ envoyant le point générique de $\mathscr{C}$ dans G_F et dont le multidegré $\mathbf{deg}_{\underline{\mathscr{L}}}(\sigma)$ est égal à δ.

Comme dans [**CLL16**, **Bil23**], on peut élargir l'étude à des points dits "S-entiers" de la façon suivante. On se donne $\mathscr{C}_0$ un ouvert dense de $\mathscr{C}$ ainsi qu'un ouvert U de X, stable sous l'action de G_F et dont le complémentaire est noté D. On suppose que $\mathscr{U}$ est un ouvert de $\mathscr{X}$ dont la restriction à F est U. Par ailleurs, quitte à remplacer $\mathscr{X}$, on peut supposer que $\mathscr{X}$ est lisse et que le complémentaire de $\mathscr{U}$ est un diviseur à croisements normaux.

In fine, on écrit $D = \sum_{\alpha\in\mathscr{A}_D} D_\alpha$ pour un certain sous-ensemble $\mathscr{A}_D \subset \mathscr{A}$. Un diviseur *log-anticanonique* pour la paire (X, D) est alors

$$-K_X - D = \sum_{\alpha\in\mathscr{A}} \rho'_\alpha D_\alpha$$

où

$$\rho'_\alpha = \begin{cases} \rho_\alpha & \text{si } \alpha \in \mathscr{A}_U = \mathscr{A} \setminus \mathscr{A}_D \\ \rho_\alpha - 1 & \text{si } \alpha \in \mathscr{A}_D. \end{cases}$$

Un modèle de celui-ci est alors donné par la combinaison linéaire

$$\mathscr{L}_{\rho'} = \sum_{\alpha\in\mathscr{A}} \rho'_\alpha \mathscr{L}_\alpha.$$

On fixe un ouvert dense $\mathscr{C}_0 \subset \mathscr{C}$ et on se restreint au sous-ensemble constructible de $\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{X})_G$ composé des sections envoyant $\mathscr{C}_0$ dans $\mathscr{U}$. La suite d'espaces de modules correpondantes est notée

$$\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{C}_0 \to \mathscr{U})_G.$$

1.5.4.2. *Résultats.* Dans [**Bil23**, Chap. 6], M. Bilu étudie les pôles de la fonction zeta des hauteurs motivique

$$Z_{\mathscr{X}}(t) = \sum_{d\in\mathbf{Z}} \left[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{C}_0 \to \mathscr{U})_G\right] t^d$$

associée au fibré anticanonique. Dans [**Fai22**] on a raffiné cette étude de deux façons.

Premièrement, comme exposé plus haut, on considère l'espace de module des sections induisant le même *multidegré* $\delta \in \mathrm{Pic}(V)^\vee$, et non pas indifféremment les sections de même *degré anticanonique* d. Cette décomposition de l'espace de modules $\mathrm{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_G$ en les $\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$ tels que $\delta \cdot \omega_V^{-1} = d$ s'inscrit dans la philosophie de l'heuristique de Batyrev décrivant les composantes irréductibles de cet espace de modules. On donne alors la contribution précise de chacune des composantes.

Deuxièmement, on a produit un premier traitement complet sur un exemple de ce que serait un principe de Batyrev-Manin-Peyre motivique, en déduisant des démonstrations de [**CLL16**, **Bil23**] la convergence du rapport

$$\left[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{C}_0 \to \mathscr{U})_G\right] \mathbf{L}_k^{-\delta\cdot\omega_X(D)^{-1}}$$

et en donnant une interprétation précise de la limite. Ce travail a nécessité une analyse fine des arguments exposés dans [**CLL16**, **Bil23**], tout en travaillant avec une couche de technicité additionnelle.

Notamment, la condition d'intégralité "casse" le degré par rapport aux composantes de D en autant d'invariants numériques qu'il y a de points dans $\mathsf{S} = \mathscr{C} \setminus \mathscr{C}_0$, et donc en autant de composantes. Lorsque l'on passe à la limite, le multidegré arbitrairement grand "force" les courbes à passer infiniment près des $\mathscr{D}_\alpha$ pour $\alpha \in \mathscr{A}_D$ et la limite dépend donc d'une donnée combinatoire décrite par un certain *complexe de Clemens*. Pour tout $s \in \mathsf{S}$, on appellera *face maximale* et on notera M_s tout sous-ensemble maximal de $\mathscr{A}_D$ tel que

$$\bigcap_{\alpha\in\mathsf{M}_s} D_\alpha(F_s) \neq \varnothing.$$

On note alors

$$\mathscr{D}_{\mathsf{M}_s} = \bigcap_{\alpha \in \mathsf{M}_s} \mathscr{D}_{\alpha,s}.$$

Une S-face maximale M sera la donnée pour tout $s \in \mathsf{S}$ d'une face maximale. À toute S-face maximale M est attaché un S-uplet δ_{S} de multidegrés locaux, le support de δ_{S} en $s \in \mathsf{S}$ étant exactement M_s. Fixer la valeur de $\delta_{\mathsf{S}} \in (\mathrm{Pic}(X)^\vee)^{\mathsf{S}}$ définit un ensemble constructible de $\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{C}_0 \to \mathscr{U})_G$ noté

$$\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{C}_0 \to \mathscr{U} \mid \delta_{\mathsf{S}})_G.$$

On appellera *choix de composantes verticales* et on notera β la donnée en tout point fermé p de $\mathscr{C}$ d'une composante verticale E_{β_p} de la fibre de $\mathscr{V} \to \mathscr{C}$ au-dessus dudit point. Une telle fibre étant irréductible au-dessus de presque tout point, le nombre de choix possibles est fini. Etant donné un point fermé p et une composante E_{β_p}, on notera $E^{\circ}_{\beta_p}$ le complémentaire des autres composantes dans celle-ci.

Par abus de notation on désignera par ω_X l'unique differentielle méromorphe G_F-équivariante sur X. On la voit comme une section méromorphe de $\Omega^n_{\mathscr{X}/\mathscr{C}}$. On a alors

$$\varepsilon_{\mathscr{L}_\rho - (\Omega^n \mathscr{X}/\mathscr{C})^\vee} = \mathrm{ord}_{\mathscr{L}_\rho} + \mathrm{ord}_{\mathrm{div}(\omega_X)}$$

(où l'on voit à gauche $\mathscr{L}_\rho$ comme un modèle du fibré anticanonique de X, et à droite comme un diviseur de Cartier sur $\mathscr{X}$ étendant K_X) et on notera dans la suite $|\omega_X| = \mathbf{L}^{-\mathrm{ord}_{\mathrm{div}(\omega_X)}}$ (voir [**CLL16**, §6.1] concernant le lien entre intégrales de Haar et mesures motiviques justifiant cette notation).

Théorème D ([**Fai22**]). *Avec les notations précédentes, pour tout choix β de composantes irréductibles, la classe normalisée*

$$\left[\mathrm{Hom}^{\delta}_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{C}_0 \to \mathscr{U} \mid \delta_{\mathsf{S}})^{\beta}_G\right] \mathbf{L}_k^{-\delta \cdot \omega_X(D)^{-1}}$$

tend vers le nombre de Tamagawa motivique

$$\frac{\mathbf{L}_k^n[\mathrm{Pic}^0(\mathscr{C})]}{\left(1-\mathbf{L}_k^{-1}\right)^r} \prod_{p \in \mathscr{C}_0} \left(1 - \mathbf{L}_{\kappa(p)}^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(U))} \int_{\mathrm{Gr}_\infty(\mathscr{U}_{R_p}, E^{\circ}_{\beta_v})} \mathbf{L}_p^{-\mathrm{ord}_{\mathscr{L}_{\rho'}}} |\omega_X|$$
$$\times \prod_{s \in \mathsf{S}} \left(1 - \mathbf{L}_{\kappa(p)}^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(U))} \mathrm{res}_{t = \mathbf{L}_{\kappa(s)}^{-1}} (1 - \mathbf{L}_{\kappa(s)} t)^{|\mathsf{M}_s|} \int_{\mathrm{Gr}_\infty(\mathscr{X}_{R_s}, E^{\circ}_{\beta_v} \cap \mathscr{D}_{\mathsf{M}_s})} t^{-\mathrm{ord}_{\mathscr{L}_{\rho'}}} |\omega_X|$$

lorsque
- $\delta \in \mathrm{Eff}(X)^\vee_{\mathbf{Z}}$, *ainsi que*
- $\delta_{\mathsf{S}}(s) \in \mathrm{Eff}(X)^\vee_{\mathbf{Z}}$, *pour tout* $s \in \mathsf{S}$,

s'éloignent du bord de $\mathrm{Eff}(X)^\vee$.

D'une certaine manière le facteur local au-dessus des points de S traduit le fait que l'on force les sections (qui pour rappel correspondent à des F-points de G) de degré arbitrairement grand (lequel ne peut-être réalisé qu'au-dessus de S) à s'approcher arbitrairement près de $\prod_{s \in \mathsf{S}} \mathscr{D}_{\mathsf{M}_s}(F_s)$, ce qui n'est pas le cas au-dessus de l'ouvert $\mathscr{C}_0$.

Dans ce manuscrit de thèse, on reprend [**Fai22**] et on étend le résultat précédent dans deux directions. D'abord, une extension peu coûteuse consiste à traiter la question de l'équidistribution, généralisant les "conditions β" précédemment considérées tend vers concernant les composantes verticales non-irréductibles.

Théorème E (Theorem 4.4.15). *Avec les notations précédentes, pour tout produit* $W = \prod_{p\in|\mathscr{C}|} W_p$ *d'ensembles constructibles de* $\mathrm{Gr}_\infty(\mathscr{X}^\circ_{R_p})$ *indexé par les points fermés de* $\mathscr{C}$ *et tel que* $W_p = \mathrm{Gr}_\infty(\mathscr{X}^\circ_{R_p})$ *pour tout point fermé* p *en dehors d'un ensemble fini, la classe normalisée*

$$[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid W)_G]\mathbf{L}_k^{-\delta\cdot\omega_X^{-1}}$$

tend vers le symbole

$$\frac{\mathbf{L}_k^n[\mathrm{Pic}^0(\mathscr{C})]^r}{\left(1-\mathbf{L}_k^{-1}\right)^r}\prod_{p\in\mathscr{C}}\left(1-\mathbf{L}_{\kappa(p)}^{-1}\right)^r\int_{W_p}\mathbf{L}_p^{-\operatorname{ord}_{\mathscr{L}_\rho}}|\omega_X|$$

lorsque $\delta\in\mathrm{Eff}(V)^\vee_{\mathbf{Z}}$ *s'éloigne arbitrairement du bord de* $\mathrm{Eff}(V)^\vee$.

Ensuite, on étend l'étude des points S-entiers à celle des points de Campana, obtenant ainsi un résultat d'un nouveau type.

Soit $(m_\alpha)_{\alpha\in\mathscr{A}_U}$ une famille d'entiers strictement positifs et $(\epsilon_\alpha)_{\alpha\in\mathscr{A}}$ la famille de rationnels donnée par

$$\epsilon_\alpha = \begin{cases}1-\frac{1}{m_\alpha} & \text{si } \alpha\in\mathscr{A}_U\\ 1 & \text{si } \alpha\in\mathscr{A}_D.\end{cases}$$

Il est par ailleurs assez naturel et commode de poser $m_\alpha=\infty$ si $\alpha\in\mathscr{A}_D$. On note

$$D_\epsilon = \sum_{\alpha\in\mathscr{A}}\epsilon_\alpha D_\alpha$$

le $\mathbf{Q}$-diviseur associé à ϵ et $\mathscr{D}_\epsilon=\sum_{\alpha\in\mathscr{A}}\epsilon_\alpha\mathscr{D}_\alpha$ son adhérence dans $\mathscr{X}$. Soit $r\in\mathbf{N}^*$ donné par

$$r=\prod_{\substack{\alpha\in\mathscr{A}\\ m_\alpha<\infty}} m_\alpha = \prod_{\substack{\alpha\in\mathscr{A}\\ \epsilon_\alpha<1}}\frac{1}{1-\epsilon_\alpha}.$$

On considère alors les sections de $\mathscr{V}\to\mathscr{C}$ dont le degré d'intersection local avec $\mathscr{D}_\alpha$ en un point fermé en dehors de S est nul ou au moins égal à $m_\alpha=\frac{1}{1-\epsilon_\alpha}$. On les appellera dans la suite courbes ou sections (S,ϵ)-Campana. Notons que l'on retrouve le cas des points S-entiers en choisissant $\epsilon_\alpha=0$ si $\alpha\in\mathscr{A}_U$. Dès lors, $D_\epsilon=D$.

Soit $\mathscr{L}_{\rho-\epsilon}$ le modèle de $-(K_X+D_\epsilon)$ donné par

$$\mathscr{L}_{\rho-\epsilon}=\sum_{\alpha\in\mathscr{A}}(\rho_\alpha-\epsilon_\alpha)\mathscr{L}_\alpha.$$

Soit $\mathrm{Gr}^\epsilon_\infty(\mathscr{X}_{R_p})$ le sous-ensemble constructible de $\mathrm{Gr}_\infty(\mathscr{X}_{R_p})$ composé des arcs vérifiant les conditions de Campana pour ϵ, c'est-à-dire le lieu

$$\operatorname{ord}_{\mathscr{D}_\alpha}\geqslant m_\alpha\quad\forall\,\alpha\in\mathscr{A}.$$

Notons que $\mathrm{Gr}^\epsilon_\infty(\mathscr{X}_{R_p})=\mathrm{Gr}_\infty(\mathscr{X}_{R_p})$ si $p\in\mathsf{S}$.

Théorème F (Theorem 4.4.15). *Avec les notations précédentes, la classe normalisée de l'espace de module des* (S,ϵ)*-sections de Campana sur* $\mathscr{X}$

$$\left[\mathrm{Hom}^\delta_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid(\mathsf{S},\epsilon)\text{–Camp.}\mid\delta_{\mathsf{S}})_G\right]\mathbf{L}_k^{-\delta\cdot\omega_X(D_\epsilon)^{-1}}\in\mathscr{M}_{k,r}$$

tend vers

$$\begin{aligned}&\frac{\mathbf{L}_k^{n(1-g)}\left[\mathrm{Pic}^0(\mathscr{C})\right]^r}{(1-\mathbf{L}_k^{-1})^r}\\&\times\prod_{p\in\mathscr{C}_0}(1-\mathbf{L}_p^{-1})^{\mathrm{rg}(\mathrm{Pic}(U))}\int_{\mathrm{Gr}_\infty^\epsilon(\mathscr{X}_{R_p})}\mathbf{L}_p^{-\mathrm{ord}_{\mathscr{L}_{\rho-\epsilon}}}|\omega_X|\\&\times\prod_{s\in\mathsf{S}}\left(1-\mathbf{L}_{\kappa(p)}^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(U))}\mathrm{res}_{t=\mathbf{L}_{\kappa(s)}^{-1}}(1-\mathbf{L}_{\kappa(s)}t)^{|\mathsf{M}_s|}\int_{\mathrm{Gr}_\infty(\mathscr{X}_{R_p}^\circ,\mathscr{D}_{\mathsf{M}_s})}t^{-\mathrm{ord}_{\mathscr{L}_{\rho-\epsilon}}}|\omega_X|\end{aligned}$$

dans $\widehat{\mathscr{M}_{k,r}}^w$ *lorque* $\delta\in\mathrm{Eff}(V)_{\mathbf{Z}}^\vee$ *et* $\delta_{\mathsf{S}}\in(\mathrm{Eff}(V)_{\mathbf{Z}}^\vee)^{\mathsf{S}}$ *s'éloignent arbitrairement du bord du cône effectif de* X*.*

1.5.4.3. *Techniques employées.* Naturellement, les démonstrations des résultats précédents reposent beaucoup sur les outils et analyses développés dans les travaux de Chambert-Loir-Loeser [**CLL16**] et Bilu [**Bil23**].

Tandis que le travail sur les variétés toriques passe par une montée au torseur universel, celui sur les compactifiés d'espaces vectoriels passe par l'analyse des pôles de la fonction zeta des hauteurs motivique

$$Z(\mathbf{t})=\sum_{\delta\in\mathrm{Pic}(X)^\vee}\left[\mathrm{Hom}_{\mathscr{C}}^\delta(\mathscr{C},\mathscr{X}\mid\mathscr{C}_0\to\mathscr{U})_G\right]\mathbf{t}^\delta$$

via des techniques d'analyse harmonique motivique.

Dans sa thèse, Bilu construit une transformée de Fourier motivique vérifiant une formule de Poisson "en famille" [**Bil23**, Chap. 5] pour des fonctions de Schwartz-Bruhat motiviques, généralisant celle introduite par Hrushovski-Kazhdan [**HK09**] et compatible avec la notion de produit eulérien motivique. Or $Z(\mathbf{t})$ admet une réécriture en un tel produit, il est donc possible de lui appliquer une telle formule de Poisson, et le pôle dominant correspond alors au caractère trivial.

Une réécriture similaire peut être faite pour les fonctions zeta sous contraintes

$$Z_{|W}(\mathbf{t})=\sum_{\delta\in\mathrm{Pic}(X)^\vee}\left[\mathrm{Hom}_{\mathscr{C}}^\delta(\mathscr{C},\mathscr{X}\mid W)_G\right]\mathbf{t}^\delta$$

ainsi que pour la généralisation de $Z(\mathbf{t})$ aux sections de Campana, donnée par

$$\mathscr{Z}_{\mathrm{Camp}}(\mathbf{u})=\sum_{\substack{\delta\in\oplus_{\alpha\in\mathscr{A}_U}D_\alpha^\vee\\\delta_{\mathsf{S}}\in(\oplus_{\alpha\in\mathscr{A}_D}D_\alpha^\vee)^{\mathsf{S}}}}\left[\mathrm{Hom}_{\mathscr{C}}^\delta(\mathscr{C},\mathscr{X}\mid(\mathsf{S},\epsilon)\text{–Camp.}\mid\delta_{\mathsf{S}})_G\right]\mathbf{u}^{\delta+\delta_{\mathsf{S}}}.$$

Ici $\mathbf{u}$ est une famille d'indéterminées indexée par $\mathscr{A}_U\sqcup\mathscr{A}_D^{\mathsf{S}}$ permettant de prendre en compte le phénomène de démultiplication des degrés évoqué plus haut pour les points S-entiers. La spécialisation à $\mathbf{t}$ est donnée par $\mathbf{u}_\alpha=\mathbf{t}_\alpha$ si $\alpha\in\mathscr{A}_U$ et $\prod_{s\in\mathsf{S}}\mathbf{u}_{\alpha,s}=\mathbf{t}_\alpha$ si $\alpha\in\mathscr{A}_D$. Comme dans [**Bil23**], la formule de Poisson motivique fournit une décomposition de la forme

$$\mathscr{Z}_{\mathrm{Camp}}(\mathbf{u})=\mathbf{L}_k^{n(1-g)}\sum_{\xi\in k(\mathscr{C})^n}\mathscr{Z}_{\mathrm{Camp}}(\mathbf{u},\xi)$$

où chaque $\mathscr{Z}_{\mathrm{Camp}}(\mathbf{u},\xi)$ admet une expression en tant que produit eulérien motivique dont le facteur local est une intégrale d'Igusa motivique avec exponentielles. De l'analyse précise du pôle de $\mathscr{Z}_{\mathrm{Camp}}(\mathbf{t},0)$ en $t_\alpha=\mathbf{L}^{-\rho_\alpha(1-\epsilon_\alpha)}$ et des contributions des caractères non-triviaux, combinée à un développement explicite de son terme de degré δ, tout en contrôlant localement uniformément la contribution de chaque terme dans la sommation de Poisson, on en déduit la convergence attendue.

En pratique, on traite d'un seul coup les contraintes constructibles au-dessus d'un nombre fini de points et les contraintes type Campana.

1.6. Organisation de la thèse

Le Chapitre 2 est consacré à des rappels préliminaires. Sa première section concerne les anneaux de variétés et leurs variantes, sa deuxième section la transformée de Fourier motivique en famille et la formule de Poisson associée. On donne dans la troisième section les définitions et lemmes d'existence des espaces de morphismes à multidegré fixé, des degrés locaux et des modèles de Néron. Enfin on rappelle en détails la construction du produit eulérien motivique en quatrième section, laquelle est accompagnée de critères de convergence dans la cinquième et dernière section du chapitre.

La définition des nombres de Tamagawa motiviques associés à des modèles et du principe d'équidistribution de courbes occupe le Chapitre 3. On y démontre notamment l'indépendance de cette notion vis-à-vis du choix de modèle.

Dans le Chapitre 4 on vérifie la validité de ces principes pour les compactifications équivariantes d'espaces vectoriels lorsque le corps de base absolu est algébriquement clos de caractéristique nulle.

On fait de même concernant les variétés toriques déployées projectives et lisses dans le Chapitre 5, cette fois-ci pour un corps de base absolu quelconque, et on applique les résultats d'équidistribution aux produits tordus de variétés toriques.

CHAPTER 2

Preliminaries

Abstract. We recall some definitions and constructions from motivic integration, motivic Fourier transforms and Poisson formula, weak Néron models, moduli spaces of morphisms, filtration by the weight from mixed Hodge modules, and finally motivic Euler products.

2.1. Rings of varieties and motivic integration

2.1.1. Grothendieck ring of varieties. Let S be a scheme. The *Grothendieck group of S-varieties*

$$K_0(\mathbf{Var}_S)$$

is defined as the abelian group generated by the isomorphism classes of *S-varieties* (by this, we mean S-schemes of finite presentation), together with *scissors relations*

$$X - Y - U$$

whenever X is an S-variety, Y is a closed subscheme of X and U is its open complement in X. The class of an S-variety X is denoted by $[X]$. The class of the affine line $\mathbf{A}^1_S$ is denoted by $\mathbf{L}_S$ and when the base scheme is clear from the context we may drop the index. Any constructible subset X of an S-variety admits a class $[X]$ in such a group [**CLNS18**, p. 59]. In our case, a constructible subset is a finite union of locally closed subset of an S-variety.

The product $[X][Y] = [X \times_S Y]$ defines a ring structure on $K_0\mathbf{Var}_S$ with unit element the class of S over itself with natural structural map. The localised Grothendieck ring of varieties $\mathscr{M}_S$ is by definition the ring $K_0\mathbf{Var}_S$ localised at the class $\mathbf{L}_S$ of the affine line.

2.1.1.1. *Filtrations.* The ring $\mathscr{M}_S$ admits a decreasing filtration by the virtual dimension: for $m \in \mathbf{Z}$, let $\mathcal{F}^m\mathscr{M}_S$ be the subgroup of $\mathscr{M}_S$ generated by elements of the form

$$[X]\mathbf{L}_S^{-i}$$

where X is an S-variety and i an integer such that $\dim_S(X) - i \leqslant -m$. The completion of $\mathscr{M}_S$ with respect to this decreasing dimensional filtration is the projective limit

$$\widehat{\mathscr{M}_S}^{\dim} = \varprojlim \mathscr{M}_S/\mathcal{F}^m\mathscr{M}_S$$

which comes with a surjective morphism $\mathscr{M}_S \to \widehat{\mathscr{M}_S}^{\dim}$. The dimensional filtration is one of the filtrations we are going to use, another one being the filtration by the weight of the Hodge realisation, see Section 2.4.1.

2.1.2. A modified ring. In positive characteristic, we will work with modified versions of $K_0\mathbf{Var}_S$ and $\mathscr{M}_S$ (see §4.4 of [**CLNS18**, Chap. 2]). An S-morphism $f : X \to Y$ between S-varieties is called a *universal homeomorphism* if for any S-morphism $Y' \to Y$ the induced morphism $X \times_Y Y' \to Y'$ is a homeomorphism. Then the modified ring of varieties $K_0^{\mathrm{uh}}(\mathbf{Var}_S)$ is the quotient of $K_0\mathbf{Var}_S$ by the ideal given by classes $[X]$ and $[Y]$ of S-varieties such that there exists a S-morphism $X \to Y$ which is a universal homeomorphism. If S is a $\mathbf{Q}$-scheme, this ideal is trivial so that $K_0^{\mathrm{uh}}\mathbf{Var}_S \simeq K_0\mathbf{Var}_S$ [**CLNS18**, Chap. 2, Cor. 4.4.7]. An equivalent description is given by the quotient of $K_0\mathbf{Var}_S$ by *radicial surjective morphisms*, see [**BH21**, Remark 2.1.4]. Note that we will systematically drop the "uh" exponent in this thesis.

2.1.3. Ring of varieties with exponentials. The Grothendieck ring of varieties with exponentials $K_0\mathbf{ExpVar}_R$ is defined in a similar way. Its generators are pairs of S-varieties X together with morphisms $f : X \to \mathbf{A}^1 = \mathrm{Spec}(\mathbf{Z}[T])$. Relations are the isomorphism relation

$$(X, f) - (Y, f \circ u)$$

whenever X, Y are S-varieties, $f : X \to \mathbf{A}^1$ a morphism and $u : Y \to X$ an isomorphism of S-varieties; the scissors relation

$$(X, f) - (Y, f_{|Y}) - (U, f_{|U})$$

whenever X is an S-variety, Y a closed subscheme of X, U its complement in X and $f : X \to \mathbf{A}^1$ a morphism; and finally the relation

$$(X \times_{\mathbf{Z}} \mathbf{A}^1, \mathrm{pr}_2)$$

whenever X is an S-variety and pr_2 is the second projection. If X is a constructible set and $f : X \to \mathbf{A}^1$ a piecewise morphism (given by the datum of morphisms $f_i : X_i \to \mathbf{A}^1$ on a partition $(X_i)_{1\leqslant i\leqslant m}$ of X into locally closed subsets) then the class $[X, f]$ is well-defined. The class $[\mathbf{A}^1_S, 0]$ is again denoted $\mathbf{L}$. Sending a class of a S-variety X to the class $[X, 0]$ defines a morphism of Abelian groups $\iota : K_0\mathbf{Var}_S \to K_0\mathbf{ExpVar}_S$ sending $\mathbf{L}$ to $\mathbf{L}$. The product

$$[X, f][Y, g] = [X \times_S Y, f \circ \mathrm{pr}_1 + g \circ \mathrm{pr}_2]$$

defines a ring structure on $K_0\mathbf{ExpVar}_S$, with unit element $[\mathrm{Spec}(R), 0]$. Then the morphism ι is actually a morphism of rings. Localising at $\mathbf{L}$, one gets the localised ring $\mathscr{E}xp\mathscr{M}_S$ of varieties with exponentials , together with a ring morphism $\iota : \mathscr{M}_S \to \mathscr{E}xp\mathscr{M}_S$.

Any morphism $u : R \to S$ of k-varieties induces a morphism of group

$$u_! : K_0\mathbf{ExpVar}_R \to K_0\mathbf{ExpVar}_S$$

sending any effective element $[X, f]_R$ to $[X, f]_S$, where we view X as an S-variety through u. If u is a immersion, $u_!$ is a morphism of rings. The morphism u induces as well a morphism of rings in the other direction

$$u^* : K_0\mathbf{ExpVar}_S \to K_0\mathbf{ExpVar}_R$$

sending any effective element $[X, f]_S$ to $[X \times_S R, f \circ \mathrm{pr}_X]_R$. If T is another S-variety, combining pull-backs and product, one obtains an exterior product

$$\boxtimes : K_0\mathbf{ExpVar}_R \times K_0\mathbf{ExpVar}_T \xrightarrow{\mathrm{pr}_R^*\,\mathrm{pr}_T^*} K_0\mathbf{ExpVar}_{R\times_S T}.$$

We conclude this subsection by introducing an analogue of the exponential sums of characters over finite field. Assume for a while that k is a finite field and $\psi : k \to \mathbf{C}^*$ is a non-trivial character. Then the exponential sum associated to a pair $[X,f]_k = [X,f]_{\mathrm{Spec}(k)}$ is

$$\sum_{x\in X(k)} \psi(f(x)).$$

Let S be a k-variety, together with a morphism $u : S \to \mathbf{A}^1$, and $g : X \to S$ a variety over S together with a morphism $f : X \to \mathbf{A}^1$.

$$\begin{tikzcd}
 & \mathbf{A}^1 & \mathbf{A}^1 \\
X \arrow[ur, "f"] \arrow[r, "g"] \arrow[dr] & S \arrow[ur, "u"] \arrow[d] & \\
 & \mathrm{Spec}(k) &
\end{tikzcd}$$

We write $\theta = [X,f]_S$. Then over a point s we can introduce the sum over the fibre

$$\theta(s) = \sum_{\substack{x\in X(k)\\ g(x)=s}} \psi(f(x)).$$

Using the additivity of ψ we decompose fibre by fibre the sum

$$\sum_{s\in S(k)} \theta(s)\psi(u(s)) = \sum_{s\in S(k)} \sum_{\substack{x\in X(k)\\ g(x)=s}} \psi(f(x))\psi(u(s)) = \sum_{x\in X(k)} \psi(f(x)+u(g(x)))$$

which is the exponential sum associated to the pair $[X\times_S S, f+u\circ g]_S = [X,f]_S[S,u]_S$ viewed in $K_0\mathbf{ExpVar}_k$.

$$\begin{tikzcd}
X\times_S S \arrow[r, "\sim"] \arrow[d] & X \arrow[r, "f"] \arrow[d, "g"] & \mathbf{A}^1 \\
S \arrow[r, equal] & S \arrow[r, "u"] & \mathbf{A}^1
\end{tikzcd}$$

Thus in general we define the *sum over rational points*

$$\sum_{x\in S} \theta(s)\psi(u(s)) \tag{2.1.3.8}$$

for any $\theta \in \mathscr{E}xp\mathscr{M}_S$ and $u : S \to \mathbf{A}^1$ as the image by $K_0\mathbf{ExpVar}_S \to K_0\mathbf{ExpVar}_k$ of the class $\theta\cdot[S,u]_S$ (we may sometimes omit the exponential factor $\psi(u(s))$ when writing the sum). This class is explicitly given by $[X, f+u\circ g]$ when $\theta = [X,f]_S$ and $g : X \to S$ is a S-variety. This notation easily extends to the relative setting when $v : S \to T$ is a morphism of varieties over k : we write

$$v_!\theta = \sum_{s\in S/T} \theta(s)\psi(u(s))$$

where $v_! : K_0\mathbf{ExpVar}_S \to K_0\mathbf{ExpVar}_T$ is the morphism of groups induced by v.

2.2. Motivic Fourier transform and Poisson formula in families

In this subsection we recall the motivic analogue of a bunch of Fourier analysis tools used in the sixth chapter of [**Bil23**], where an easy-to-handle expression of the motivic height zeta function is obtained by using a motivic Poisson formula. We will explicitly need this construction at the end of our proof (Section 4.4.3), where we will have to check an uniform convergence. We follow [**CLL16**] and the fifth chapter of [**Bil23**], which extends the scope of Hrushovski and Kazhdan's motivic Poisson formula [**HK09**].

2.2.1. Local motivic Schwartz-Bruhat functions. The first building block of a *motivic* Poisson formula is a motivic analogue of classical Scwhart-Bruhat functions on the non-archimedian local field $K = k((t))$. One should keep in mind that this field can be thought about as the completion of the field of rational functions of a curve, for a valuation given by a closed point. The letter t denotes an uniformiser of this completion; then the ring of integers of K is $\mathcal{O}_K = k[[t]]$.

Recall that a classical Schwartz-Bruhat function φ on a locally compact, non-archimedian local field L is a locally constant and compactly supported function on L. If ϖ is an uniformiser for L and $\mathcal{O}_L$ is the ring of integers of L, then there exist integers $M \leqslant N$ such that φ is zero outside $\varpi^M \mathcal{O}_L$ and invariant modulo $\varpi^N \mathcal{O}_L$. The pair (M, N) is called the level of φ and the function φ on L can be seen as a function on the quotient $\varpi^M \mathcal{O}_L / \varpi^N \mathcal{O}_L$.

The motivic analogue of such a function, as it has been introduced by Hrushovski and Kazhdan in [**HK09**], takes values in the Grothendieck ring $\mathscr{E}xp\mathscr{M}_k$. In order to deal with the fact that $k((t))$ is not locally compact for the topology induced by the valuation, one takes as a first definition the previous properties concerning invariance and compact support: a motivic Schwartz-Bruhat function of level (M, N) is a function $t^M \mathcal{O}_K / t^N \mathcal{O}_K \to \mathscr{E}xp\mathscr{M}_k$. Through the identification

$$\begin{array}{rcl} t^M \mathcal{O}_K / t^N \mathcal{O}_K & \longrightarrow & \mathbf{A}_k^{M-N} \\ a_M t^M + a_{M+1} t^M + \ldots + a_{N-1} t^{N-1} & \longmapsto & (a_M, \ldots, a_{N-1}) \end{array}$$

which endows $t^M \mathcal{O}_K / t^N \mathcal{O}_K$ with the structure of a k-variety, such a motivic Schwartz-Bruhat function on K is basically seen as an element of $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{N-M}}$. In what follows we will freely use the notation

$$\mathbf{A}_k^{n(M,N)} = \mathbf{A}_k^{n(N-M)}.$$

for any $M \leqslant N$ and non-negative integer n. The previous identification naturally extends to K^n, leading to the following definition.

Definition 2.2.1. Let $M \leqslant N$ and $n \geqslant 1$ be integers. A (local) motivic Schwartz-Bruhat function of level (M, N) on K^n is an element of $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(M,N)}}$.

The *extension-by-zero* morphism

$$t^M \mathcal{O}_K / t^N \mathcal{O}_K \to t^{M-1} \mathcal{O}_K / t^N \mathcal{O}_K$$

and the *cutting* morphism

$$t^M \mathcal{O}_K / t^{N+1} \mathcal{O}_K \to t^M \mathcal{O}_K / t^N \mathcal{O}_K$$

induce respectively a closed immersion

$$\mathbf{A}_k^{(M,N)} \to \mathbf{A}_k^{(M-1,N)}$$

and a trivial fibration

$$\mathbf{A}_k^{(M,N+1)} \to \mathbf{A}_k^{(M,N)}$$

(with fibre $\mathbf{A}_k^1$). Such morphisms in turn provide homomorphisms at the level of Grothendieck rings with exponentials, turning

$$(\mathscr{E}xp\mathscr{M}_{\mathbf{A}^{n(M,N)}})_{M \leqslant N}$$

into a direct system, the direct limit of which is by definition the set of all motivic Schwartz-Bruhat functions on K^n.

2.2.1.1. *Fourier kernel and transform.* Recall that we see $K = k((t))$ as the completion F_v at a closed point v of the field of fractions $F = k(\mathscr{C})$ of a smooth projective curve over k, together with the choice of an uniformiser t. If one fixes a non-zero rational differential form $\omega \in \Omega_{K/k}$, then one obtains a non-zero k-linear map $r_v : K \to k$ defined by sending any element $a \in K$ to the residue at v of the rational form $a\omega$:

$$r_v(a) = \mathrm{res}_v(a\omega).$$

There exists a smallest integer ν such that r_v vanishes on $t^\nu \mathcal{O}_K$, given by the order of the pole of ω at v (the *conductor* of r_v). In particular, r_v is invariant modulo $t^N\mathcal{O}_K$ for all $N \geqslant \nu$ and can be seen as a linear function $r^{(M,N)} : \mathbf{A}_k^{(M,N)} \to \mathbf{A}_k^1$ for every $M \leqslant N$ such that $N \geqslant \nu$.

The product $K \times K \to K$ restricts to a morphism $\mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \to \mathbf{A}_k^{(M+M',N'')}$ for every $M \leqslant N$, $M' \leqslant N'$ and $N'' = \min(M'+N, M+N')$. If $N, N', N'' \geqslant \nu$ it can be composed with $r_v^{(M+M',N'')}$ and one obtains an element

$$r_v : \mathbf{A}_k^{(M,N)} \times \mathbf{A}_k^{(M',N')} \xrightarrow{\cdot\times\cdot} \mathbf{A}_k^{(M+M',N'')} \xrightarrow{r_v^{(M+M',N'')}} \mathbf{A}_k^1$$

of $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{(M,N)}\times_k\mathbf{A}_k^{(M',N')}}$, called the *Fourier kernel on* K and written $\mathbf{e}(xy)$. This notation is the analogue of the exponential factor $e^{2i\pi xy}$ of the integrand in classical Fourier analysis.

More generally, the pairing $\langle x, y\rangle = \sum_{i=1}^n x_i y_i$ on K^n provides a morphism

$$\mathbf{A}_k^{n(M,N)} \times \mathbf{A}_k^{n(M',N')} \xrightarrow{\langle\cdot,\cdot\rangle} \mathbf{A}_k^{(M+M',N'')}$$

which can be composed with $r_v^{(M+M',N'')}$ to give the *Fourier kernel* $\mathbf{e}(\langle x, y\rangle)$ *on* K^n.

The Fourier transform of a motivic Schwart-Bruhat function $\varphi \in \mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(M,N)}}$ of level (M, N) on K^n is defined as

$$\mathscr{F}\varphi(y) = \int_{K^n} \varphi(x)\mathbf{e}(\langle x, y\rangle)\mathbf{d}x \tag{2.2.1.9}$$

which is a notation for the class of $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(\nu-N,\nu-M)}}$ given by

$$\mathscr{F}\varphi = \mathbf{L}^{-Nn}\varphi \times_{\mathbf{A}_k^{n(M,N)}} \left[\mathbf{A}_k^{n(M,N)} \times_k \mathbf{A}_k^{n(\nu-N,\nu-M)}, r\right]$$

where $r = r^{(M+\nu-N,N+\nu-M)} \circ \langle\cdot,\cdot\rangle$. The formal variable y is thus *living in* $\mathbf{A}_k^{n(\nu-N,\nu-M)}$. We can be more explicit when $\varphi = [U, f]$ with $g : U \to \mathbf{A}_k^{n(M,N)}$ and $f : U \to \mathbf{A}^1$; in this case $\mathscr{F}\varphi$ is the class

$$\mathscr{F}\varphi = \mathbf{L}^{-Nn}\left[U \times_k \mathbf{A}_k^{n(\nu-N,\nu-M)}, f\circ \mathrm{pr}_1 + r^{(M+\nu-N,N+\nu-M)}(\langle g\circ\mathrm{pr}_1, \mathrm{pr}_2\rangle)\right]$$

in $\mathscr{E}xp\mathscr{M}_{\mathbf{A}_k^{n(\nu-N,\nu-M)}}$.

$$\begin{tikzcd}
U \times_k \mathbf{A}_k^{n(\nu-N,\nu-M)} \arrow[r, "\sim"] \arrow[ddr, bend right] & U \times_g \mathbf{A}_k^{n(M,N)} \times_k \mathbf{A}_k^{n(\nu-N,\nu-M)} \arrow[r] \arrow[d] & U \arrow[r, "f"] \arrow[d, "g"] & \mathbf{A}^1 \\
\mathbf{A}^1 & \mathbf{A}_k^{n(M,N)} \times_k \mathbf{A}_k^{n(\nu-N,\nu-M)} \arrow[l, "r"'] \arrow[r] \arrow[d] & \mathbf{A}_k^{n(M,N)} \arrow[d] & \\
 & \mathbf{A}_k^{n(\nu-N,\nu-M)} \arrow[r] & \mathrm{Spec}(k) &
\end{tikzcd}$$

Notation (2.2.1.9) can be seen as a variant of the exponential sum notation of Section 2.1.3.

2.2.1.2. *From local functions to global ones and summation over rational points.* Such a construction easily extends to finite products $\prod_{s\in S} F_s$ of completions of $F = k(\mathscr{C})$ at a finite number of closed points $s \in S$ of the curve $\mathscr{C}$. The motivic Schwartz-Bruhat functions of level $(M_s, N_s)_{s\in S}$ are the elements of the ring

$$\mathscr{E}xp\mathscr{M}_{\prod_{s\in S} \mathrm{Res}_{\kappa(s)/k} \mathbf{A}_{\kappa(s)}^{n(M_s,N_s)}}$$

where $\mathrm{Res}_{\kappa(s)/k}$ denotes the Weil restriction functor. The set of global motivic Schwartz-Bruhat functions on K^n is defined as a direct limit over the set S of closed points of $\mathscr{C}$ and one can easily extend the previous Fourier kernel and transform to this setting; we will not be more explicit about this construction for now, since it will be generalised in the next paragraph.

Nevertheless it might be enlightening for the reader to define the *sum over rational points* of a global motivic Schwartz-Bruhat is this simple setting. Let φ be a global Schwartz-Bruhat function of level $(M_s, N_s)_{s\in S}$ where $S \subset \mathscr{C}$ is a finite set of closed points of $\mathscr{C}$. We choose an uniformiser t_s of F_s for every $s \in S$. Consider the divisor $D = -\sum_{s\in S} M_s[s]$ on $\mathscr{C}$ and remark that the embeddings $F \hookrightarrow F_s$ map the Riemann-Roch space

$$L(D) = H^0(\mathscr{C}, \mathscr{O}_{\mathscr{C}}(D))$$

to $t_s^{M_s}\mathcal{O}_s$ for every $s \in S$. This mapping provides a morphism of varieties $\theta : \mathcal{L}(D)^n \to \mathrm{Res}_{\kappa(s)/k} \mathbf{A}_{\kappa(s)}^{n(M_s,N_s)}$. The sum over rational points of $\mathbf{A}^n$

$$\sum_{x\in k(C)^n} \varphi(x) = \sum_{x\in L(D)^n} \theta^*\varphi$$

is by definition the image in $\mathscr{E}xp\mathscr{M}_k$ of the pull-back $\theta^*\varphi_S$, that is, the image of φ through the map

$$\mathscr{E}xp\mathscr{M}_{\prod_{s\in S} \mathrm{Res}_{\kappa(s)/k} \mathbf{A}_{\kappa(s)}^{n(M_s,N_s)}} \xrightarrow{\theta^*} \mathscr{E}xp\mathscr{M}_{\mathcal{L}(D)^n} \longrightarrow \mathscr{E}xp\mathscr{M}_k$$

which can be seen as another variant of the exponential sum notation of Section 2.1.3. In this context, one gets a motivic Poisson formula

$$\sum_{x\in k(\mathscr{C})^n} \varphi(x) = \mathbf{L}^{(1-g)n} \sum_{y\in k(\mathscr{C})^n} \mathscr{F}\varphi(y)$$

(see e.g. [**CLL16**, Theorem 1.3.10] for a proof).

2.2.2. Constructible families of Schwartz-Bruhat functions. Until the end of this subsection we follow [Bil23, Chapter 5]. Recall that a function $X \xrightarrow{f} \mathbf{Z}$ on a k-variety is said to be *constructible* if $f^{-1}(\{m\})$ is a constructible subset of X for any $m \in \mathbf{Z}$.

Again $\mathscr{C}$ is a smooth projective connected curve over an algebraically closed field k of characteristic zero. Let $M, N : \mathscr{C} \to \mathbf{Z}$ be constructible functions such that $M \leqslant N$. Such functions are constant respectively equal to M_0 and N_0 on a dense open subset U of the curve $\mathscr{C}$. Then $\mathbf{A}_{\mathscr{C}}^{(M,N)}$ stands for the $\mathscr{C}$-variety isomorphic to $U \times \mathbf{A}_k^{(M_0,N_0)}$ over U and with fibre over $u \notin U$ equal to $\mathbf{A}_k^{(M_u,N_u)}$. Let n be a positive integer. One can as well define the n-th product over $\mathscr{C}$

$$\mathbf{A}_{\mathscr{C}}^{n(M,N)} = \underbrace{\mathbf{A}_{\mathscr{C}}^{(M,N)} \times_{\mathscr{C}} \dots \times_{\mathscr{C}} \mathbf{A}_{\mathscr{C}}^{(M,N)}}_{n \text{ times}}.$$

Until the end of this section we fix two constructible functions $a, b : C \to \mathbf{Z}$ such that $a \leqslant 0 \leqslant b$ and we assume that there exists a dense open subset U of $\mathscr{C}$ such that $a_{|U} = b_{|U} = 0$. Let Σ be the complement of U in $\mathscr{C}$, it is a finite set of points. Let $M = (M_\iota)$ and $N = (N_\iota)$ be families of non-negatives integers indexed by $\mathbf{N}^p$, such that $M_{\mathbf{0}} = N_{\mathbf{0}} = 0$. This data provides a family

$$\mathscr{A}_{\mathbf{n}}(a,b,M,N) = \operatorname{Sym}^{\mathbf{n}}_{\mathscr{C}/k}\left(\left(\mathbf{A}_{\mathscr{C}}^{n(a-M_\iota, b+N_\iota)}\right)_{\iota \in \mathbf{N}^p}\right)$$

for any p-tuple $\mathbf{n}$ of non-negative integers, which is a variety over $\operatorname{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$. Let us quickly describe the fibre of $\mathscr{A}_{\mathbf{n}}(a,b,M,N) \to \operatorname{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$ over a schematic point $D \in \operatorname{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$. The latter can be decomposed formally as

$$D = \sum_{v \in \mathscr{C}} \iota_v v = \sum_{v \in U} \iota_v v + \sum_{v \in \Sigma} \iota_v v = D_U + D_\Sigma$$

where the ι_v are p-tubles of non-negative integers (almost all equal to $\mathbf{0} \in \mathbf{N}^p$),

$$D_U = \sum_{v \in U} \iota_v v \in S^\pi U$$

for some partition $\pi = (n_\iota^U)_{\iota \in \mathbf{N}^p}$ of some $\mathbf{n}^U \leqslant \mathbf{n}$ and $D_\Sigma = \sum_{v \in \Sigma} \iota_v v$. Then, the fibre over D can be seen as the domain of definition of a Schwartz-Bruhat function Φ_D, up to a finite extension of the residue field $\kappa(D)$ of D (seen as a point of $\operatorname{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$) [Bil23, §5.3.2]: it is of the form

$$\prod_{\iota \in \mathbf{N}^p} \mathbf{A}_{\kappa(D)}^{n_\iota^U \times n(M_\iota + N_\iota)} \times_{\kappa(D)} \prod_{\substack{v \in \Sigma \\ \mathbf{i}_v \neq \mathbf{0}}} \mathbf{A}_{\kappa(D)}^{n(a_v - M_{\mathbf{i}_v}, b_v + N_{\mathbf{i}_v})}. \tag{2.2.2.10}$$

Note that the first product is actually finite, since $(n_\iota^U)_{\iota \in \mathbf{N}^p}$ has finite support as a partition.

For this reason, elements of $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,M,N)}$ are called *constructible families of Schwartz-Bruhat functions of level* $\mathbf{n}$ [Bil23, §5.3.4.1]. Two special cases arise. On one hand, if all the integers N_ι are zero, the family is said to be *uniformly smooth*. On the other hand, if all the integers M_ι are zero, the family is said to be *uniformly compactly supported* [Bil23, §5.3.5.1].

2.2.3. Motivic Fourier transform in families. It is possible to define the Fourier transform of a family $\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,M,N)}$ of Schwartz-Bruhat functions, independently of the choice of b and N. Let us fix a non-zero rational differential form

$$\omega \in \Omega_{k(\mathscr{C})/k}$$

and define ν by

$$\nu_v = -\operatorname{ord}_v \omega$$

for every closed point $v \in \mathscr{C}$. This allows one to define a constructible Fourier kernel morphism

$$\mathbf{A}_{\mathscr{C}}^{n(a-M_\iota,b+N_\iota)} \times_{\mathscr{C}} \mathbf{A}_{\mathscr{C}}^{n(\nu-b-N_\iota,\nu-a+M_\iota)} \to \mathbf{A}^1$$

which induces morphisms on the symmetric products

$$r_{\mathbf{n}} : \mathscr{A}_{\mathbf{n}}(a,b,M,N) \times_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \mathscr{A}_{\mathbf{n}}(v-b,v-a,N,M) \to \mathbf{A}^1$$

hence an element of $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,M,N)\times_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}}\mathscr{A}_{\mathbf{n}}(v-b,v-a,N,M)}$ for any $\mathbf{n} \in \mathbf{N}^p$ [**Bil23**, §5.4.2]. Then one can show that the class $[\mathrm{Sym}^{\mathbf{n}}_{\mathscr{C}/k}((\mathbf{A}_{\mathscr{C}}^{n(b,N_\iota)})_{\iota\in\mathbf{N}^p})]$ has an inverse in $\mathscr{E}xp\mathscr{M}_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}}$. We finally define a motivic Fourier transform

$$\mathscr{F} : \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,M,N)} \longrightarrow \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(\nu-b,\nu-a,N,M)}$$

given for any $\Phi = [V,f] \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,M,N)}$, with $g : V \to \mathscr{A}_{\mathbf{n}}(a,b,M,N)$ the structure morphism, by the class in $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(\nu-b,\nu-a,N,M)}$

$$[V \times_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \mathscr{A}_{\mathbf{n}}(\nu-b,\nu-a,N,M), f\circ \mathrm{pr}_1 + r_{\mathbf{n}}(g\circ\mathrm{pr}_1\cdot\mathrm{pr}_2)][\mathrm{Sym}^{\mathbf{n}}_{\mathscr{C}/k}((\mathbf{A}_{\mathscr{C}}^{n(b,N_\iota)})_{\iota\in\mathbf{N}^p})]^{-1}$$

where $\cdot$ denotes the product in the ring $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,M,N)\times_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}}\mathscr{A}_{\mathbf{n}}(\nu-b,\nu-a,N,M)}$ [**Bil23**, §5.4.3]. This Fourier transform is independent of the choice of b and N, and is compatible with symmetric products [**Bil23**, Proposition 5.4.4.2]. It generalises the local and global Fourier transforms of a Schwartz-Bruhat function we presented in the first paragraph of this subsection.

2.2.4. Local Poisson formula. Let D'_U be a pull-back of $D_U \in S^\pi U$ via the quotient map

$$\left(\prod_{\iota\in\mathbf{N}^p\setminus\{\mathbf{0}\}} U^{m_\iota}\right)_* \longrightarrow S^\pi U$$

defining $S^\pi U$, and $v_{\iota,j}$ the projection of D'_U on the j-th copy of U in U^{m_ι}. A different choice of D'_U only permutes the order of the $v_{\iota,j}$'s. Let E_D be the effective $\kappa(D)$-zero-cycle on the curve $\mathscr{C}_{\kappa(D)}$ defined by

$$E_D = \sum_{\iota\in\mathbf{N}^p\setminus\{0\}} M_\iota(v_{\iota,1} + \dots + v_{\iota,m_\iota}) - \sum_{v\in\Sigma}(a_v - M_{\iota_v})v.$$

The divisor E_D does not depend on the choice of D'_U since it is invariant under the action of $\prod_{\iota\in\mathbf{N}^p\setminus\{0\}} \mathfrak{S}_{m_\iota}$ on the product $\prod_{\iota\in\mathbf{N}^p\setminus\{0\}} U^{m_\iota}$ [**Bil23**, Remark 5.5.1.1]. It can be rewritten as

$$\sum_{v\in\mathscr{C}} (M_{\iota_v} - a_v)\, v$$

since a is zero on U. One then defines a constructible morphism of $\kappa(D)$-varieties

$$\theta_D : L_{\kappa(D)}(E_D)^n \longrightarrow \mathscr{A}_{\mathbf{n}}(a,b,M,N)_D$$

where $L_{\kappa(D)}(E_D)$ is the Riemann-Roch space over E_D

$$L_{\kappa(D)}(E_D) = \Gamma(\mathscr{C}_{\kappa(D)}, \mathscr{O}_{\mathscr{C}_{\kappa(D)}}(E_D)).$$

Pointwise, the morphism θ_D sends a function $f \in L_{\kappa(D)}(E_D)$ to its v-adic expansions in the ranges given by the exponents defining $\mathscr{A}_{\mathbf{n}}(a,b,M,N)_D$ in (2.2.2.10). For example, its image in

$$\prod_{v\in\Sigma}\left(\mathfrak{m}_v^{a_v-M_{\iota_v}}\mathcal{O}_v/\mathfrak{m}_v^{b_v+N_{\iota_v}}\mathcal{O}_v\right)$$

may be understood as the coefficients of its v-adic expansion in the range $a_v - M_{\iota_v}, \ldots, b_v + N_v - 1$ (see [**Bil23**, page 164] for an explicit definition).

The *summation over rational points* of Φ_D is by definition the class in $\mathscr{E}xp\mathscr{M}_{\kappa(D)}$ of

$$\sum_{x\in\kappa(D)(\mathscr{C})^n}\Phi_D(x) = \sum_{x\in L_{\kappa(D)}(E_D)^n}(\theta_D^*\Phi_D)(x). \tag{2.2.4.11}$$

Here the sum of the right side is the exponential sum notation (2.1.3.8) corresponding to the morphism

$$\mathscr{E}xp\mathscr{M}_{L_{\kappa(D)}(E_D)^n} \longrightarrow \mathscr{E}xp\mathscr{M}_{\kappa(D)}$$

induced by the projection on $\kappa(D)$.

For a proof of the following, see [**CLL16**, Theorem 1.3.10] and [**Bil23**, Lemma 5.5.1.4].

Proposition 2.2.2 (Poisson formula)**.** *Let D be a schematic point of* $\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$ *and* Φ_D *an element of the fibre* $\mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,M,N)_D}$. *Then*

$$\sum_{x\in\kappa(D)(\mathscr{C})^n}\Phi_D(x) = \mathbf{L}^{(1-g)n}\sum_{x\in\kappa(D)(\mathscr{C})^n}\mathscr{F}\Phi_D(x).$$

2.2.5. Poisson formula in families. If the family Φ is uniformly compactly supported, the zero cycle E_D is actually equal to the k-zero-cycle $D_a = -\sum_v a_v v$ for any schematic point $D \in \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$. Its Riemann-Roch space over k is by definition

$$L(D_a) = \{f \in k(\mathscr{C}) \mid f = 0 \text{ or } \operatorname{div}(f) + D_a \geqslant 0\}.$$

By flat base change, there is a $\kappa(D)$-linear canonical isomorphism $L_{\kappa(D)}(D_a) \simeq L(D_a)\otimes_k \kappa(D)$. Then one can prove the existence of a constructible morphism

$$\theta_{\mathbf{n}} : L(D_a) \times \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C} \longrightarrow \mathscr{A}_{\mathbf{n}}(a,b,0,N)$$

over $\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$, whose restriction to the fibres above a schematic point $D \in \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$ induces θ_D [**Bil23**, §5.5.2.1]. Given an uniformly compactly supported family

$$\Phi \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(a,b,0,N)},$$

the image in $\mathscr{E}xp\mathscr{M}_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}}$ of the pullback $\theta_{\mathbf{n}}^*\Phi$ is called the *uniform summation over rational points*. It is denoted by

$$\left(\sum_{x\in\kappa(D)(\mathscr{C})^n}\Phi_D(x)\right)_{D\in\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}}.$$

This is an example of what Bilu calls an *uniformly summable family*, that is, a constructible family of functions $\Phi \in \mathscr{A}_{\mathbf{n}}(a,b,M,N)$ such that there exists a function $\Sigma \in \mathscr{E}xp\mathscr{M}_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}}$ on $\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}$ whose pullback $D^*\Sigma$ in $\mathscr{E}xp\mathscr{M}_{\kappa(D)}$ is the sum over rational points $\sum_{x\in\kappa(D)(\mathscr{C})^n}\Phi_D(x)$, for any $D \in \mathrm{Sym}^{\mathbf{n}}_{/k}(\mathscr{C})$.

Now one may remark that there is a commutative diagram of group morphisms

$$\begin{array}{ccc} \mathscr{E}xp\mathscr{M}_{L(D_\alpha)^n \times \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} & \longrightarrow & \mathscr{E}xp\mathscr{M}_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \\ \downarrow & & \downarrow \\ \mathscr{E}xp\mathscr{M}_{L(D_\alpha)^n} & \longrightarrow & \mathscr{E}xp\mathscr{M}_k \end{array}$$

which means that it is possible to permute the sums as follows:

$$\sum_{D \in \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \sum_{x \in \kappa(D)(\mathscr{C})^n} \Phi_D(x) = \sum_{x \in k(\mathscr{C})^n} \sum_{D \in \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \Phi_D(x).$$

Proposition 2.2.3 (Poisson formula in families). *If Φ is an uniformly smooth family of Schwartz-Bruhat functions, then its Fourier transform $\mathscr{F}\Phi$ is uniformly compactly supported and one has*

$$\sum_{D \in \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \sum_{x \in \kappa(D)(\mathscr{C})^n} \Phi_D(x) = \mathbf{L}^{(1-g)n} \sum_{y \in k(\mathscr{C})^n} \sum_{D \in \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \mathscr{F}\Phi_D(y).$$

in $\mathscr{E}xp\mathscr{M}_k$.

2.3. Models and moduli spaces of sections

2.3.1. Global models and degrees. Let $\mathscr{C}$ be a smooth projective and geometrically integral curve over a field k, with function field F, and let V be a smooth F-variety. As in the introduction, a *model of V over $\mathscr{C}$* is a separated, faithfully flat and finite type $\mathscr{C}$-scheme $\mathscr{V}$ whose generic fibre is isomorphic to V.

Example 2.3.1. Let $V \hookrightarrow \mathbf{P}^N_F$ be an embedding of V in some projective space. Take $\mathscr{V}$ to be the Zariski closure of V in $\mathbf{P}^N_{\mathscr{C}}$. Then the composition $\mathscr{V} \to \mathbf{P}^N_{\mathscr{C}} \to \mathscr{C}$ is a projective model of V.

Remark 2.3.2. If $\pi : \mathscr{V} \to \mathscr{C}$ is a proper model, the functors $\pi_!$ and π_* from the category of sheaves on $\mathscr{V}$ to the ones on $\mathscr{C}$ coincide [Mil80, Chap. 6, §3]. Since $\mathscr{C}$ is integral, by [Kle05, Theorem 4.18.2] there exists a nonempty Zariski open subset $\mathscr{C}' \subset \mathscr{C}$ such that the Picard scheme $\mathrm{Pic}_{\mathscr{V}_{\mathscr{C}'}/\mathscr{C}'}$ representing the Picard functor $\mathbf{Pic}_{(\mathscr{V}_{\mathscr{C}'}/\mathscr{C}')_{(\mathrm{fppf})}}$ exists and is a disjoint union of open quasi-projective subschemes. Here we recall that $\mathbf{Pic}_{(X/S)_{(\mathrm{fppf})}}$ is the sheaf associated to the functor

$$(T/S) \mapsto \mathrm{Pic}(X \times_S T)/\mathrm{Pic}(T)$$

in the fppf (faithfully flat of finite type) topology, given a separated map of finite type $X \to S$ between locally Noetherian schemes [Kle05, Definition 2.2].

Moreover, we assume that π has (local) sections, so that the Picard functor $\mathbf{Pic}_{(\mathscr{V}_{\mathscr{C}}/\mathscr{C})_{(\mathrm{fppf})}}$ is actually

$$\mathbf{Pic}_{\mathscr{V}/\mathscr{C}}(T) = H^0(T, \mathcal{R}^1\pi_*(\mathbf{G}_m))$$

for the Zariski topology [BLR90, p.204].

Remark 2.3.3. Assume there exists a closed point $p_0 \in \mathscr{C}$ such that $H^1(\mathscr{V}_{p_0}, \mathscr{O}_{\mathscr{V}_{p_0}}) = H^2(\mathscr{V}_{p_0}, \mathscr{O}_{\mathscr{V}_{p_0}}) = 0$. By the Semicontinuity Theorem [Gro63, (7.7.5-I)], there exists an open neighborhood $\mathscr{C}''$ of p_0 such that $H^1(\mathscr{V}_p, \mathscr{O}_{\mathscr{V}_p}) = H^2(\mathscr{V}_p, \mathscr{O}_{\mathscr{V}_p}) = 0$ for all $p \in \mathscr{C}''$.

This shows in particular that $(R^1\pi_!\mathscr{O}_{\mathscr{V}})_{|\mathscr{C}''} = (R^2\pi_!\mathscr{O}_{\mathscr{V}})_{|\mathscr{C}''} = 0$. By flat base-change [**Sta21**, Lemma 02KH] applied to the generic fibre,

$$\begin{array}{ccc} V & \longrightarrow & \mathscr{V} \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ \mathrm{Spec}(F) & \longrightarrow & \mathscr{C} \end{array}$$

we have $(R^i\pi_!\mathscr{O}_{\mathscr{V}})_\eta = H^i(V,\mathscr{O}_V)$ for all i and in particular $H^1(V,\mathscr{O}_V) = H^2(V,\mathscr{O}_V) = 0$. Conversely, if $H^1(V,\mathscr{O}_V)$ and $H^2(V,\mathscr{O}_V)$ are both trivial, then there exists a non-empty open subset of $\mathscr{C}$ above which $R^1\pi_!\mathscr{O}_{\mathscr{V}}$ and $R^2\pi_!\mathscr{O}_{\mathscr{V}}$ are both trivial. This argument actually shows that the assumptions on the first and second cohomology groups of the structure sheaf of a Fano-like variety, in Définition A, can be done indifferently with respect to V or to a special fibre of $\mathscr{V}$.

Moreover, by [**Kle05**, Proposition 5.19], $\mathrm{Pic}_{\mathscr{V}_{\mathscr{C}'}/\mathscr{C}'}$ is smooth over $\mathscr{C}'\cap\mathscr{C}''$ and by [**Kle05**, Corollary 5.13] each fibre $\mathrm{Pic}_{\mathscr{V}_p/\kappa(p)}$ above $p\in\mathscr{C}'\cap\mathscr{C}''$ is discrete, given by $H^2(\mathscr{V}_p,\mathbf{Z})$. From this point of view, $\mathbf{Pic}_{\mathscr{V}_{\mathscr{C}'}/\mathscr{C}'}$ is a constructible sheaf on $\mathscr{C}$ and can be seen as a variation of mixed Hodge structure, see the proof of Proposition 2.4.6 page 62.

Setting 2.3.4. Let $\mathscr{V}\to\mathscr{C}$ be a proper model of a Fano-like F-variety V. We fix a finite set $L_1,\dots,L_r$ of invertible sheaves on V whose linear classes form a basis of the torsion-free $\mathbf{Z}$-module $\mathrm{Pic}(V)$, as well as invertible sheaves $\mathscr{L}_1,\dots,\mathscr{L}_r$ on $\mathscr{V}$ extending respectively $L_1,\dots,L_r$.

Definition 2.3.5 (Multidegree)**.** Let $\mathscr{V}\to\mathscr{C}$ and $\underline{\mathscr{L}} = (\mathscr{L}_1,\dots,\mathscr{L}_r)$ be as in Setting 2.3.4.

A section $\sigma:\mathscr{C}\to\mathscr{V}$ defines an element of the dual $\mathrm{Pic}(V)^\vee$ by sending an effective invertible sheaf of the form

$$\otimes_{i=1}^r L_i^{\otimes\lambda_i}$$

to the linear combination of degree

$$\mathbf{deg}_{\underline{\mathscr{L}}}(\sigma) = \sum_{i=1}^r \lambda_i \deg(\sigma^*\mathscr{L}_i).$$

This element is called the *multidegree of σ with respect to the model $\underline{\mathscr{L}}$*.

2.3.2. Moduli spaces of curves.

Still, let $\mathscr{V}\to\mathscr{C}$ and $\underline{\mathscr{L}} = (\mathscr{L}_1,\dots,\mathscr{L}_r)$ be as in Setting 2.3.4. For every class $\delta\in\mathrm{Pic}(V)^\vee$, we consider the functor

$$\mathbf{Hom}^\delta_{\mathscr{C}}(\mathscr{C},\mathscr{V})$$

sending a k-scheme T to the set of maps $\sigma\in\mathrm{Hom}_{\mathrm{Sch}/T}(\mathscr{C}\times_k T,\mathscr{V}\times_k T)$ such that

$$\begin{aligned} &\pi_T\circ\sigma = \mathrm{id}_{\mathscr{C}\times_k T} \\ &\text{for all } t\in T,\ \mathbf{deg}_{\underline{\mathscr{L}}}(\sigma_t) = \delta. \end{aligned}$$

If U is a dense open subset of the generic fibre V, we define

$$\mathbf{Hom}^\delta_{\mathscr{C}}(\mathscr{C},\mathscr{V})_U$$

to be the subfonctor of $\mathbf{Hom}^\delta_{\mathscr{C}}(\mathscr{C},\mathscr{V})$ sending a k-scheme T to the T-families of maps sending the generic point of $\mathscr{C}$ into $U(F)$.

The moduli space of sections of a proper model $\mathscr{V}\to\mathscr{C}$ is well-defined: adapting the ideas of [**Bou09**, Lemme 4.1] and [**CLL16**, Proposition 2.2.2] we get the following general representability lemma. Here we assume that $\mathscr{V}$ is projective over the base field k.

Lemma 2.3.6. *For all non-empty open subset $U \subset V$ and all class $\delta \in \mathrm{Pic}(V)^\vee$, the functor $\mathbf{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$ is representable by a quasi-projective scheme.*

Proof. Let $\mathscr{L}$ be an ample invertible sheaf on $\mathscr{V}$. For every $d \geqslant 1$, let

$$\mathbf{Hom}^d_k(\mathscr{C}, \mathscr{V})$$

be the functor of morphisms $\varsigma : \mathscr{C} \to \mathscr{V}$ such that $\deg(\varsigma^*\mathscr{L}) = d$ and

$$\mathbf{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})$$

be the functor of sections $\sigma : \mathscr{C} \to \mathscr{V}$ such that $\deg(\sigma^*\mathscr{L}) = d$.

By the existence theorems of Hilbert schemes [Gro60, 4.c], there exists a quasi-projective k-scheme $\mathrm{Hom}^d_k(\mathscr{C}, \mathscr{V})$ representing $\mathbf{Hom}^d_k(\mathscr{C}, \mathscr{V})$. The condition $\pi \circ \sigma = \mathrm{id}_{\mathscr{C}}$ is closed and thus $\mathbf{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})$ is a closed subfunctor of $\mathbf{Hom}^d_k(\mathscr{C}, \mathscr{V})$. Therefore it is represented by a quasi-projective scheme $\mathrm{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})$.

Let U be a non-empty open subset of V and let $\mathrm{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$ be the complement of the closed subscheme of $\mathrm{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})$ defined by the condition $\sigma(\mathscr{C}) \subset |\mathscr{V} \setminus U|$. This open subscheme $\mathrm{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$ parametrises sections $\sigma : \mathscr{C} \to \mathscr{V}$ such that $f(\eta_{\mathscr{C}}) \in U(F)$ and $\deg(\sigma^*(\mathscr{L})) = d$. It is again a quasi-projective scheme, since $\mathrm{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})$ is.

The restriction L of $\mathscr{L}$ to V is isomorphic to a linear combination

$$\otimes_{i=1}^r L_i^{\otimes \lambda_i}$$

of the L_i's. Let $\mathscr{L}'$ be the invertible sheaf

$$\otimes_{i=1}^r \mathscr{L}_i^{\otimes \lambda_i}$$

on $\mathscr{V}$.

Let $\delta \in \mathrm{Pic}(V)^\vee$ be a multidegree and $\sigma : \mathscr{C} \to \mathscr{V}$ a section such that $\mathbf{deg}_{\underline{\mathscr{L}}} = \delta$ and $\sigma(\eta_{\mathscr{C}}) \in U(F)$. Then

$$\begin{aligned}\deg(\sigma^*\mathscr{L}) &= \deg(\sigma^*\mathscr{L}) - \deg(\sigma^*\mathscr{L}') + \deg(\sigma^*\mathscr{L}') \\ &= \deg(\sigma^*\mathscr{L}) - \deg(\sigma^*\mathscr{L}') + \delta \cdot L.\end{aligned}$$

Since the restriction of $\mathscr{L} \otimes (\mathscr{L}')^{-1}$ to the generic fibre is trivial, there exist vertical divisors E and E' such that

$$\mathscr{L} \otimes (\mathscr{L}')^{-1} \simeq \mathscr{O}_{\mathscr{V}}(E' - E).$$

In particular, the difference $\deg(\sigma^*\mathscr{L}) - \deg(\sigma^*\mathscr{L}')$ only takes a finite number of values, since σ has intersection degree one with any fibre of π. Let a be its maximal value. Moreover, by flatness, the r conditions given by $\mathbf{deg}_{\underline{\mathscr{L}}} = \delta$ are open and closed in the Hilbert scheme of $\mathscr{V}$. Therefore $\mathbf{Hom}^\delta_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$ can be identified with an open subfunctor of

$$\coprod_{0 \leqslant d \leqslant a + \delta \cdot L} \mathbf{Hom}^d_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U$$

hence the lemma. $\square$

2.3.3. Greenberg schemes and motivic integrals. A reference for this subsection is given by chapters 4 to 6 of [**CLNS18**].

If R is a complete discrete valuation ring, with field of fractions F and residue field κ, such that F and κ have equal characteristic, then the choice of an uniformizer π of R together with a section of $R \to \kappa$ provides a morphism of κ-algebras

$$\begin{array}{rcl} \kappa[[t]] & \to & R \\ t & \mapsto & \pi \end{array}$$

which is an isomorphism by Theorem 2 of [**Bou83**, Chap. IX, §3].

Example 2.3.7. If p is a closed point of the smooth projective k-curve $\mathscr{C}$, the previous result applies to the completed local ring $R_p = \widehat{\mathscr{O}_{\mathscr{C},p}}$.

Definition 2.3.8 (Greenberg schemes)**.** Let R be a complete discrete valuation ring, with maximal ideal $\mathfrak{m}$ and residue field $\kappa = R/\mathfrak{m}$. Assume that R have equal characteristic and fix a section of $R \to \kappa$.

Let $\mathscr{X}$ be an R-variety. For any non-negative integer m, the Greenberg scheme of order m of $\mathscr{X}$ is the κ-scheme $\mathrm{Gr}_m(\mathscr{X})$ representing the functor

$$A \mapsto \mathrm{Hom}_R(\mathrm{Spec}(R_m \otimes_\kappa A), \mathscr{X})$$

on the category of κ-algebras [**CLNS18**, Chap. 4, §2.1], where $R_m = R/\mathfrak{m}^{m+1}$ for all $m \geqslant 0$. There are canonical affine projection morphisms

$$\theta_m^{m+1} : \mathrm{Gr}_{m+1}(\mathscr{X}) \to \mathrm{Gr}_m(\mathscr{X}),$$

given by truncation, and the Greenberg scheme is the κ-pro-scheme

$$\mathrm{Gr}_\infty(\mathscr{X}) = \varprojlim \mathrm{Gr}_m(\mathscr{X})$$

(or more concisely $\mathrm{Gr}(\mathscr{X})$) which represents the functor

$$A \mapsto \mathrm{Hom}_\kappa(\mathrm{Spec}(A \otimes_\kappa R), X)$$

on the category of κ-algebras. This scheme carries a canonical projection

$$\theta_m^\infty : \mathrm{Gr}_\infty(\mathscr{X}) \to \mathrm{Gr}_m(\mathscr{X})$$

for every non-negative integer m, called the truncation of level m.

Example 2.3.9. If $\mathscr{V} \to \mathscr{C}$ is a model of a projective variety V over $F = k(\mathscr{C})$, let $\mathscr{V}_{R_p}$ be the schematic fibre over the completed local ring $R_p = \widehat{\mathscr{O}_{\mathscr{C},p}}$. If $\mathfrak{m}_p$ is the maximal ideal of R_p, then the $\kappa(p)$-points respectively of $\mathrm{Gr}_m(\mathscr{V}_{R_p})$ and $\mathrm{Gr}(\mathscr{V}_{R_p})$ are in bijection respectively with the sets of points $\mathscr{V}_{R_p}(R_p/\mathfrak{m}_p^{m+1})$ and $\mathscr{V}_{R_p}(R_p)$. Moreover, if $\mathscr{V} \to \mathscr{C}$ is proper, by the valuative criterion of properness the set $\mathscr{V}_{R_p}(R_p)$ is in one-to-one correspondance with the set $V_{F_p}(F_p)$, where F_p is the completion of F at p.

By [**CLNS18**, Chap. 4, Lemma 4.2.2], the constructible subsets of $\mathrm{Gr}(\mathscr{X})$ are exactly the subsets C of the form

$$C = (\theta_m^\infty)^{-1}(C_m)$$

for a certain level m and a constructible subset C_m of $\mathrm{Gr}_m(\mathscr{X})$. Moreover, if C is Zariski-closed, respectively Zariski-open, then C_m can be chosen to be closed, respectively open. Then, a map

$$f : C \to \mathbf{Z} \cup \{\infty\}$$

on a constructible subset C of $\mathrm{Gr}(\mathscr{X})$ is said to be constructible if $f^{-1}(n)$ is constructible for every $n \in \mathbf{Z}$.

By [**CLNS18**, Chap. 6, §2], there is an additive motivic measure

$$\mu_{\mathscr{X}} : \mathrm{Cons}_{\mathrm{Gr}(\mathscr{X})} \to \widehat{\mathscr{M}_{\mathscr{X}_0}}^{\dim}$$

(where $\mathscr{X}_0$ is the special fibre of $\mathscr{X}$) called the *motivic volume* or *motivic density*, which extends to a countably additive motivic measure $\mu^*_{\mathscr{X}}$ on a class $\mathrm{Cons}^*_{\mathrm{Gr}(\mathscr{X})}$ of measurable subsets of $\mathrm{Gr}(\mathscr{X})$, see [**CLNS18**, Chap. 6, §3]. For example, if $\mathscr{X}$ is smooth of pure relative dimension d over R, then

$$\mu_{\mathscr{X}}(\mathrm{Gr}(\mathscr{X})) = [\mathscr{X}_0]\mathbf{L}^{-d}.$$

If A is a measurable subset of $\mathrm{Gr}(\mathscr{X})$ and $f : A \to \mathbf{Z} \cup \{\infty\}$ has measurable fibres (by this we mean that $f^{-1}(n)$ is measurable for all $n \in \mathbf{Z}$) such that the series

$$\sum_{n\in\mathbf{Z}} \mu^*_{\mathscr{X}}(f^{-1}(n))\mathbf{L}^{-n}$$

is convergent in $\widehat{\mathscr{M}_{\mathscr{X}_0}}^{\dim}$, then the motivic integral of $\mathbf{L}^{-f}$

$$\int_A \mathbf{L}^{-f} \mathrm{d}\mu^*_{\mathscr{X}} = \sum_{n\in\mathbf{Z}} \mu^*_{\mathscr{X}}(f^{-1}(n))\mathbf{L}^{-n}$$

is well-defined.

Remark 2.3.10. By the quasi-compactness of the constructible topology, a constructible function f which does not reach infinity is bounded and thus only takes a finite number of values. In particular, if f is bounded constructible, then f is measurable and $\mathbf{L}^{-f}$ is integrable, see Example 4.1.3 in [**CLNS18**, Chap. 6].

Definition 2.3.11 (§3.1 of [**CLNS18**, Chap. 5])**.** Let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism of R-schemes of finite type and pure relative dimension d. Let R' be an extension of R and $q \in \mathscr{Y}(R')$. Consider the morphism

$$\alpha(q) : \left((f\circ q)^*\Omega^d_{\mathscr{X}/R}\right)/(\text{torsion}) \to \left(q^*\Omega^d_{\mathscr{Y}/R}\right)/(\text{torsion})$$

of free R'-modules of finite rank induced by f.

The *order of the Jacobian of f along q* is defined by

$$\mathrm{ordjac}_f(q) = \mathrm{length}_{R'} \operatorname{coker}(\alpha(q)).$$

This provides a function

$$\mathrm{ordjac}_f : \mathrm{Gr}(\mathscr{Y}) \to \mathbf{N}.$$

Note that by Proposition 3.1.4 of [**CLNS18**, Chap. 5], if $\mathscr{Y}$ is smooth over R and the restriction of f to generic fibres is étale, then ordjac_f is constructible and bounded.

Proposition 2.3.12 (Smooth change of variable)**.** *Let $\mathscr{X}$ and $\mathscr{Y}$ be two R-schemes of finite type and pure relative dimension d, with singular loci respectively $\mathscr{X}_{sing}$ and $\mathscr{Y}_{sing}$. Let $f : \mathscr{Y} \to \mathscr{X}$ be a morphism of R-schemes. Let A and B be constructible subsets respectively of $\mathrm{Gr}(\mathscr{X}) - \mathrm{Gr}(\mathscr{X}_{sing})$ and $\mathrm{Gr}(\mathscr{Y}) - \mathrm{Gr}(\mathscr{Y}_{sing})$. Assume that f induces a bijection*

$$B(\kappa') \to A(\kappa')$$

for every extension κ' of κ, and that $B \cap \mathrm{ordjac}_f^{-1}(+\infty)$ is empty.

Let $\alpha : A \to \mathbf{Z}$ be a constructible function on A. Then the function

$$\beta : y \in B \longmapsto (\alpha \circ \mathrm{Gr}(f))(y) + \mathrm{ordjac}_f(y)$$

is constructible and

$$\int_A \mathbf{L}^{-\alpha} \mathrm{d}\mu_{\mathscr{X}} = f_{0!} \int_B \mathbf{L}^{-\beta} \mathrm{d}\mu_{\mathscr{Y}}$$

in $\mathscr{M}_{\mathscr{X}_0}$.

Proof. See Theorem 1.2.5 in [**CLNS18**, Chap. 6]. □

Remark 2.3.13. Here the symbol $\mathbf{L}^{-\operatorname{ordjac}_f}$ plays the role of the absolute value of the determinant $\left|\det\left(\left(\frac{\partial x_i}{\partial y_j}\right)_{i,j}\right)\right|$ in the usual change-of-variable formula.

2.3.4. Local intersection degrees. Let L be an invertible sheaf on V. A coherent sheaf on $\mathscr{V}$ whose restriction to V is isomorphic to L is call a *model* of V. In this paragraph we define local intersection degrees on L of a section $\mathscr{C} \to \mathscr{V}$ at a closed point $p \in \mathscr{C}$, with respect to a model of L, and study the difference between the degrees given by two different models of L. This is a reformulation, in the framework of Greenberg schemes and motivic integration [**CLNS18**], of the p-adic and adelic metrics of [**Pey12**, §1.2].

Let us first remind the local definition of such metrics. Note that in order to define functions at the level of Greenberg schemes, it is enough to define them on R'-points, for any extension R' of ramification index one over the complete valuation ring R, by [**CLNS18**, Chap. 4, (3.3.7)].

Definition 2.3.14. Let R be a complete discrete valuation ring with fraction field K. Let $\mathscr{X}$ be an R-scheme of pure dimension, X its generic fibre, L an invertible sheaf on X and $\mathscr{L}$ a model of L on $\mathscr{X}$ (not necessarily invertible).

For any extension R' of R, of ramification index one over R, with uniformizer ϖ and field of fractions K', we define order functions on $\mathscr{X}(R')$ as follows: let $\tilde{q} : \mathrm{Spec}(R') \to \mathscr{X}$ be an R'-point of $\mathscr{X}$, q its restriction to the generic fibre and y a point of the K'-vector space $q^*L = (\tilde{q}^*\mathscr{L}) \otimes_{R'} K'$. Then we set, if $q^*y \neq 0$,

$$\mathrm{ord}_{\tilde{q}}(y) = \max\{n \in \mathbf{Z} \mid \varpi^{-n} y \in \tilde{q}^*\mathscr{L}/(\mathrm{torsion})\}$$

(the unique integer ℓ such that $\varpi^{-\ell} y$ is a generator of the R'-lattice $\tilde{q}^*\mathscr{L}/(\mathrm{torsion})$), and

$$\mathrm{ord}_{\tilde{q}}(y) = \infty$$

if $q^*y = 0$.

In other words, $\mathrm{ord}_{\tilde{q}}(y)$ is the valuation of y with respect to the lattice $\tilde{q}^*\mathscr{L}/(\mathrm{torsion})$: given a generator y_0 of $\tilde{q}^*\mathscr{L}/(\mathrm{torsion})$,

$$\mathrm{ord}_{\tilde{q}}(y) = v_{R'}(y/y_0) \in \mathbf{Z} \cup \{\infty\}$$

where $v_{R'}$ is the normalized discrete valuation extended to K'. In particular, this definition does not depend on the choice of the uniformizer ϖ nor on the choice of the generator y_0.

If s is a section of L on an open set containing q, then

$$\mathrm{ord} \circ s : \mathscr{X}(R') \to \mathbf{Z} \cup \{\infty\}$$

is the function sending $\tilde{q} \in \mathscr{X}(R')$ to $\mathrm{ord}_{\tilde{q}}(s(q))$.

Remark 2.3.15. We are going to apply the previous definition as follows. Let $\mathscr{V}$ be a proper model of a smooth F-variety V. We fix a closed point $p \in \mathscr{C}$ and work with the R_p-scheme $\mathscr{V}_{R_p}$, identifying the sets $V(F_p)$ and $\mathscr{V}(R_p)$ by properness. We take L to be an invertible sheaf on V and $\mathscr{L}$ a model of L on $\mathscr{V}$.

If q is a F_p-point of V, the function $\mathrm{ord}_{\tilde{q}}$ is well-defined on $L_q(F_p)$. Of course, any F_p-point of L belongs to the fibre of a unique F_p-point of V, so that this function extends to

$$\begin{array}{rcl} L(F_p) & \to & \mathbf{Z} \cup \{\infty\} \\ y & \mapsto & \mathrm{ord}_{\tilde{q}}(y) \text{ whenever } y \in L_q. \end{array}$$

If $s \in \Gamma(U, L)$ is a section of L above an open subset $U \subset V$, by composition one gets a map

$$\begin{array}{rcl} \mathrm{ord}_p \circ s : U(F_p) & \to & \mathbf{Z} \cup \{\infty\} \\ x & \mapsto & \mathrm{ord}_{\tilde{x}}(s(x)). \end{array}$$

If $\mathscr{D}$ is a Cartier divisor on $\mathscr{V}$ given by a rational section of $\mathscr{L}$ and s is the restriction of this section to V, defined on an open subset $U \subset V$, then $\mathrm{ord}_p(s(x))$ coincides with the intersection number $(x, \mathscr{D})_p = \deg(\tilde{x}^*\mathscr{D})$ for all $x \in U(F_p)$ not in $\mathscr{D}$. Besides, both take an infinite value whenever x lies in $\mathscr{D}$.

By the product formula, for all $x \in V(F)$ the (finite) sum on closed points

$$\sum_{p \in |\mathscr{C}|} \mathrm{ord}_p(s(x)) \in \mathbf{Z}$$

does not depend on the choice of a local section s of L such that $s(x) \neq 0$, and if $\sigma : \mathscr{C} \to \mathscr{V}$ is the section of the proper model $\mathscr{V}$ given by the point $x \in V(F)$, then

$$\deg(\sigma^*\mathscr{L}) = \sum_{p \in |\mathscr{C}|} \mathrm{ord}_p(s(x)).$$

We go back to the notations of Definition 2.3.14.

Lemma 2.3.16. *Let $\mathscr{X}$ and $\mathscr{L}$, $\mathscr{L}'$ be two models of L, and ord, ord' the corresponding order functions given by Definition 2.3.14. For all $\tilde{x} \in \mathscr{X}(R')$, the difference*

$$y \mapsto \mathrm{ord}'_{\tilde{x}}(y) - \mathrm{ord}_{\tilde{x}}(y)$$

is constant on the stalks of L and induces a function $\varepsilon_{\mathscr{L}'-\mathscr{L}}$ on $\mathrm{Gr}_\infty(\mathscr{X})$ with values in $\mathbf{Z}$.

Proof. The difference of the two corresponding valuations is

$$\mathrm{ord}_y(\tilde{x})' - \mathrm{ord}_y(\tilde{x}) = v_{R'}(y/y_0') - v_{R'}(y/y_0) = v_{R'}(y_0/y_0')$$

for every $\tilde{x} \in \mathscr{X}(R')$ and $y \in x^*L$, where y_0 and y_0' are generators respectively of $\tilde{x}^*\mathscr{L}$ and $\tilde{x}^*\mathscr{L}'$ in x^*L. Consequently, this difference induces a map

$$\tilde{x} \in \mathscr{X}(R') \mapsto v_{R'}(y_0/y_0')$$

which does not depend on the choices of the generators y_0 and y_0' of $\tilde{x}^*\mathscr{L}$ and $\tilde{x}^*\mathscr{L}'$, since the quotient of two generators have valuation zero. □

Remark 2.3.17. Note that if $\mathscr{L}$, $\mathscr{L}'$ and $\mathscr{L}''$ are three models of L, we have the relation

$$\varepsilon_{\mathscr{L}''-\mathscr{L}} = \varepsilon_{\mathscr{L}''-\mathscr{L}'} + \varepsilon_{\mathscr{L}'-\mathscr{L}}.$$

Remark 2.3.18. If $\mathscr{I}$ is a coherent sheaf of ideals on $\mathscr{X}$, an order function

$$\mathrm{ord}_{\mathscr{I}} : \mathrm{Gr}_\infty(\mathscr{X}) \to \mathbf{N} \cup \{+\infty\}$$

can be obtained by taking

$$\mathrm{ord}_{\mathscr{I}}(\tilde{x}) = \inf_{f \in \mathscr{I}_{\tilde{x}}} v_{R'}(f(\tilde{x}))$$

for all point $\tilde{x} \in \mathscr{X}(R')$, where $v_{R'}(f(\tilde{x})) = v_{R'}(\tilde{x}^* f)$, see (4.4.3) of [**CLNS18**, Chap. 4]. By Corollary 4.4.8 of [**CLNS18**, Chap. 4] this defines a constructible function $\mathrm{Gr}_\infty(\mathscr{X}) \to \mathbf{N} \cup \{+\infty\}$.

The affine local description of this function [**CLNS18**, Chap. 4, Example 4.4.4] shows that $\mathrm{ord}_{\mathscr{I}}(\tilde{x})$ is given by the smallest $v_{R'}(f(\tilde{x}))$ for f belonging to a finite set of generators of the ideal corresponding to $\mathscr{I}$. In particular, if $\mathscr{I}$ and $\mathscr{I}'$ are two coherent sheaves of ideals, with local generators respectively $y_0 \in \mathscr{I}_{\tilde{x}}$ and $y_0' \in \mathscr{I}'_{\tilde{x}}$, and whose restrictions I and I' to X are invertible and isomorphic, then $\mathrm{ord}_{\mathscr{I}}(\tilde{x}) = v_{R'}(\tilde{x}^* y_0)$ and for all $y \in I_x \simeq I'_x$

$$\begin{aligned}
\mathrm{ord}_{\tilde{x}}(y)' - \mathrm{ord}_{\tilde{x}}(y) &= v_{R'}(y/(1/y_0')) - v_{R'}(y/(1/y_0)) = v_{R'}(y_0'/y_0) \\
&= v_{R'}(\tilde{x}^* y_0') - v_{R'}(\tilde{x}^* y_0) \\
&= \mathrm{ord}_{\mathscr{I}'}(\tilde{x}) - \mathrm{ord}_{\mathscr{I}}(\tilde{x}).
\end{aligned}$$

The $(1/y_0)$ here comes from the fact that the ideal sheaf associated to an effective Cartier divisor D on X is $\mathscr{O}_X(-D)$.

Lemma 2.3.19. *The difference $\varepsilon_{\mathscr{L}'-\mathscr{L}}$ is a contructible function on $\mathrm{Gr}_\infty(\mathscr{X})$. By the quasi-compactness of the constructible topology, it takes only finitely many values.*

Proof. Let $n \in \mathbf{Z}$. Assume first that $n \geqslant 0$. Then $\varepsilon_{\mathscr{L}'-\mathscr{L}}(\tilde{x}) > n$ if and only if y_0/y_0' belongs to the n-th power of the maximal ideal of R, if and only its class in R_n is zero.

We claim that there exists $n_0 \in \mathbf{Z}$ such that $\varepsilon_{\mathscr{L}'-\mathscr{L}}(\tilde{x}) \geqslant n_0$ for all $\tilde{x} \in \mathscr{X}(R')$. We choose generators y_0 and y_0' of $\tilde{x}^*\mathscr{L}$ and $\tilde{x}^*\mathscr{L}'$. Then the rational section $\frac{y_0}{y_0'} \in K'$ of $(\mathscr{L}')^\vee \otimes \mathscr{L}$ has a vertical divisor of poles E which is the pull-back of a formal multiple of the closed point of R', and $(\mathscr{L}')^\vee \otimes \mathscr{L} \otimes \mathscr{O}_{\mathscr{X}}(-E)$ is effective. Let z_0 a generator of E, then $z_0\frac{y_0}{y_0'} \in R'$ and

$$\begin{aligned}
\varepsilon_{\mathscr{L}'-\mathscr{L}}(\tilde{x}) = v_{R'}(y_0/y_0') = v_{R'}\left(\frac{y_0 z_0}{y_0'} \cdot \frac{1}{z_0}\right) &= v_{R'}\left(\frac{y_0 z_0}{y_0'}\right) - v_{R'}(z_0) \\
&\geqslant -v_{R'}(z_0).
\end{aligned}$$

We take $n_0 = -v_{R'}(z_0)$ so that

$$\varepsilon_{\mathscr{L}'-\mathscr{L}}(\tilde{x}) = v_{R'}(\varpi^{-n_0} y_0'/y_0) + n_0$$

for all $\tilde{x}$. Then for a given $n \geqslant n_0$, one has $\varepsilon_{\mathscr{L}'-\mathscr{L}}(\tilde{x}) > n$ if and only if $\varpi^{-n_0} y_0'/y_0$ belongs to the $(n-n_0)$-th power of the maximal ideal of R, if and only if its class in R_{n-n_0} is zero. Thus

$$\{\xi \in \mathrm{Gr}(\mathscr{X}) \mid \varepsilon_{\mathscr{L}'-\mathscr{L}}(\xi) > n\}$$

is constructible of level $\leqslant (n-n_0)$.

Moreover it follows from the definition that $\varepsilon_{\mathscr{L}'-\mathscr{L}}$ does not reach infinity. Thus it takes only a finite number of values by the quasi-compactness of the constructible topology (see for example Theorem 1.2.4 in [**CLNS18**, Appendix A]). ☐

Remark 2.3.20. Note that this difference is trivial if $\mathscr{L}$ and $\mathscr{L}'$ are already isomorphic above R. In particular, if $\mathscr{L}$ and $\mathscr{L}'$ are two different models on $\mathscr{V} \to \mathscr{C}$ of the same L on V, there exists a dense open subset of $\mathscr{C}$ above which they are isomorphic. Its complement S is a finite set of closed points of $\mathscr{C}$ and

$$\deg_{\mathscr{L}'} - \deg_{\mathscr{L}} = \sum_{p \in \mathsf{S}} \varepsilon_{\mathscr{L}'_{R_p} - \mathscr{L}_{R_p}}$$

is bounded.

Definition 2.3.21 (Motivic density associated to a model of the anticanonical sheaf)**.** Let $\mathscr{X}$ be an R-scheme of pure relative dimension n.

Assume that the generic fibre X is smooth over K and take a model $\mathscr{L}_{\mathscr{X}}$ of the anticanonical sheaf ω_X^{-1} over $\mathscr{X}$.

The sheaf $\Omega^1_{\mathscr{X}/R}$ of relative differentials of $\mathscr{X}$ over R [**Har77**, p. 175] is a coherent sheaf of $\mathscr{O}_{\mathscr{X}}$-modules and the dual of its determinant $(\Lambda^n \Omega^1_{\mathscr{X}/R})^\vee$ is a model of ω_X^{-1}.

By Lemma 2.3.19 the local difference function $\varepsilon_{\mathscr{L}_{\mathscr{X}} - (\Lambda^n \Omega^1_{\mathscr{X}/R})^\vee}$ is a constructible function.

Its motivic integral over any measurable subset A avoiding the singular locus of $\mathscr{X}$ will be written

$$\mu^*_{\mathscr{L}_{\mathscr{X}}}(A) = \int_A \mathbf{L}^{-\varepsilon_{\mathscr{L}_{\mathscr{X}} - (\Lambda^n \Omega^1_{\mathscr{X}/R})^\vee}} \mathrm{d}\mu^*_{\mathscr{X}}$$

and $\mu^*_{\mathscr{L}_{\mathscr{X}}}$ will be called *motivic density associated to $\mathscr{L}_{\mathscr{X}}$*.

2.3.5. Weak Néron models and smoothening. In order to prove our result about invariance by change of model (see Theorem 3.2.6 below), we need to collect a few additional definitions and results about weak Néron models and relations between them. References for this subsection are the third chapter of the book of Bosch, Lütkebohmert and Raynaud [**BLR90**], together with the reminder of §7.1 in [**CLNS18**, Chap. 7] as well as §3.4 in [**CLNS18**, Chap. 3].

2.3.5.1. *Local models.* We keep the notations of the previous paragraph, except that we do not need to assume that R is complete.

Definition 2.3.22. Let X be a K-variety.

A model for X is a flat separated R-scheme of finite type $\mathscr{X}$ together with an isomorphism $\mathscr{X}_K \to X$.

A *weak Néron model* [1] of X is a model of X such that $\mathscr{X}$ is smooth over R and every K'-point of X extends to an R'-point of $\mathscr{X}$, for every unramified extension R' of R with fraction field K'. Since by definition $\mathscr{X}$ is separated, such an R'-point is unique.

Let $\mathscr{Y}$ be a flat separated R-scheme of finite type with smooth generic fibre. A *Néron smoothening* of $\mathscr{Y}$ is a smooth R-scheme $\mathscr{X}$ of finite type together with an R-morphism $\mathscr{X} \to \mathscr{Y}$ inducing an isomorphism $\mathscr{X}_K \to \mathscr{Y}_K$ and such that $\mathscr{X}(R') \to \mathscr{Y}(R')$ is bijective for every unramified extension R' of R.

Given $\mathscr{X}$ and $\mathscr{X}'$ two weak Néron models of X, a morphism of weak Néron models is a R-morphism $\mathscr{X}' \to \mathscr{X}$ whose restriction to the generic fibre commutes with the isomorphisms with X. In that case we say that $\mathscr{X}'$ dominates $\mathscr{X}$.

1. We adopt the terminology used by Chambert-Loir, Nicaise and Sebag in [**CLNS18**]. For the comparison of this definition of weak Néron models with the one given by Bosch, Lütkebohmert and Raynaud in [**BLR90**], see Remark 7.1.6 in [**CLNS18**, Chap. 7]. The main difference is a properness assumption.

Then a short reformulation of Theorem 3 and Corollary 4 of [**BLR90**, p. 61] is the following fundamental fact.

THEOREM 2.3.23. *Every R-scheme of finite type whose generic fibre is smooth over K admits a Néron smoothening, given by the R-smooth locus of a composition of admissible blow-ups.*

Even if the result is formulated in the language of formal schemes, the proof of Proposition 3.4.7 in [**CLNS18**, Chap. 3], which is a variant of the Néron smoothening algorithm of [**BLR90**], gives the following useful proposition.

Proposition 2.3.24. *Let X be a smooth K-variety. If $\mathscr{X}'$ and $\mathscr{X}''$ are two models of X, then there exists another model $\mathscr{X}''$ of X above $\mathscr{X}$ and $\mathscr{X}'$ whose R-smooth locus $(\mathscr{X}'')^\circ$ is a Néron smoothening of both models.*

$$
\begin{tikzcd}
 & (\mathscr{X}'')^\circ \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & \mathscr{X}'' \arrow[dl] \arrow[dr] \arrow[dd] & \\
\mathscr{X} \arrow[dr] & & \mathscr{X}' \arrow[dl] \\
 & \operatorname{Spec}(R) &
\end{tikzcd}
$$

In the equal characteristic case, we have the following correspondance of points of Greenberg schemes, see Proposition 3.5.1 of [**CLNS18**, Chap. 4]. It allows one to apply the motivic change of variable formula, Proposition 2.3.12, to Néron smoothenings.

Proposition 2.3.25. *Let $\mathscr{Y} \to \mathscr{X}$ be a morphism of separated flat R-schemes of finite type, restricting to an immersion on generic fibres and such that $\mathscr{Y}$ is smooth.*

Then $\mathscr{Y} \to \mathscr{X}$ is a Néron smoothening if and only if the induced map

$$\mathrm{Gr}(\mathscr{Y})(\kappa') \to \mathrm{Gr}(\mathscr{X})(\kappa')$$

is a bijection for every separable extension κ' of κ.

Proposition 2.3.26. *Let $\mathscr{X}$ be an R-model of a smooth K-variety X, of pure relative dimension n, and $f : \mathscr{Y} \to \mathscr{X}$ a Néron smoothening of $\mathscr{X}$.*

Then

$$\varepsilon^{\left(\Omega^n_{\mathscr{Y}/R}\right)^\vee - f^*\left(\Omega^n_{\mathscr{X}/R}\right)^\vee} = \mathrm{ordjac}_f$$

on $\mathrm{Gr}(\mathscr{Y})$.

PROOF. This is given by the argument of the chain rule (5.2.2) in [**CLNS18**, Chap. 7] and the fact that in our situation the function ordjac_f coincides with the order function of the Jacobian ideal of f. See Lemma 3.1.3 in [**CLNS18**, Chap. 5] as well. $\square$

2.3.5.2. *From local models to global ones.* We will need the following gluing result, which is a variant of [**BLR90**, p.18, Proposition 1].

Proposition 2.3.27. *Let $\mathscr{V} \to \mathscr{C}$ be a model of V above $\mathscr{C}$. Let $\mathscr{C}_0$ be a dense open subset of $\mathscr{C}$ and $\mathscr{V}'_0, \mathscr{V}'_1, ..., \mathscr{V}'_s$ a finite number of models of V respectively over $\mathscr{C}_0$ and over the local rings of the closed points $p_1, ..., p_s$ not in $\mathscr{C}_0$. Assume that these models dominate respectively the restriction of $\mathscr{V}$ to $\mathscr{C}_0$ and to these local rings. In particular, they induce isomorphisms on generic fibres.*

Then there exists a model $\mathscr{V}' \to \mathscr{C}$ of V extending $\mathscr{V}_0, ..., \mathscr{V}_s$ as well as a $\mathscr{C}$-morphism $\mathscr{V}' \to \mathscr{V}$ extending the local ones. If moreover the local models are smooth, then $\mathscr{V}' \to \mathscr{C}$ is smooth as well.

Proof. Let R_i be the local ring of $\mathscr{C}$ at the point $p_i \in \mathscr{C} \setminus \mathscr{C}_0$ for $i = 1, ..., s$. The morphism $\mathscr{V}'_i \to \mathscr{V}_{R_i}$ uniquely extends to a $\mathscr{C}_i$-morphism for a certain open neighbourhood $\mathscr{C}_i$ of p_i by [**Gro66**, Théorème 8.8.2]. Since $(\mathscr{V}'_i)_F \simeq (\mathscr{V}_{R_i})_F \simeq V$, such a model coincides with $\mathscr{V}'_0 \to \mathscr{C}_0$ above a non-empty open subset $\mathscr{C}'_0 \subset \mathscr{C}_0$. Then, up to removing a finite number of points of $\mathscr{C}'_i$ so that $\mathscr{C}_i \cap (\mathscr{C} - \mathscr{C}'_0) = \{s_i\}$, we can assume that they coincide above $\mathscr{C}_i \cap \mathscr{C}'_0$ and glue them above each $\mathscr{C}_i \cap \mathscr{C}'_0$, obtaining a model extending the starting data and dominating $\mathscr{V} \to \mathscr{C}$. □

For us, a weak Néron model of V above $\mathscr{C}$ will be a smooth $\mathscr{C}$-scheme $\mathscr{V}$ of finite type, together with an isomorphism $\mathscr{V}_F \to V$ and satisfying the following property concerning étale integral points: for any closed point p and any étale local $\mathscr{O}_{\mathscr{C},p}$-algebra R' with field of fractions F', the canonical map $\mathscr{V}(R') \to \mathscr{V}_F(F')$ is surjective (see p.7, Definition 1, as well as the end of p.12, and p.60-61 in [**BLR90**]).

Corollary 2.3.28. *Let $\mathscr{V}$ and $\mathscr{V}'$ be two models of V above $\mathscr{C}$. Then there exists a third model $\mathscr{V}''$ of V above $\mathscr{C}$ whose $\mathscr{C}$-smooth locus is a Néron smoothening of both $\mathscr{V}$ and $\mathscr{V}'$ above $\mathscr{C}$.*

$$\begin{tikzcd} & \mathscr{V}'' \arrow[dl] \arrow[dd] \arrow[dr] & \\ \mathscr{V} \arrow[dr] & & \mathscr{V}' \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

Moreover, if $\mathscr{V}$ and $\mathscr{V}'$ are proper, then $\mathscr{V}''$ can be taken proper as well.

Proof. By spreading-out [**Gro66**, Théorème 8.8.2] applied to the generic point of $\mathscr{C}$, we know the existence of a non-empty open subset $\mathscr{C}' \subset \mathscr{C}$ such that the restrictions of $\mathscr{V}$ and $\mathscr{V}'$ above $\mathscr{C}'$ are isomorphic as $\mathscr{C}'$-schemes.

$$\begin{tikzcd} \mathscr{V}_{\mathscr{C}'} \arrow[rr, "\sim"] \arrow[d] \arrow[dr] & & \mathscr{V}'_{\mathscr{C}'} \arrow[dl] \arrow[d] \\ \mathscr{V} \arrow[dr] & \mathscr{C}' \arrow[d] & \mathscr{V}' \arrow[dl] \\ & \mathscr{C} & \end{tikzcd}$$

Then, for each closed point p in the complement of $\mathscr{C}'$, by Proposition 2.3.24 we can find a weak Néron model which dominates both restrictions of $\mathscr{V}$ and $\mathscr{V}'$ to $\mathrm{Spec}(\mathscr{O}_{\mathscr{C},p})$. One can use Proposition 2.3.27 to glue $\mathscr{V}_{\mathscr{C}'}$ together with these models and get the desired new weak Néron model. These operations preserve properness, by the valuative criterion. $\square$

2.3.6. Piecewise trivial fibrations. In this subsection S is a Noetherian scheme. We recall definitions and properties of (classes in $K_0\mathbf{Var}_S$) of piecewise trivial fibrations, following [**CLNS18**, Chap. 2, §2.3].

Definition 2.3.29. Let F be an S-variety. A *piecewise trivial fibration with fibre F* is a morphism of schemes $f : X \to Y$ between two S-varieties such that there exists a finite partition $(Y_i)_{i\in I}$ of Y into locally closed subsets together with an isomorphism of $(Y_i)_{\mathrm{red}}$-schemes between $(X \times_Y Y_i)_{\mathrm{red}}$ and $(F \times_S Y_i)_{\mathrm{red}}$ for all $i \in I$.

Proposition 2.3.30. *Let F be an S-variety and $f : X \to Y$ a piecewise trivial fibration with fibre F. Then*

$$[X] = [F][Y]$$

in $K_0\mathbf{Var}_S$.

Proof. See Corollary 1.4.9 and Proposition 2.3.3 of [**CLNS18**, Chap. 2]. $\square$

We will make extensive use of the following criterion.

Proposition 2.3.31. *Let F be an S-variety and $f : X \to Y$ a morphism of S-varieties. Then f is a piecewise trivial fibration with fibre F if and only if, for every point $y \in Y$, the $\kappa(y)$-schemes $f^{-1}(y)_{red}$ and $(F \otimes_k \kappa(y))_{red}$ are isomorphic.*

Proof. See Proposition 2.3.4 of [**CLNS18**, Chap. 2] (one proceeds by Noetherian induction and applies [**Gro66**, 8.10.5]). $\square$

2.4. Motivic Euler products

2.4.1. The weight filtration. In this paragraph S is a variety over a subfield k of the field $\mathbf{C}$ of complex numbers. We fix once and for all an embedding of k in $\mathbf{C}$ and consider that S is actually defined over $\mathbf{C}$ by extension of scalars. We briefly recall the construction of a weight filtration on the Grothendieck ring of varieties over S. We use [**PS08**] as a general reference for Mixed Hodge Modules, as well as the summaries of [**Bil23**, Chapter 4] and [**CLNS18**, Chap. 2, §3.1-3.3].

2.4.1.1. *Mixed Hodge modules.* The category $\mathbf{MHM}_S$ of mixed Hodge modules over S was introduced by Saito in [**Sai88**, **Sai90**]. It is an abelian category which provides a cohomological realization of the Grothendieck group $K_0\mathbf{Var}_S$ of S-varieties. Its derived category is endowed with a six-functors formalism *à la Grothendieck*. In case $S = \mathrm{Spec}(\mathbf{C})$ is a point, mixed Hodge modules over S coincides with polarizable Hodge structures as defined by Deligne [**Del71**], see [**PS08**, Lemma 14.8].

The Grothendieck group $K_0(\mathbf{MHM}_S)$ of mixed Hodge modules over S is the quotient of the free abelian group of isomorphism classes of mixed Hodge modules over S by the relations

$$[E] - [F] + [G]$$

whenever there is a split exact sequence

$$0 \to E \to F \to G \to 0$$

for E, F and G objects of $\mathbf{MHM}_S$. There is a notion of weight of a mixed Hodge module, morphisms are strict for the weight filtration and the Grothendieck group $K_0(\mathbf{MHM}_S)$ is generated by the classes of pure Hodge modules.

The tensor product operation in the bounded derived category of $\mathbf{MHM}_S$ provides a multiplicative structure on $K_0(\mathbf{MHM}_S)$ as follows. The Grothendieck group $K_0^{\mathrm{tri}}(D^b(\mathbf{MHM}_S))$ is defined similarly to $K_0(\mathbf{MHM}_S)$ by taking distinguished triangles

$$E^\bullet \to F^\bullet \to G^\bullet \to E^\bullet[1]$$

of complexes as relations, in the place of exact sequences [**Gro77**, Exposé VIII]. By the theorem of decomposition of mixed Hodge modules [**PS08**, Cor. 14.4], there is an isomorphism of groups

$$K_0(\mathbf{MHM}_S) \to K_0^{\mathrm{tri}}(D^b(\mathbf{MHM}_S))$$

sending the class of a mixed Hodge module M to the class of the complex with M in degree zero. Indeed, the inverse is given by the morphism

$$[M^\bullet] \mapsto \sum_{i\in\mathbf{Z}} (-1)^i [\mathcal{H}^i(M^\bullet)]$$

from $K_0^{\mathrm{tri}}(D^b(\mathbf{MHM}_S))$ to $K_0(\mathbf{MHM}_S)$. The tensor product on $D^b(\mathbf{MHM}_S)$ induces a ring structure on $K_0^{\mathrm{tri}}(D^b(\mathbf{MHM}_S))$ and on $K_0(\mathbf{MHM}_S)$ through the previous isomorphism.

The faithful and exact functor

$$\mathrm{rat}_S : \mathbf{MHM}_S \to \mathrm{Perv}_S$$

to perverse sheaves on S sends the six functors formalism of $\mathbf{MHM}_S$ to the one of the bounded derived category of constructible sheaves $D^b_c(S)$. In order to prove an isomorphism between two mixed Hodge modules, it will be enough to check it at the level of perverse sheaves. Indeed, a mixed Hodge module M is given by the data of filtrations on a $\mathcal{D}$-module isomorphic to $\mathbf{C} \otimes_{\mathbf{Q}} \mathrm{rat}_S(M)$ by the Riemann-Hilbert correspondence (comparison isomorphism, [**PS08**, §14.1]). Moreover, the Verdier duality functor $\mathbf{D}_S$ on $D^b_c(S)$ lift to $\mathbf{MHM}_S$ so that $\mathrm{rat}_S \circ \mathbf{D}_S = \mathbf{D}_S \circ \mathrm{rat}_S$.

2.4.1.2. *The Hodge realisation of $K_0\mathbf{Var}_S$.* For every integer d we denote by $\mathbf{Q}_S^{\mathrm{Hdg}}(-d)$ the complex of mixed Hodge modules obtained by pulling back to S the Hodge structure $\mathbf{Q}_{\mathrm{pt}}^{\mathrm{Hdg}}(-d)$ of type (d,d) through the structure morphism $S \to \mathrm{Spec}(\mathbf{C})$. If $p : X \to S$ is an S-variety, let $\mathbf{Q}_X^{\mathrm{Hdg}}$ be the complex $p^*\mathbf{Q}_S^{\mathrm{Hdg}}$ of mixed Hodge modules.

Definition 2.4.1. The *Hodge realisation*

$$\chi_S^{\mathrm{Hdg}} : K_0\mathbf{Var}_S \to K_0\,\mathbf{MHM}_S,$$

sometimes called the *motivic Hodge-Grothendieck characteristic*, sends a class $\left[X \xrightarrow{p} S\right]$ to

$$\left[p_!\mathbf{Q}_X^{\mathrm{Hdg}}\right] = \sum_{i\in\mathbf{Z}} (-1)^i \left[\mathcal{H}^i\left(p_!\mathbf{Q}_X^{\mathrm{Hdg}}\right)\right]$$

where the equality comes from the isomorphism

$$K_0(\mathbf{MHM}_S) \xrightarrow{\sim} K_0^{\mathrm{tri}}(D^b(\mathbf{MHM}_S))$$

described in the previous paragraph.

The perverse realisation $K_0\mathbf{Var}_S \to K_0(D^b(\mathrm{Perv}_S))$ factors through this morphism as $\mathrm{rat}_S \circ \chi_S^{\mathrm{Hdg}}$ [**CLNS18**, Chap. 2, Proposition 3.3.7]. If $S = \mathrm{Spec}(\mathbf{C})$ this is the class, in the Grothendieck group of mixed Hodge structures, of the cohomology with compact support of X and rational coefficients, together with its Hodge structure. This homomorphism is well-defined (see [**PS08**, Lemma 16.61] or Proposition 3.3.7 in [**CLNS18**, Chap. 2] for a proof). Since χ_S^{Hdg} sends $\mathbf{L}_S$ to $\left[\mathbf{Q}_S^{\mathrm{Hdg}}(-1)\right]$, which is invertible in $K_0(\mathbf{MHM}_S)$, it is a morphism of rings compatible with the localisation $K_0\mathbf{Var}_S \to \mathscr{M}_S$.

2.4.1.3. *The weight filtration on $K_0\mathbf{Var}_S$.* In this paragraph we collect definitions and properties from [Bil23, §4.6] and develop a few useful examples about weights.

Definition 2.4.2. The weight function $w_S : \mathscr{M}_S \to \mathbf{Z}$ is given by the composition of χ_S^{Hdg} together with the weight function on mixed Hodge modules.

Proposition 2.4.3. *Let S be a complex variety. The weight function $w_S : \mathscr{M}_S \to \mathbf{Z}$ satisfies the following properties.*

(1) $w_S(0) = -\infty$.

(2) $w_S(\mathfrak{a} + \mathfrak{a}') \leqslant \max(w_S(\mathfrak{a}), w_S(\mathfrak{a}'))$ *with equality if* $w_S(\mathfrak{a}) \neq w_S(\mathfrak{a}')$, *for any* $\mathfrak{a}, \mathfrak{a}' \in \mathscr{M}_S$.

(3) *If $\mathscr{Y} \to S$ is a variety over S then*
$$w_S(\mathscr{Y}) = 2\dim_S(\mathscr{Y}) + \dim(S).$$

For proofs of these properties, see Lemmas 4.5.1.3, 4.6.2.1 and 4.6.3.1 of [Bil23, Chap. 4]. This weight function induces a filtration $(W_{\leqslant n}\mathscr{M}_S)_{n\in\mathbf{Z}}$ on $\mathscr{M}_S$ given by
$$W_{\leqslant n}\mathscr{M}_S = \{\mathfrak{a} \in \mathscr{M}_S \mid w_S(\mathfrak{a}) \leqslant n\}$$
for all $n \in \mathbf{Z}$.

Definition 2.4.4. The *completion of $\mathscr{M}_S$ with respect to the weight topology* is the projective limit
$$\widehat{\mathscr{M}_S}^{w} = \varprojlim \left(\mathscr{M}_S / W_{\leqslant n}\mathscr{M}_S\right).$$

2.4.1.4. *Useful examples and vanishing properties.* Let n be the dimension of the complex variety $S \xrightarrow{a_S} \mathrm{Spec}(\mathbf{C})$. If S is smooth, the only non-trivial graded part of $\mathbf{Q}_S^{\mathrm{Hdg}}$ is the one of degree n, which is the pure Hodge module given by the constant one dimensional variation of Hodge structure on S, with weight zero. Furthermore, we have the relation $a_S^! \cong a_S^*(n)[2n]$, in particular $\mathbf{Q}_S^{\mathrm{Hdg}}(n)[2n] \cong a_S^! \mathbf{Q}_{\mathrm{Spec}(\mathbf{C})}^{\mathrm{Hdg}}$.

The class of $\mathbf{A}_S^d = \mathbf{A}_{\mathbf{C}}^d \times_{\mathbf{C}} S$ is sent by χ_S^{Hdg} to
$$\chi_S^{\mathrm{Hdg}}\left(\mathbf{L}_S^d\right) = \left(\chi_S^{\mathrm{Hdg}}(\mathbf{L}_S)\right)^{\otimes d} = \left(\mathrm{pr}_! \, \mathbf{Q}_{\mathbf{A}_S^1}^{\mathrm{Hdg}}\right)^{\otimes d} = \mathbf{Q}_S^{\mathrm{Hdg}}(-d).$$
More generally, we have the following proposition on top-graded parts.

Proposition 2.4.5 ([Bil23, Lemma 4.6.3.4])**.** *Let S be a smooth and connected complex variety of dimension n. Let $p : \mathscr{Y} \to S$ and $\mathscr{Z} \to S$ be two smooth S-varieties with irreducible fibres of dimension $d \geqslant 0$. Then*
$$w_S([\mathscr{Y}] - [\mathscr{Z}]) \leqslant 2d + n - 1.$$

Proof. Since p is smooth, one has $p^! \simeq p^*(d)[2d]$ and there is a morphism of mixed Hodge modules
$$\mathcal{H}^{2d+n}\left(p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}\right) \to \mathcal{H}^{2d+n}\left(\mathbf{Q}_S^{\mathrm{Hdg}}(-d)[-2d]\right)$$
which induces an isomorphism on the $(2d+n)$-th graded parts [Bil23, Remark 4.1.5.5]. This means that if $\mathscr{Z} \to S$ is another smooth S-variety with irreducible fibres of dimension d, the corresponding top-weight graded parts cancel out and the weight of $\chi_S^{\mathrm{Hdg}}([\mathscr{Y}] - [\mathscr{Z}])$ is at most $2d + n - 1$. □

Proposition 2.4.6. *Let S be a smooth and connected complex variety of dimension n. Let $p : \mathscr{Y} \to S$ be a proper smooth morphism whose fibres are smooth projective varieties of dimension $\dim_S(\mathscr{Y}) = d$. Assume that*

$$\mathcal{R}^1 p_! \mathscr{O}_{\mathscr{Y}} = \mathcal{R}^2 p_! \mathscr{O}_{\mathscr{Y}} = 0$$

and that p has local sections.

Then there exists an open subset $S' \subset S$ above which the relative Picard scheme exists, is smooth with discrete fibres of rank r, and such that the class

$$\begin{aligned}&\chi^{\mathrm{Hdg}}_{S'}(\mathscr{Y}) - \chi^{\mathrm{Hdg}}_{S'}(\mathbf{L}^d_{S'}) - \chi^{\mathrm{Hdg}}_{S'}(r\mathbf{L}^{d-1}_{S'})\\ &= \left[p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}_{|S'}}\right] - \left[\mathbf{Q}^{\mathrm{Hdg}}_{S'}(-d)[2d]\right] - \left[\mathbf{Q}^{\mathrm{Hdg}}_{S'}(-(d-1))[2(d-1)]^{\oplus r}\right]\end{aligned}$$

has S'-weight at most $2d+n-3$.

Proof. Since S is smooth and connected, the complex $\mathbf{Q}^{\mathrm{Hdg}}_S$ is concentrated in degree $n = \dim(S)$ and

$$\mathcal{H}^{2d+n}(\mathbf{Q}^{\mathrm{Hdg}}_S(-d)[-2d])$$

is a pure Hodge module of weight $2d+n$. As complexes, $\mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}}$ and $\mathbf{Z}^{\mathrm{Hdg}}_{\mathscr{Y}}$ are both concentrated in degree $d+n$, which will explain the shift in what follows. The morphism p induces functors $p_! : D^b(\mathscr{Y}) \to D^b(S)$ and p_* between the bounded derived categories of sheaves respectively over $\mathscr{Y}$ and S, compatible with the ones on mixed hodge modules. Since p is proper, p_* and $p_!$ coincide. The first one increases weights while the second one decrease weights; thus $p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}}$, as a complex of mixed Hodge modules, has weight exactly 0.

The exponential exact sequence of sheaves of abelian groups over $\mathscr{Y}$

$$0 \to \mathbf{Z}_{\mathscr{Y}}(1) \to \mathbf{G}_{a,\mathscr{Y}} \to \mathbf{G}_{m,\mathscr{Y}} \to 0$$

gives rise to an exact sequence of cohomology sheaves over S

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{H}^n(p_!\mathbf{Z}_{\mathscr{Y}}(1)) \arrow[r] & \mathcal{H}^n(p_!\mathscr{O}_{\mathscr{Y}}[-n]) \arrow[r] & \mathcal{H}^n(p_!\mathscr{O}^*_{\mathscr{Y}}[-n]) \arrow[dll] \\
& \mathcal{H}^{n+1}(p_!\mathbf{Z}_{\mathscr{Y}}(1)) \arrow[r] & \mathcal{H}^{n+1}(p_!\mathscr{O}_{\mathscr{Y}}[-n]) \arrow[r] & \mathcal{H}^{n+1}(p_!\mathscr{O}^{\times}_{\mathscr{Y}}[-n]) \arrow[dll] \\
& \mathcal{H}^{n+2}(p_!\mathbf{Z}_{\mathscr{Y}}(1)) \arrow[r] & \mathcal{H}^{n+2}(p_!\mathscr{O}_{\mathscr{Y}}[-n]) \arrow[r] & \cdots
\end{tikzcd} \tag{2.4.1.12}$$

which on stalks specialises to the well-known exact sequence of abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{Z}(1) \arrow[r] & \mathbf{C} \arrow[r] & \mathbf{C}^* \arrow[dll, rounded corners] \\
& H^1(\mathscr{Y}_s(\mathbf{C}), \mathbf{Z}(1)) \arrow[r] & H^1(\mathscr{Y}_s, \mathscr{O}_{\mathscr{Y}_s}) \arrow[r] & H^1(\mathscr{Y}_s, \mathscr{O}_{\mathscr{Y}_s}^\times) \arrow[dll, rounded corners] \\
& H^2(\mathscr{Y}_s(\mathbf{C}), \mathbf{Z}(1)) \arrow[r] & H^2(\mathscr{Y}_s, \mathscr{O}_{\mathscr{Y}_s}) \arrow[r] & \cdots
\end{tikzcd}$$

where $\mathbf{Z}(1)$ is the $\mathbf{Z}$-Hodge structure with underlying $\mathbf{Z}$-module $2i\pi\mathbf{Z}$ and Hodge type $(-1,-1)$. Since $\mathcal{H}^{n+i}(p_!\mathscr{O}_{\mathscr{Y}}[n]) = \mathcal{R}^i p_! \mathscr{O}_{\mathscr{Y}} = 0$ for $i = 1, 2$, the map

$$\mathrm{Pic}_{\mathscr{Y}/S} = \mathcal{H}^{n+1}(p_! \mathscr{O}_{\mathscr{Y}}^\times[-n]) \to \mathcal{H}^{n+2}(p_! \mathbf{Z}_{\mathscr{Y}}(1))$$

is an isomorphism. In particular, the map it induces on stalks

$$\mathrm{Pic}(\mathscr{Y}_s) = H^1(\mathscr{Y}_s, \mathscr{O}_{\mathscr{Y}_s}^\times) \to H^2(\mathscr{Y}_s(\mathbf{C}), \mathbf{Z}(1))$$

is an isomorphism. Since we assumed that S and $\mathscr{Y} \to S$ are smooth, this means that

$$\mathrm{Pic}_{\mathscr{Y}/C} \otimes \mathbf{Q}_S^{\mathrm{Hdg}} \simeq \mathcal{H}^{n+2}(p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}(1)) \tag{2.4.1.13}$$

is a variation of Hodge structure, of rank r, above S.

By surjectivity of the exponential map, the arrow $\mathbf{C}^* \to H^1(\mathscr{Y}_s(\mathbf{C}), \mathbf{Z}(1))$ is trivial. Thus $H^1(\mathscr{Y}_s(\mathbf{C}), \mathbf{Z}(1))$ injects into $H^1(\mathscr{Y}_s, \mathscr{O}_{\mathscr{Y}_s})$ which is trivial by assumption, thus $H^1(\mathscr{Y}_s(\mathbf{C}), \mathbf{Z}(1))$ is trivial as well for all $s \in S$, which means that $\mathcal{H}^{n+1}(p_! \mathbf{Z}_{\mathscr{Y}}(1))$ is trivial.

We use the exact involutive dual functor

$$\mathbf{D} : \mathbf{MHM} \to \mathbf{MHM}^{\mathrm{opp}}$$

of Verdier duality on mixed Hodge module, which extends to the derived bounded category $D^b_c(\mathbf{MHM})$, by the formula

$$\mathbf{D}M^\bullet = \mathcal{H}om(M^\bullet, \mathbf{D}\mathbf{Q}^{\mathrm{Hdg}})$$

in $D^b_c(\mathbf{MHM})$, where $\mathbf{D}\mathbf{Q}^{\mathrm{Hdg}}$ is the dualyzing complex. Here since S is smooth,

$$\mathbf{D}_S \mathbf{Q}_S^{\mathrm{Hdg}} \simeq \mathbf{Q}_S^{\mathrm{Hdg}}(n)[2n],$$

see for example [**Sai16**, Appendix A]. It sends mixed Hodge modules of weight w to mixed Hodge modules of weight $-w$, and interchanges p_* and $p_!$. In our situation, it gives

$$\mathbf{D}_S(p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}) = p_*(\mathbf{D}_{\mathscr{Y}} \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}) = p_!(\mathbf{D}_{\mathscr{Y}} \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}) = p_!(\mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}(n+d)[2(n+d)])$$

thus

$$\mathrm{Gr}_i^W \mathcal{H}^j(p_! \mathbf{D}_{\mathscr{Y}} \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}) = \mathrm{Gr}_{i+2(n+d)}^W \mathcal{H}^{j+2(n+d)}(p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}})$$

for all integers i and j (recall that the $(n+d)$th Tate twist translates into a double shift $2(n+d)$ of the weight). On the other hand the decomposition theorem [**PS08**, Corollary 14.4]

$$p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}} \simeq \oplus_k \mathcal{H}^k(p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}})[-k]$$

in $D^b(\mathbf{MHM}_S)$, gives

$$\mathbf{D}_S(p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}) \simeq \oplus_k (\mathbf{D}_S \mathcal{H}^k(p_! \mathbf{Q}_{\mathscr{Y}}^{\mathrm{Hdg}}))[k].$$

We apply [Sai90, Prop. 2.6], which says that

$$\mathbf{D}_S \operatorname{Gr}^W_i M = \operatorname{Gr}^W_{-i} \mathbf{D}_S M$$

for all $M \in \mathbf{MHM}_S$, which gives

$$\begin{aligned}\operatorname{Gr}^W_{i+2(n+d)} \mathcal{H}^{j+2(n+d)}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}}) &= \operatorname{Gr}^W_i \mathcal{H}^j(\mathbf{D}_S(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}}))\\ &\simeq \operatorname{Gr}^W_i \mathbf{D}_S(\mathcal{H}^{-j} p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}})\\ &\simeq \mathbf{D}_S \operatorname{Gr}^W_{-i} \mathcal{H}^{-j}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}})\end{aligned}$$

which, in turn, for $j \in \{-(n+1), -(n+2)\}$ and $i = j$, specializes to

$$\begin{aligned}\operatorname{Gr}^W_{n+2d-1} \mathcal{H}^{n+2d-1}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}}) &\simeq \mathbf{D}_S \operatorname{Gr}^W_{n+1} \mathcal{H}^{n+1}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}})\\ \operatorname{Gr}^W_{n+2(d-1)} \mathcal{H}^{n+2(d-1)}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}}) &\simeq \mathbf{D}_S \operatorname{Gr}^W_{n+2} \mathcal{H}^{n+2}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}})\end{aligned}$$

We previously showed that $\mathcal{H}^{n+1}(p_! \mathbf{Z}^{\mathrm{Hdg}}_{\mathscr{Y}}(1))$ is trivial, we have in particular

$$\mathcal{H}^{n+1}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}}) = 0$$

and the first line is zero. Finally, since by (2.4.1.13)

$$\mathcal{H}^{n+2}(p_! \mathbf{Q}_{\mathscr{Y}}) \simeq \operatorname{Pic}_{\mathscr{Y}/S} \otimes \mathbf{Q}^{\mathrm{Hdg}}_S(-1)$$

is pure, the local rank of $\operatorname{Gr}^W_{n+2(d-1)} \mathcal{H}^{n+2(d-1)}(p_! \mathbf{Q}^{\mathrm{Hdg}}_{\mathscr{Y}})$ is given by the one of $\operatorname{Pic}_{\mathscr{Y}/S'} \otimes \mathbf{Q}^{\mathrm{Hdg}}_{S'}$ above an open subset $S' \subset S$, hence the result. □

2.4.2. Motivic Euler product. Formal motivic Euler products have been introduced by Margaret Bilu [Bil23], as a *notation* generalizing the Kapranov zeta function and *behaving like a product*. For our purpose we will only need a particular case of this construction, but we will state a few useful properties of this object in a general framework. We mostly follow the exposition one can find in the third and sixth section of [BH21].

2.4.2.1. *Symmetric products and configuration spaces.* Let S be a k-variety and X an S-variety. The m-th symmetric product of X relatively to S is by definition the quotient

$$\operatorname{Sym}^m_{/S}(X) = (\underbrace{X \times_S \dots \times_S X}_{m \text{ times}})/\mathfrak{S}_m.$$

Let $\mathscr{X} = (X_i)_{i\in I}$ be a family of quasi-projective varieties above X, where I is an arbitrary set. Let $\mu = (m_i)_{i\in I} \in \mathbf{N}^{(I)}$ be a family of non-negative integers with finite support, which we call a *partition* (if $I = \mathbf{N}^*$, then a partition of a non-negative integer n is a family $(m_i)_{i\geqslant 1}$ such that $\sum_{i\geqslant 1} i m_i = n$).[2] For such a partition, we define

$$\operatorname{Sym}^\mu_{/S}(X) = \prod_{i\in I} \operatorname{Sym}^{m_i}_{/S}(X)$$

as well as

$$\operatorname{Sym}^\mu_{X/S}(\mathscr{X}) = \prod_{i\in I} \operatorname{Sym}^{m_i}_{X/S}(X_i)$$

which is a variety over $\operatorname{Sym}^\mu_{/S}(X)$. These constructions extend to elements of $K_0\mathbf{Var}_X$, using Cauchy products; for details, see for example [BH21, §6.1.1].

2. Note that such partitions admit holes and that this set $\mathbf{N}^{(I)}$ of generalised partitions is denoted by $\mathcal{P}(I)$ in [VW15, How19, BDH22, BH21], while the set of partitions with no hole is written $\mathcal{Q}(I)$. We will adopt the notations $\mathcal{P}$ and $\mathcal{Q}$ only for partitions of integers, that is to say elements of $\mathbf{N}^{(\mathbf{N}^*)}$.

Given a partition $\mu \in \mathbf{N}^{(I)}$, one can construct the restricted μ-th symmetric product

$$\mathrm{Sym}^{\mu}_{X/S}(\mathscr{X})_*$$

as follows. If we write $(\prod_{i\in I} X^{m_i})_{*,X/S}$ for the complement of the diagonal (points having at least two equal coordinates) in $\prod_{i\in I} X^{m_i}$, then the restricted symmetric product

$$\mathrm{Sym}^{\mu}_{/S}(X)_*$$

sometimes abbreviated

$$\mathrm{S}^{\mu}_{/S}(X)_*$$

is by definition the image of $(\prod_{i\in I} X^{m_i})_{*,X/S}$ in $\mathrm{Sym}^{\mu}_{/S} X$. Furthermore, there is a Cartesian diagram

$$\begin{tikzcd} (\prod_{i\in I} X_i^{m_i})_{*,X/S} \arrow[r, hook] \arrow[d] & \prod_{i\in I} X_i^{m_i} \arrow[d] \\ (\prod_{i\in I} X^{m_i})_{*,X/S} \arrow[r, hook] & \prod_{i\in I} X^{m_i} \end{tikzcd}$$

defining an open subset $(\prod_{i\in I} X_i^{m_i})_{*,X/S}$ of points of $\prod_{i\in I} X_i^{m_i}$ mapping to points of $\prod_{i\in I} X^{m_i}$ with pairwise distinct coordinates. Then one defines[3]

$$\mathrm{S}^{\mu}_{X/S}(\mathscr{X})_* = \mathrm{Sym}^{\mu}_{X/S}(\mathscr{X})_* = \left(\prod_{i\in I} X_i^{m_i}\right)_{*,X/S} \Big/ \prod_{i\in I} \mathfrak{S}_{m_i}$$

that is to say, the image of $(\prod_{i\in I} X_i^{m_i})_{*,X/S}$ in $\mathrm{Sym}^{\mu}_{X/S}(\mathscr{X})$.

Example 2.4.7. In the case where I is a singleton, and $\mathscr{X} = (Y \to X)$, then any partition μ is given by a non-negative integer n and $\mathrm{Sym}^{\mu}_{X/S}(\mathscr{X})_* = \mathrm{Sym}^{n}_{X/S}(Y)_*$ is the scheme parametrizing étale zero-cycles of degree n above X, with labels in Y.

Example 2.4.8. If $I = \mathbf{N}^r \setminus \{\mathbf{0}\}$, then for any $\mathbf{n} \in \mathbf{N}^r \setminus \{\mathbf{0}\}$ the disjoint union

$$\mathrm{Sym}^{\mathbf{n}}_{X/S}(\mathscr{X})_* = \coprod_{\substack{\mu=(n_{\mathbf{m}})\in\mathbf{N}^{((\mathbf{N}^r)^*)} \\ \sum_{\mathbf{m}} n_{\mathbf{m}}\mathbf{m}=\mathbf{n}}} \mathrm{Sym}^{\mu}_{X/S}(\mathscr{X})_*$$

parametrizes r-tuples of zero-cycles of degree $\mathbf{n}$ with labels in $\mathscr{X}$.

As well, this construction extends to families of elements of $K_0\mathbf{Var}_X$ and $\mathscr{M}_X$ [BH21, Definition 6.1.7]: if $\mathscr{A} = (\mathfrak{a}_i)_{i\in I}$ is such a family, then

$$\mathrm{Sym}^{\mu}_{X/S}(\mathscr{A}) = \boxtimes_{i\in I} \mathrm{Sym}^{m_i}_{X/S}(\mathfrak{a}_i) \in K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{/S} X}$$

and

$$\mathrm{S}^{\mu}_{X/S}(\mathscr{A})_* = \mathrm{Sym}^{\mu}_{X/S}(\mathscr{A})_* \in K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{/S}(X)_*}$$

is the restriction to $\mathrm{S}^{\mu}_{/S}(X)_* = \mathrm{Sym}^{\mu}_{/S}(X)_* \subset \mathrm{Sym}^{\mu}_{/S}(X)$ of $\mathrm{Sym}^{\mu}_{X/S}(\mathscr{A})$. More generally, if K is a class in $K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{/S} X}$, we will denote by K_* its image in $K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{/S}(X)_*}$ by the restriction morphism.

3. Denoted by $S^{\mu}(\mathscr{X}/S)$ in [Bil23] and $C^{\mu}_{X/S}(\mathscr{X})$ or $\left(\prod_{i\in I} \mathrm{Sym}^{m_i} X_i\right)_{*,X/S}$ in [BH21, BDH22].

2.4.2.2. *Formal and effective motivic Euler products.*

Notation 2.4.9 (Formal motivic Euler product)**.** Let X be a variety over S and $\mathscr{X} = (X_i)_{i\in I}$ be a family of elements of $K_0\mathbf{Var}_X$ or $\mathscr{M}_X$ indexed by a set I. Let $(t_i)_{i\in I}$ be a family of indeterminates. Then the product

$$\prod_{x\in X/S}\left(1+\sum_{i\in I} X_{i,x}t_i\right)$$

is defined as a *notation* for the formal series

$$\sum_{\mu\in\mathbf{N}^{(I)}} [\mathrm{Sym}^{\mu}_{X/S}(\mathscr{X})_*]\mathbf{t}^{\mu}$$

where $\mathbf{t}^{\mu} = \prod_{i\in I} t_i^{m_i}$ whenever $\mu = (m_i)_{i\in I} \in \mathbf{N}^{(I)}$.

Example 2.4.10 (Formal motivic Euler product with one indeterminate over a curve)**.** The simplest kind of motivic Euler products we are going to use is the following. Assume that I is made of a single element. Then a family $\mathscr{X}$ is given by a single class Y and $\mu = [1, ..., 1]$ is the only relevant partition type for a given integer n. In this setting,

$$\prod_{p\in\mathscr{C}} (1+Yt)$$

is the formal series

$$\sum_{m\in\mathbf{N}} [\mathrm{Sym}^{m}_{\mathscr{C}}(Y)_*]t^m$$

where $\mathrm{Sym}^{m}_{\mathscr{C}}(Y)_*$ parametrises étale zero cycles of $\mathscr{C}$, of degree m with labels in Y (whenever Y is a variety).

Proposition 2.4.11 ([**Bil23**, §3.8.1])**.** *The Euler product notation is compatible with the cut-and-past relations: if $X = U \cup Y$ with Y a closed subscheme of X and U its complement, then for any family $\mathscr{X} = (X_i)_{i\in I}$ of elements of $K_0\mathbf{Var}_X$ or $\mathscr{M}_X$*

$$\prod_{x\in X/S}\left(1+\sum_{i\in I} X_{i,x}t_i\right) = \left(\prod_{u\in U/S}\left(1+\sum_{i\in I} X_{i,u}t_i\right)\right)\left(\prod_{y\in Y/S}\left(1+\sum_{i\in I} X_{i,y}t_i\right)\right)$$

when considering the motivic Euler products of the restrictions

$$\mathscr{Y} = (X_i \times_X Y)_{i\in I}$$

and

$$\mathscr{U} = (X_i \times_X U)_{i\in I}.$$

We will need the following generalisation of [**Bil23**, Proposition 3.9.2.4].

Proposition 2.4.12. *We assume that I is of the form $I_0 \setminus \{0\}$ where I_0 is a commutative monoid, and that $(t_i)_{i\in I}$ is a collection of indeterminates, such that*[4]

(1) I_0 is endowed with a total order $<$ such that $p + q = i$, where $q \neq 0$, implies $p < i$;

(2) for all $i \in I$, the set $\{p \in I \mid p < i\}$ is finite;

(3) the collection of indeterminates $(t_i)_{i\in I}$ satisfies $t_p t_q = t_{p+q}$.

4. We refer the reader to the notion of *algèbre large d'un monoïde* in [**Bou07**, Chap. III, p. 27].

Let S be a variety, X a variety over S, $\mathscr{A} = (A_i)_{i\in I}$ and $\mathscr{B} = (B_i)_{i\in I}$ any pair of families of elements of $K_0\mathbf{Var}_X$ or $\mathscr{M}_X$. Then, under the above hypotheses on I and $(t_i)_{i\in I}$,

$$\begin{aligned}&\prod_{x\in X/S}\left(\left(1+\sum_{i\in I}A_{i,x}t_i\right)\left(1+\sum_{i\in I}B_{i,x}t_i\right)\right)\\&=\prod_{x\in X/S}\left(1+\sum_{i\in I}A_{i,x}t_i\right)\prod_{x\in X/S}\left(1+\sum_{i\in I}B_{i,x}t_i\right).\end{aligned}$$

Remark 2.4.13. In [**Bil23**, Proposition 3.9.2.4], the family $\mathscr{A}$ is assumed to be made of *effective* elements. In order to check that the motivic Tamagawa number of a product of two Fano-like varieties is the product of the two motivic Tamagawa numbers, or, more generally, to compute the motivic Tamagawa number of a fibration, we need to drop the effectiveness assumption.

Proof of Proposition 2.4.12. I thank Margaret Bilu for having pointed out that it is actually a direct application of [**Bil23**, Corollaire 3.9.2.5]. One proceeds exactly as in [**Bil23**, p. 89-90], where Proposition 2.4.12 is proved for families of varieties.

Indeed, [**Bil23**, Corollaire 3.9.2.5] tells us that if X' is a variety over X, and $\mathscr{X}' = (X'_i)_{i\in I}$ is a family of classes in $K_0\mathbf{Var}_{X'}$, then

$$\prod_{x\in X/S}\left(\prod_{x'\in X'/X}\left(1+\sum_{i\in I}X'_{i,x'}t_i\right)\right)_x=\prod_{x'\in X'/X}\left(1+\sum_{i\in I}X'_{i,u}t_i\right)$$

(see [**Bil23**, §3.9.1] for the definition of this double-product notation). Taking X' to be the disjoint union of two copies of X, and $\mathscr{A}$, $\mathscr{B}$ to be the restrictions of $\mathscr{X}$ respectively to the first and to the second copy, we get the expected identity. □

Example 2.4.14 ([**VW15**, Proposition 3.7])**.** The Kapranov zeta function of X/S is defined as

$$Z^{\mathrm{Kapr}}_{X/S}(t)=\sum_{m\in\mathbf{N}}\left[\mathrm{Sym}^m_{/S}X\right]t^m.$$

If the characteristic of the base field is zero, it can be rewritten

$$Z^{\mathrm{Kapr}}_{X/S}(t)=\prod_{x\in X/S}\left((1-t)^{-1}\right)$$

and then by Proposition 2.4.12

$$Z^{\mathrm{Kapr}}_{X/S}(t)^{-1}=\prod_{x\in X/S}(1-t)\,.$$

In positive characteristic, this equality only holds in the modified Grothendieck ring $K_0\mathbf{Var}^{\mathrm{uh}}_S$, see [**BH21**, Example 6.1.11].

Notation 2.4.15 (Effective motivic Euler product)**.** Let Y be an element of $K_0\mathbf{Var}_X$ or $\mathscr{M}_X$. The motivic Euler product

$$\prod_{x\in X/S}(1+Y_x)$$

is by definition the series

$$\sum_{m\geqslant 0}\left[\mathrm{Sym}^m_{X/S}(Y)_*\right].$$

In case this series converges in a convenient completion of $K_0\mathbf{Var}_X$ or $\mathscr{M}_X$, its sum is written $\prod_{x\in X/S}(1+Y_x)$ as well.

Remark 2.4.16. Since abstract motivic Euler products are compatible with changes of variable of the form $t'=\mathbf{L}^a t$ [**Bil23**, §3.6.4], this notation can be seen as the specialization of

$$\prod_{x\in X/S}(1+Y_x\mathbf{L}_x^a t)=\sum_{m\geqslant 0}[\mathrm{Sym}^m_{X/S}(Y\times_X\mathbf{A}^a_X)_*]t^m=\sum_{m\geqslant 0}[\mathrm{Sym}^m_{X/S}(Y)_*]\mathbf{L}_S^{ma}t^m$$

at $t=\mathbf{L}_S^{-a}$ for any non-negative integer a.

2.4.2.3. *Specialising products.* The previous multiplicative property specialises.

Proposition 2.4.17. *Assume that $\prod_{x\in X/S}(1+A_x)$ and $\prod_{x\in X/S}(1+B_x)$ converge. Then*

$$\prod_{x\in X/S}(1+A_x)(1+B_x)$$

converges and

$$\prod_{x\in X/S}(1+A_x)\prod_{x\in X/S}(1+B_x)=\prod_{x\in X/S}(1+A_x)(1+B_x).$$

Proof. Let us consider the two formal motivic Euler products

$$P(t)=\prod_{x\in X/S}\left(1+(A_x+B_x+A_xB_x)t^2\right)$$

and

$$P_{1,2}(t_1,t_2)=\prod_{x\in X/S}(1+t_1A_x)(1+t_2B_x)=\prod_{x\in X/S}(1+t_1A_x+t_2B_x+t_1t_2A_xB_x)\,.$$

We have to check that

$$P(1)=P_{1,2}(1,1).$$

We introduce the following intermediate Euler product:

$$\begin{aligned}P_{0,1,2}(t_0,t_1,t_2)&=\prod_{x\in X/S}(1+t_0t_1A_x+t_0t_2B_x+t_1t_2A_xB_x)\\&=\prod_{x\in X/S}(1+t_0(t_1A_x+t_2B_x)+t_1t_2A_xB_x)\,.\end{aligned}$$

Then by Proposition 2.4.12,

$$P_{1,2}(t_1,t_2)=\prod_{x\in X/S}(1+t_1A_x)\prod_{x\in X/S}(1+t_2B_x)$$

and by [**BH21**, Proposition 6.4.5]

$$P_{0,1,2}\,(1,t_1,t_2)=\prod_{x\in X/S}(1+t_1A_x+t_2B_x+t_1t_2A_xB_x)=P_{1,2}(t_1,t_2).$$

Moreover, by [**BH21**, Lemma 6.5.1]

$$P_{0,1,2}(t,t,t)=P(t)$$

Taking $t=1$ everywhere one gets

$$P\,(1)=P_{0,1,2}\,(1,1,1)=P_{1,2}\,(1,1)$$

as expected. □

2.4.2.4. *Convergence criterion with respect to the weight filtration.* In case k is a subfield of $\mathbf{C}$, we have a convergence criterion for motivic Euler products of families over a curve $\mathscr{C}$. It is a particular case of [**Fai22**, Proposition 2.6] which itself is a multi-variable variant of [**Bil23**, Proposition 4.7.2.1].

Proposition 2.4.18. *Fix an integer $r \geqslant 1$ and an r-tuble $\rho \in (\mathbf{N}^*)^r$. For any tuple $\mathbf{m}$ of non-negative integers, we write $\langle \rho, \mathbf{m} \rangle = \sum_{i=1}^r m_i \rho_i$.*

Assume that $\mathscr{X} = (X_{\mathbf{m}})_{\mathbf{m} \in (\mathbf{N}^r)^*}$ is a family of elements of $\mathscr{M}_{\mathscr{C}}$ such that there exist an integer $M \geqslant 0$ and real numbers $\alpha < 1$ and β such that

— *$w_{\mathscr{C}}(X_{\mathbf{m}}) \leqslant 2\langle \rho, \mathbf{m} \rangle - 2$ whenever $1 \leqslant \langle \rho, \mathbf{m} \rangle \leqslant M$;*

— *$w_{\mathscr{C}}(X_{\mathbf{m}}) \leqslant 2\alpha\langle \rho, \mathbf{m} \rangle + \beta - 1$ whenever $\langle \rho, \mathbf{m} \rangle > M$.*

Then there exists $\delta \in \,]0,1]$ and $\delta' > 0$ such that

$$w_{\mathbf{C}}\left(\mathrm{Sym}^{\pi}_{\mathscr{C}/k}(\mathscr{X})_* \cdot \mathfrak{a}_1^{\rho_1 m_1} \cdots \mathfrak{a}_r^{\rho_r m_r}\right) \leqslant -\delta' \langle \rho, \mathbf{m} \rangle$$

for all $\mathbf{m} \in (\mathbf{N}^r)^*$, all partition π of $\mathbf{m}$ and all elements $\mathfrak{a}_1, \ldots, \mathfrak{a}_r \in \mathscr{M}_{\mathbf{C}}$ such that $w(\mathfrak{a}_i) < -2 + \delta - \frac{\beta}{M+1}$ for all $1 \leqslant i \leqslant r$.

If we specialize the previous proposition to the polynomial $F(T) = 1 + Y\mathbf{L}T$ and consider the convergence at $T = \mathbf{L}^{-1}$ we get the following criterion.

Proposition 2.4.19. *Assume that $Y \in \mathscr{M}_{\mathscr{C}}$ is such that $w_{\mathscr{C}}(Y) \leqslant -2$. Then the series $\sum_{m \geqslant 0} [\mathrm{Sym}^m_{\mathscr{C}}(Y)_*]$ converges in $\widehat{\mathscr{M}_k}^w$ and the Euler product*

$$\prod_{p \in \mathscr{C}} (1 + Y_p)$$

is well-defined.

Example 2.4.20. Recall that $w_{\mathscr{C}}\left(\mathbf{L}_{\mathscr{C}}^{-2}\right) = -2\dim_{\mathscr{C}}(\mathbf{A}^2_{\mathscr{C}}) + \dim(\mathscr{C}) = -3$ thus

$$\prod_{p \in \mathscr{C}} \left(1 - \mathbf{L}_p^{-2}\right)$$

converges in $\widehat{\mathscr{M}_k}^w$. Moreover, one can show that this convergence actually holds in $\widehat{\mathscr{M}_k}^{\dim}$ for any k.

Let $p : Y \to \mathscr{C}$ be a smooth $\mathscr{C}$-variety with irreducible fibres of dimension $d \geqslant 0$. Then,

$$\prod_{p \in \mathscr{C}} \left(1 + \left([Y_p] - \mathbf{L}_p^d\right) \mathbf{L}_p^{-(d+1)}\right)$$

converges in $\widehat{\mathscr{M}_k}^w$, since

$$w_{\mathscr{C}}\left([Y] - \mathbf{L}^d_{\mathscr{C}}\right) \leqslant 2d$$

by Proposition 2.4.5, thus $w_{\mathscr{C}}\left(\left([Y] - \mathbf{L}^d_{\mathscr{C}}\right)\mathbf{L}_{\mathscr{C}}^{-(d+1)}\right) \leqslant -2$.

The following little lemma will help us to explicitly control error terms when studying the case of toric varieties. One can replace the dimension by the weight or any other filtration compatible with finite sums.

Lemma 2.4.21. *Let $(\mathfrak{c}_{\mathbf{m}})_{\mathbf{m} \in \mathbf{N}^r}$ be a family of elements of $\mathscr{M}_k$. Assume that there exists a constant $a > 0$ such that*

$$\dim(\mathfrak{c}_{\mathbf{m}}) \leqslant -a|\mathbf{m}|$$

for all $\mathbf{m} \in \mathbf{N}^r$, where $|\mathbf{m}| = \sum_{i=1}^r m_i$.

Then, for all non-empty subset $A \subset \{1, ..., r\}$, and non-negative integer b,

$$\dim \left(\sum_{\mathbf{m}' \leqslant \mathbf{m} - \mathbf{b}} \mathfrak{c}_{\mathbf{m}'} \mathbf{L}^{\mathbf{m}'_A - \mathbf{m}_A} \right) \leqslant -\frac{\min(1, a)}{2} \min_{\alpha \in A}(m_\alpha) - \frac{b}{2} \min(1, 2|A| - a)$$

for all $\mathbf{m} \in \mathbf{N}^r_{\geqslant b}$, where $\mathbf{m}_A$ is the restriction of $\mathbf{m}$ to A and $\mathbf{b} = (b, ..., b)$.

Proof. If $\mathbf{m}'_A \not\leqslant \frac{1}{2}(\mathbf{m}_A - \mathbf{b})$ then the coarse upper bound

$$\dim(\mathfrak{c}_{\mathbf{m}'} \mathbf{L}^{\mathbf{m}'_A - \mathbf{m}_A}) \leqslant -a|\mathbf{m}'| - b|A|$$

for $\mathbf{m}' \leqslant \mathbf{m} - b$ gives

$$\dim(\mathfrak{c}_{\mathbf{m}'} \mathbf{L}^{\mathbf{m}'_A - \mathbf{m}_A}) < -\frac{a}{2} \min_{\alpha \in A}(m_\alpha - b) - b|A| = -\frac{a}{2} \min_{\alpha \in A}(m_\alpha) - \frac{b}{2}(2|A| - a)$$

while if $\mathbf{m}'_A \leqslant \frac{1}{2}(\mathbf{m}_A - \mathbf{b})$ then $\mathbf{m}'_A - \mathbf{m}_A \leqslant -\frac{1}{2}(\mathbf{m}_A + b)$ and

$$\dim(\mathfrak{c}_{\mathbf{m}'} \mathbf{L}^{\mathbf{m}'_A - \mathbf{m}_A}) \leqslant -a|\mathbf{m}'| - \frac{1}{2}|\mathbf{m}_A| - \frac{b}{2} \leqslant -\frac{1}{2} \min_{\alpha \in A}(m_\alpha) - \frac{b}{2}.$$

□

2.4.3. More about convergence of Motivic Euler products. In this section we give an effective criterion, Proposition 2.4.29, which is a refinement of Proposition 4.7.2.1 in [**Bil23**]. It ensures the weight-linear convergence, in the corresponding completed ring, of multivariate motivic Euler products. We also include Lemma 2.4.27 and Lemma 2.4.28, dealing respectively with multiplicativity of weight-linear convergence and what one might call negligible convolution products.

Definition 2.4.22. Let $F(t) = \sum_{i \geqslant 0} X_i t^i$ be a formal power series with coefficients in $\mathscr{E}xp\mathscr{M}_X$. The radius of convergence of F is defined by

$$\sigma_F = \limsup_{i \geqslant 1} \frac{w_X(X_i)}{2i}.$$

We say that F is *converges* for $|t| < \mathbf{L}^{-r}$ if $r \geqslant \sigma_F$.

This terminology can be explained by the fact that if F converges for $|T| < \mathbf{L}^{-r}$ then for all $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ with weight $w(\mathfrak{a}) < -2r$ the evaluation $F(\mathfrak{a})$ exists as an element of $\widehat{\mathscr{E}xp\mathscr{M}}_X$. Indeed, by [**Bil23**, Lemma 4.5.1.3] one has

$$w_X(X_i \boxtimes \mathfrak{a}^i) \leqslant i(2r + w(\mathfrak{a}))$$

for all i sufficiently large, so $F(\mathfrak{a}) = \sum_{i \geqslant 0} X_i \boxtimes \mathfrak{a}^i$ can be thought as an element of $\widehat{\mathscr{E}xp\mathscr{M}}_X$ by considering partial sums with respect to the weight (here $\boxtimes$ denotes the exterior product defined in Section 2.1.3).

In what follows we will say that a formal series $\sum_{i \geqslant 0} \mathfrak{c}_i t^i$ with terms in $\mathscr{E}xp\mathscr{M}_X$ converges *weight-linearly* at $\mathfrak{a} \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ if there exists $\delta > 0$ such that $w_X(\mathfrak{c}_i \mathfrak{a}^i) < -\delta i$ for i sufficiently large. We endow $\mathbf{N}^r$ with its poset structure

$$\mathbf{m}' \leqslant \mathbf{m} \text{ if and only if } m'_\alpha \leqslant m_\alpha \text{ for all } \alpha$$

and extend this definition to families indexed by $\mathbf{N}^r$

Definition 2.4.23. Fix a $\rho \in \mathbf{Q}^r \setminus \{\mathbf{0}\}$. We say that a formal series

$$\sum_{\mathbf{m}\in\mathbf{N}^r} \mathfrak{c}_{\mathbf{m}}\mathbf{t}^{\mathbf{m}}$$

converges *weight-linearly with respect to* ρ, which we may sometimes abbreviate by saying ρ*-weight-linearly*, at $\mathfrak{a} = (\mathfrak{a}_1, ..., \mathfrak{a}_r) \in (\mathscr{E}xp\mathscr{M}_{\mathbf{C}})^r$ if there exists a real number $\delta > 0$ such that

$$w_X(\mathfrak{c}_{\mathbf{m}}\mathfrak{a}^{\mathbf{m}}) < -\delta\langle\rho, \mathbf{m}\rangle.$$

for every $\mathbf{m} \in \mathbf{N}^r$ outside a finite subset.

Remark 2.4.24. This property only depends on the support

$$\mathrm{Supp}(\rho) = \{1 \leqslant \alpha \leqslant r \mid \rho_\alpha > 0\}$$

of ρ. Indeed, if $\rho_1, \rho_2 \in \mathbf{Q}^r$ have same support, then

$$\min_{\alpha\in\mathrm{Supp}(\rho_1)}(\rho_{1,\alpha}/\rho_{2,\alpha})\langle\rho_1, \mathbf{m}\rangle \leqslant \langle\rho_2, \mathbf{m}\rangle \leqslant \max_{\alpha\in\mathrm{Supp}(\rho_1)}(\rho_{1\alpha}/\rho_{2,\alpha})\langle\rho_1, \mathbf{m}\rangle$$

for all $\mathbf{m} \in \mathbf{N}^r$.

In particular, for such a ρ-weight-linearly convergent series we can licitly consider the sum

$$\sum_{\mathbf{m}\in\mathbf{N}^r} \mathfrak{c}_{\mathbf{m}}\mathfrak{a}^{\mathbf{m}}$$

in $\widehat{\mathscr{E}xp\mathscr{M}}_X$ as soon as $\sum_{\mathbf{m}\in\mathbf{N}^r} \mathfrak{c}_{\mathbf{m}}\mathbf{t}^{\mathbf{m}}$ belongs to the subring of formal series with indeterminates t_α for $\alpha \in \mathrm{Supp}(\rho)$.

Then the value of the sum does not depend on the order of summation, in particular the partial sum

$$\sum_{\mathbf{m}'\leqslant\mathbf{m}} \mathfrak{c}_{\mathbf{m}'}\mathfrak{a}^{\mathbf{m}'}$$

has a limit in $\widehat{\mathscr{E}xp\mathscr{M}}_X$ when $\langle\rho, \mathbf{m}\rangle$ tends to infinity.

Definition 2.4.25. Fix a non-zero $\rho \in \mathbf{Q}^r_{\geqslant 0}$. Let $F(\mathbf{t}) = \sum_{\mathbf{m}\in\mathbf{N}^r} X_{\mathbf{m}}\mathbf{t}^{\mathbf{m}}$ be a formal power series with coefficients in $\mathscr{E}xp\mathscr{M}_X$. The ρ-radius of convergence of F is defined by

$$\sigma_F^\rho = \limsup_{\substack{\mathbf{m}\in\mathbf{N}^r\\ \langle\mathbf{m},\rho\rangle\neq 0}} \frac{w_X(X_{\mathbf{m}})}{2\langle\rho, \mathbf{m}\rangle}.$$

We say that F is ρ-convergent for $|\mathbf{t}| < \mathbf{L}^{-\gamma}$ if $\sigma_F^\rho \leqslant \gamma$.

In general, our goal will be to show that $\sigma_F^\rho < 1$. The link between the two terminologies is given by the following straight-forward lemma.

Lemma 2.4.26. *If $F(\mathbf{t})$ is ρ-convergent for $|\mathbf{t}| < \mathbf{L}^{-\gamma}$, then for all $\mathfrak{a}_1, ..., \mathfrak{a}_r \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}$ such that*

$$w_{\mathbf{C}}(\mathfrak{a}_\alpha) < -2\gamma\rho_\alpha \quad \textit{for all } \alpha \in \{1, ..., r\},$$

$F(\mathbf{t})$ converges ρ-weight-linearly at $(\mathfrak{a}_1, ..., \mathfrak{a}_r)$.

In Chapter 4, we will have to deal with weight-linearly convergent series with respect to multiple ρ's having different supports, each of these series depending only on the indeterminates t_α for $\alpha \in \mathrm{Supp}(\rho)$. In this situation, simultaneous convergence for the various ρ's will be obtained by asking $\min_{1\leqslant\alpha\leqslant r}(m_\alpha)$ to be arbitrarily large.

Lemma 2.4.27. *Fix $\rho \in \mathbf{Q}^r_{>0}$ and let $P(\mathbf{t}) = \sum_{\mathbf{m}\in\mathbf{N}^r} \mathfrak{p}_{\mathbf{m}}\mathbf{t}^{\mathbf{m}}$ and $Q(\mathbf{t}) = \sum_{\mathbf{m}\in\mathbf{N}^r} \mathfrak{q}_{\mathbf{m}}\mathbf{t}^{\mathbf{m}}$ be two formal series with coefficients in $\mathscr{E}xp\mathscr{M}_X$. Assume that there exists two positive real numbers δ and δ' such that*

$$w_X(\mathfrak{p}_{\mathbf{m}}) \leqslant (2-\delta)\langle\rho,\mathbf{m}\rangle - \frac{1}{2}\dim(X)$$
$$w_X(\mathfrak{q}_{\mathbf{m}}) \leqslant (2-\delta')\langle\rho,\mathbf{m}\rangle - \frac{1}{2}\dim(X)$$

for all $\mathbf{m} \in \mathbf{N}^r$. Then the product $R(\mathbf{t}) = P(\mathbf{t})Q(\mathbf{t})$ ρ-converges for $|\mathbf{t}| < \mathbf{L}_X^{-1+\frac{\min(\delta,\delta')}{2}}$.

More precisely, if $\mathfrak{r}_{\mathbf{m}}$ is the $\mathbf{m}$-th coefficient of $R(\mathbf{t})$ then

$$w_X(\mathfrak{r}_{\mathbf{m}}) \leqslant (2-\min(\delta,\delta'))\langle\rho,\mathbf{m}\rangle - \dim(X)$$

so that

$$w_X(\mathfrak{r}_{\mathbf{m}}\mathbf{L}_X^{-\langle\rho,\mathbf{m}\rangle}) \leqslant -\min(\delta,\delta')\langle\rho,\mathbf{m}\rangle.$$

Proof. The $\mathbf{m}$-th coefficient of $R(\mathbf{t})$ is given by

$$\mathfrak{r}_{\mathbf{m}} = \sum_{\mathbf{m}'\leqslant\mathbf{m}} \mathfrak{p}_{\mathbf{m}'}\mathfrak{q}_{\mathbf{m}-\mathbf{m}'}.$$

By Proposition 2.4.3

$$w_X(\mathfrak{r}_{\mathbf{m}}) \leqslant \max_{\mathbf{m}'\leqslant\mathbf{m}} \left(w_X\left(\mathfrak{p}_{\mathbf{m}'}\right) + w_X\left(\mathfrak{q}_{\mathbf{m}-\mathbf{m}'}\right)\right)$$

but for all $\mathbf{m}' \leqslant \mathbf{m}$ one has

$$\begin{aligned} w_X\left(\mathfrak{p}_{\mathbf{m}'}\right) + w_X\left(\mathfrak{q}_{\mathbf{m}-\mathbf{m}'}\right) &\leqslant (2-\delta)\langle\rho,\mathbf{m}\rangle - \frac{1}{2}\dim(X) + (2-\delta')\langle\rho,\mathbf{m}\rangle - \frac{1}{2}\dim(X) \\ &\leqslant (2-\min(\delta,\delta'))\langle\rho,\mathbf{m}\rangle - \dim(X) \end{aligned}$$

hence we get the bound for $w_X(\mathfrak{r}_{\mathbf{m}})$.

Since $w_X(\mathbf{L}^{-\langle\rho,\mathbf{m}\rangle}) = 2\langle\rho,\mathbf{m}\rangle + \dim(X)$ we deduce the second bound of the lemma. □

Lemma 2.4.28. *Let $\sum_{\mathbf{m}\in\mathbf{N}^r} \mathfrak{c}_{\mathbf{m}}\mathbf{t}^{\mathbf{m}}$ be a formal series with coefficients in $\mathscr{E}xp\mathscr{M}_X$.*

Assume that there exists $\delta > 0$ such that

$$w_X(\mathfrak{c}_{\mathbf{m}}) \leqslant (2-\delta)\langle\rho,\mathbf{m}\rangle - \dim(X)$$

for all $\mathbf{m} \in \mathbf{N}^r$. Let $b \in \mathbf{Q}^r$ such that

$$B = \{1 \leqslant i \leqslant r \mid \delta\rho_i \leqslant 2b_i\}$$

is non-empty. Then for every $\mathfrak{a} = (\mathfrak{a}_1, ..., \mathfrak{a}_r) \in \mathscr{E}xp\mathscr{M}_X^r$ such that $w_X(\mathfrak{a}_i) \leqslant 2\rho_i$ for all $i \in \{1, ..., r\}$, we have

$$w_X\left(\sum_{\mathbf{m}'\leqslant\mathbf{m}} \mathfrak{c}_{\mathbf{m}'}\mathfrak{a}^{\mathbf{m}'}\mathbf{L}_X^{-\langle b,\mathbf{m}-\mathbf{m}'\rangle}\right) \leqslant -\delta\langle\rho,\mathbf{m}\rangle_B$$

for all $\mathbf{m} \in \mathbf{N}^r$.

PROOF. For every $\mathbf{m}' \leqslant \mathbf{m}$ we have

$$\begin{aligned}
& w_X\left(\mathfrak{c}_{\mathbf{m}'}\mathfrak{a}^{\mathbf{m}'}\mathbf{L}_X^{-\langle b,\mathbf{m}-\mathbf{m}'\rangle}\right) \\
& \leqslant -\delta\langle \mathbf{m}',\rho\rangle - \dim(X) + 2\langle b,\mathbf{m}-\mathbf{m}'\rangle + \dim(X) \\
& = -\delta\langle \mathbf{m},\rho\rangle + \langle \delta\rho - 2b,\mathbf{m}-\mathbf{m}'\rangle \\
& = -\delta\langle \mathbf{m},\rho\rangle_B + \underbrace{\langle \delta\rho - 2b,\mathbf{m}-\mathbf{m}'\rangle_B}_{\leqslant 0} \underbrace{-\delta\langle \mathbf{m},\rho\rangle_{\mathscr{A}\setminus B} + \langle \delta\rho - 2b,\mathbf{m}-\mathbf{m}'\rangle_{\mathscr{A}\setminus B}}_{=-2\langle b,\mathbf{m}\rangle_{\mathscr{A}\setminus B}-\langle \delta\rho-2b,\mathbf{m}'\rangle_{\mathscr{A}\setminus B}\leqslant 0} \\
& \leqslant -\delta\langle \mathbf{m},\rho\rangle_B
\end{aligned}$$

and using Proposition 2.4.3 one gets the claim. □

Now we state and prove a weight-linear convergence criterion for Euler products. This result and its proof are a slight extension of Proposition 4.7.2.1 in [**Bil23**] to the framework of multivariate series.

Proposition 2.4.29. *Fix a complex variety X, an integer $r \geqslant 1$ and an r-tuple $\rho \in \mathbf{Q}^r_{>0}$. Let $\mathscr{X}$ be a family $(X_{\mathbf{m}})_{\mathbf{m}\in\mathbf{N}^r\setminus\{\mathbf{0}\}}$ of elements of $\mathscr{E}xp\mathscr{M}_X$. Assume that there exist a rational number $M \geqslant 0$ and real numbers $\varepsilon > 0$, $\alpha < 1$ and β such that*
— $w_X(X_{\mathbf{m}}) \leqslant \left(\langle \rho,\mathbf{m}\rangle - \frac{1}{2} - \varepsilon\right) w(X)$ *whenever* $1 \leqslant \langle \rho,\mathbf{m}\rangle \leqslant M$;
— $w_X(X_{\mathbf{m}}) \leqslant \left(\alpha\langle \rho,\mathbf{m}\rangle + \beta - \frac{1}{2}\right) w(X)$ *whenever* $\langle \rho,\mathbf{m}\rangle > M$.
Let $\delta \in]0,1[$ given by

$$\delta = \begin{cases} 1 - \max\left(1 - \frac{\varepsilon}{M},\alpha\right) & \text{if } M \neq 0 \\ 1-\alpha & \text{otherwise.}\end{cases}$$

Then

$$w\left(\mathrm{Sym}^{\pi}_{X/\mathbf{C}}(\mathscr{X})_*\right) \leqslant \left(1 - \delta + \frac{\beta}{M+1}\right)\langle \rho,\mathbf{m}\rangle w(X)$$

for all $\mathbf{m} \in \mathbf{N}^r \setminus \{\mathbf{0}\}$ and all partition π of $\mathbf{m}$. In particular, the Euler product

$$\prod_{x\in X}\left(1 + \sum_{\mathbf{m}\in\mathbf{N}^r\setminus\{\mathbf{0}\}} X_{\mathbf{m},x}\mathbf{t}^{\mathbf{m}}\right) \in \mathscr{E}xp\mathscr{M}_{\mathbf{C}}[[T]]$$

ρ-converges for $|\mathbf{t}| < \mathbf{L}^{-\frac{w(X)}{2}\left(1-\delta+\frac{\beta}{M+1}\right)}$ and then takes non-zero values for $|\mathbf{t}| \leqslant \mathbf{L}^{-\frac{w(X)}{2}\left(1-\eta+\frac{\beta}{M+1}\right)}$ for any $0 \leqslant \eta < \delta$.

PROOF. In substance the proof is identical to the one of [**Bil23**, Proposition 4.7.2.1], one just has to adapt it to our notations potentially involving non-integral rational numbers. We fix $\pi = (n_{\mathbf{i}})_{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}}$ a partition of $\mathbf{m}$, that is such that $\sum_{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}} n_{\mathbf{i}}\mathbf{i} = \mathbf{m}$.

Then

$$\begin{aligned}
&w(\mathrm{Sym}^{\pi}_{X/\mathbf{C}}(\mathscr{X})_*)\\
&\leqslant w_{\mathrm{Sym}^{\pi}_{/\mathbf{C}}(X)_*}(\mathrm{Sym}^{\pi}_{X/\mathbf{C}}(\mathscr{X})_*) + \dim(\mathrm{Sym}^{\pi}_{/\mathbf{C}}(X))\\
&\leqslant \sum_{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}} n_{\mathbf{i}}\left(w_X(X_{\mathbf{i}}) + \frac{1}{2}w(X)\right)\\
&\leqslant \sum_{\substack{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}\\ \langle\rho,\mathbf{i}\rangle\leqslant M}} n_{\mathbf{i}}(\langle\rho,\mathbf{i}\rangle-\varepsilon)w(X) + \sum_{\substack{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}\\ \langle\rho,\mathbf{i}\rangle> M}} (\alpha\langle\rho,\mathbf{i}\rangle+\beta)n_{\mathbf{i}}w(X)\\
&\leqslant \sum_{\substack{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}\\ \langle\rho,\mathbf{i}\rangle\leqslant M}} n_{\mathbf{i}}\left(1-\frac{\varepsilon}{M}\right)\langle\rho,\mathbf{i}\rangle w(X) + \sum_{\substack{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}\\ \langle\rho,\mathbf{i}\rangle> M}} \left(\alpha+\frac{\beta}{M+1}\right)n_{\mathbf{i}}\langle\rho,\mathbf{i}\rangle w(X)\\
&= \left(1-\frac{\varepsilon}{M}\right)\sum_{\substack{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}\\ \langle\rho,\mathbf{i}\rangle\leqslant M}} n_{\mathbf{i}}\langle\rho,\mathbf{i}\rangle w(X) + \left(\alpha+\frac{\beta}{M+1}\right)\sum_{\substack{\mathbf{i}\in\mathbf{N}^r\setminus\{\mathbf{0}\}\\ \langle\rho,\mathbf{i}\rangle> M}} n_{\mathbf{i}}\langle\rho,\mathbf{i}\rangle w(X)\\
&\leqslant \left(1-\delta+\frac{\beta}{M+1}\right)\langle\rho,\mathbf{m}\rangle w(X)
\end{aligned}$$

where $\delta\in\,]0,1[$ is given by $1-\delta = \max\left(1-\frac{\varepsilon}{M},\alpha\right)$ if $M\neq 0$ and $\delta = 1-\alpha$ otherwise. This proves the first part of the proposition.

Now remark that if $w(\mathfrak{a}_k) < -w(X)\left(1-\eta_k+\frac{\beta}{M+1}\right)$ for some $0\leqslant\eta_k<\delta$, for all integer k in $\{1,...,r\}$, then the weight of $\mathrm{Sym}^{\pi}_{X/\mathbf{C}}(\mathscr{X})_*\mathfrak{a}_1^{\rho_1 m_1}\cdots\mathfrak{a}_r^{\rho_r m_r}$ is negative for any $\mathbf{m}$ non-zero and any partition of $\mathbf{m}$. This argument shows that the product takes a non-zero value at $\mathfrak{a}$: it is equal to 1 plus some terms of negative weight. $\square$

CHAPTER 3

Motivic Tamagawa measures and equidistribution of curves

Abstract. We start this chapter with showing that the motivic Tamagawa number of a model is well-defined in characteristic zero. Then we formulate a motivic Batyrev-Manin-Peyre principle, as well as a stronger equidistribution principle. We finally show that the equidistribution property does not depend on the choice of models. This is [**Fai23**, §3 and §4].

3.1. Batyrev-Manin-Peyre principle for curves

3.1.1. A convergence lemma in characteristic zero. In this paragraph we assume that k is a subfield of $\mathbf{C}$ and choose once and for all an embedding $k \hookrightarrow \mathbf{C}$, as in Section 2.4.1.

Lemma 3.1.1. *Let $\mathscr{V} \to \mathscr{C}$ be a proper model of a Fano-like variety V. Then for all dense open subsets $\mathscr{C}' \subset \mathscr{C}$, the motivic Euler product*

$$\prod_{p\in\mathscr{C}'}\left(\frac{[\mathscr{V}_p]}{\mathbf{L}_p^{\dim(V)}}\left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rk}(\mathrm{Pic}(V))}\right)$$

converges in the completion of $\mathscr{M}_k$ with respect to the weight filtration.

Proof. By the multiplicative property of motivic Euler products Proposition 2.4.11, it is enough to find some dense open subset $\mathscr{C}' \subset \mathscr{C}$ such that the product above converges. Let r be the Picard rank of V. In $\mathscr{M}_{\mathscr{C}}$, we have

$$\begin{aligned}
&[\mathscr{V}]\mathbf{L}_{\mathscr{C}}^{-\dim_{\mathscr{C}}\mathscr{V}}(1-\mathbf{L}_{\mathscr{C}}^{-1})^r \\
&= \left(1+\left([\mathscr{V}]-\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}\right)\mathbf{L}_{\mathscr{C}}^{-\dim_{\mathscr{C}}(\mathscr{V})}\right)\left(1+\sum_{k=1}^{r}\binom{r}{k}(-1)^k\mathbf{L}_{\mathscr{C}}^{-k}\right) \\
&= 1+\frac{[\mathscr{V}]-\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}-r\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})-1}}{\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}}+\mathscr{R}
\end{aligned}$$

where

$$\mathscr{R} = -r\frac{[\mathscr{V}]-\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}}{\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})+1}}+\frac{[\mathscr{V}]}{\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}}\sum_{k=2}^{r}\binom{r}{k}(-1)^k\mathbf{L}_{\mathscr{C}}^{-k}.$$

By Notation 2.4.15 of the motivic Euler product, we are interested in the series

$$\sum_{m\geqslant 0}[S^m_*(\mathscr{Y})]$$

where

$$\mathscr{Y} = \frac{[\mathscr{V}]-\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}-r\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})-1}}{\mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}}+\mathscr{R} \in \mathscr{M}_{\mathscr{C}}$$

and in order to prove convergence, it is enough to check that $w_{\mathscr{C}}(\mathscr{Y}) \leqslant -2$. Since the motivic Euler product is compatible with finite products, up to replacing $\mathscr{C}$ by a non-empty open subset we can assume that $\mathscr{V} \to \mathscr{C}$ is smooth with irreducible fibres. Then by Proposition 2.4.5 we have a first bound on the weight

$$w_{\mathscr{C}}\left(\left([\mathscr{V}] - \mathbf{L}_{\mathscr{C}}^{\dim_{\mathscr{C}}(\mathscr{V})}\right)\mathbf{L}_{\mathscr{C}}^{-\dim_{\mathscr{C}}(\mathscr{V})-1}\right) \leqslant -2$$

and the expression of $\mathscr{R}$ shows that

$$w_{\mathscr{C}}(\mathscr{R}) \leqslant -2$$

as well. In order to show that $w_{\mathscr{C}}(\mathscr{Y} - \mathscr{R}) \leqslant -2$ we use the fact that V is Fano-like. By Remark 2.3.3, we are in the situation of Proposition 2.4.6: up to restricting to an open subset of $\mathscr{C}$, we have a decomposition

$$\chi_{\mathscr{C}}^{\mathrm{Hdg}}([\mathscr{V}]) = (\chi_{\mathscr{C}}^{\mathrm{Hdg}}(\mathbf{L}_{\mathscr{C}}))^{\dim_{\mathscr{C}}(\mathscr{V})} + r(\chi_{\mathscr{C}}^{\mathrm{Hdg}}(\mathbf{L}_{\mathscr{C}}))^{\dim_{\mathscr{C}}(\mathscr{V})-1} + \mathscr{W} \in K_0(\mathbf{MHM}_{\mathscr{C}})$$

with $w_{\mathscr{C}}(\mathscr{W}) \leqslant 2(\dim_{\mathscr{C}}(\mathscr{V}) - 1)$. Hence $w_{\mathscr{C}}(\mathscr{Y} - \mathscr{R}) \leqslant -2$ and by the property of the weight one has $w_{\mathscr{C}}(\mathscr{Y}) \leqslant -2$. By the convergence criterion of Proposition 2.4.19, this shows that the motivic Euler product we are considering converges in $\widehat{\mathscr{M}_k}^{w}$. □

3.1.2. Motivic Tamagawa number of a model. Let V be a Fano-like F-variety of dimension n and $\mathscr{V}$ a proper model of V. We are now able to give a precise meaning to the motivic analogue of the Tamagawa number.

Up to replacing $\mathscr{V}$ by a dominating model, we can always assume that it is a *good model*, that is to say a proper model of V whose smooth locus $\mathscr{V}^{\circ}$ is a weak Néron model of V, by Theorem 2.3.23 and Proposition 2.3.27.

Definition 3.1.2. The motivic constant $\tau_{\mathscr{L}}(\mathscr{V})$ of the proper model $\mathscr{V} \to \mathscr{C}$ with respect to a model $\mathscr{L}$ of ω_V^{-1} is the effective element of $\widehat{\mathscr{M}_k}^{\dim}$ or $\widehat{\mathscr{M}_k}^{w}$ given by the motivic Euler product

$$\frac{\mathbf{L}_k^{(1-g)\dim(V)}\left[\mathrm{Pic}^0(\mathscr{C})\right]^{\mathrm{rg}(\mathrm{Pic}(V))}}{\left(1-\mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))}} \prod_{p\in\mathscr{C}} \left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))} \mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}(\mathrm{Gr}(\mathscr{V}^{\circ}_{R_p}))$$

where the motivic density $\mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}$ is given by Definition 2.3.21.

In case $\mathscr{V}$ is a constant model of the form $\mathscr{V} = V \times_k \mathscr{C}$, with V a nice variety defined over k, $\mathscr{L} = \mathrm{pr}_1^*(\omega_V^{-1})$ and $\pi = \mathrm{pr}_2$, then the constant $\tau_{\mathscr{L}}(\mathscr{V})$ will be written $\tau(V/\mathscr{C})$. If moreover $\mathscr{C}$ is clear from the context, it will be simply written $\tau(V)$.

Remark 3.1.3. The use of the motivic Euler product notation in this definition is licit. Indeed, there exists a dense open subset $\mathscr{C}'$ of $\mathscr{C}$ on which π is smooth, $\left(\Omega^n_{\mathscr{V}/\mathscr{C}}\right)^{\vee}$ is invertible and isomorphic to $\mathscr{L}$. For points p in $\mathscr{C}'$,

$$\mu^*_{\mathscr{L}}(\mathrm{Gr}(\mathscr{V}^{\circ}_{R_p})) = \mu_{\mathscr{V}_{R_p}}(\mathrm{Gr}(\mathscr{V}_{R_p})) = [\mathscr{V}_{\kappa(p)}]\mathbf{L}^{-\dim(V)}.$$

Then the motivic Euler product

$$\prod_{p\in\mathscr{C}'} \left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))} \mu_{\mathscr{V}_{R_p}}(\mathrm{Gr}(\mathscr{V}_{R_p}))$$

is obtained by applying Notation 2.4.15 to the family

$$\left(1-\mathbf{L}_{\mathscr{C}'}^{-1}\right)^{\mathrm{rk}(\mathrm{Pic}(V))}[\mathscr{V}_{\mathscr{C}'}]\mathbf{L}_{\mathscr{C}'}^{-\dim(V)} - 1 \in \mathscr{M}_{\mathscr{C}'}.$$

Remark 3.1.4. By Lemma 3.1.1, when k is a subfield of $\mathbf{C}$, this class is well-defined in $\widehat{\mathscr{M}_k}^{w}$. When V is a smooth split projective toric variety, the convergence holds in $\widehat{\mathscr{M}_k}^{\dim}$, without any assumption on k (see Theorem 5.1.5).

Remark 3.1.5. The normalisation factor

$$[\mathrm{Pic}^0(\mathscr{C})]\left(1-\mathbf{L}_k^{-1}\right)^{-1}$$

can be interpreted as the residue at $t=\mathbf{L}_k^{-1}$ of the Kapranov motivic zeta function of $\mathscr{C}$

$$((1-\mathbf{L}_k t)Z_{\mathscr{C}}^{\mathrm{Kapr}}(t))(\mathbf{L}_k^{-1}).$$

It is possible to define variants of this constant, related to components of the moduli space.

Definition 3.1.6. Let β be a choice of vertical components E_β of multiplicity one, that is to say, over a finite number of closed points p, the choice of an irreducible component of $\mathscr{V}_p$ of multiplicity one.

If $E^\circ_{\beta_p}=E_{\beta_p}\cap\mathscr{V}^\circ$ for all p, then

$$\tau_{\mathscr{L}}(\mathscr{V})^\beta$$

is the motivic Tamagawa number

$$\frac{\mathbf{L}_k^{(1-g)\dim(V)}\left[\mathrm{Pic}^0(\mathscr{C})\right]^{\mathrm{rg}(\mathrm{Pic}(V))}}{\left(1-\mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))}}\prod_{p\in\mathscr{C}}\left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))}\mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}(\mathrm{Gr}(E^\circ_{\beta_p})).$$

Note that $\tau_{\mathscr{L}}(\mathscr{V})$ equals the finite sum of the $\tau_{\mathscr{L}}(\mathscr{V})^\beta$'s, for β running over the finite set of choices of vertical components E_β above each closed point of $\mathscr{C}$.

Definition 3.1.7. If $\mathscr{C}'$ is a non-empty open subset of $\mathscr{C}$, we will call restriction of $\tau_{\mathscr{L}}(\mathscr{V})^\beta$ to $\mathscr{C}'$, written $\tau_{\mathscr{L}}(\mathscr{V})^\beta_{|\mathscr{C}'}$, the element

$$\mathbf{L}_k^{(1-g)\dim(V)}(\mathrm{res}_{t=\mathbf{L}_k^{-1}}Z_{\mathscr{C}'}(t))^{\mathrm{rg}(\mathrm{Pic}(V))}\prod_{p\in\mathscr{C}'}\left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V))}\mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}(\mathrm{Gr}(E^\circ_{\beta_p})).$$

3.1.3. Motivic Batyrev-Manin-Peyre principle for curves. In order to deal with different good models of a projective variety V over F, we need a refined version of Question 1. The previous tools make the adaptation straightforward.

We use the notations introduced in Setting 2.3.4: we fix a finite set $L_1,\dots,L_r$ of invertible sheaves on V whose linear classes form a $\mathbf{Z}$-basis of $\mathrm{Pic}(V)$, as well as invertible sheaves $\mathscr{L}_1,\dots,\mathscr{L}_r$ on $\mathscr{V}$ extending respectively $L_1,\dots,L_r$.

There exists a unique r-tuple of integers $(\lambda_1,\dots,\lambda_r)$ such that

$$\omega_V^{-1}\simeq\otimes_{i=1}^r L_i^{\otimes\lambda_i}.$$

Consequently, we have a model of ω_V^{-1} given by

$$\mathscr{L}_{\mathscr{V}}=\otimes_{i=1}^r\mathscr{L}_i^{\otimes\lambda_i}. \tag{3.1.3.14}$$

For any choice β of irreducible vertical components of multiplicity one of $\mathscr{V}$, and for every non-empty open subset U of V, the space

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{V},\mathscr{C})^\beta_U$$

parametrizing curves of multidegree $\mathbf{deg}_{\underline{\mathscr{L}}} = \delta$ intersecting the components given by β, exists as a quasi-projective scheme, and the space

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{V},\mathscr{C})_U \tag{2.3.6}$$

is then the finite disjoint union over β of these subspaces.

Recall that $\mathrm{Eff}(V)^{\vee}_{\mathbf{Z}}$ is the intersection of $\mathrm{Eff}(V)^{\vee}$ and $\mathrm{Pic}(V)^{\vee}$ in $\mathrm{Pic}(V)^{\vee} \otimes_{\mathbf{Z}} \mathbf{Q}$.

QUESTION 2 (Relative Geometric Batyrev-Manin-Peyre). *Let $\mathscr{V}$ and $\underline{\mathscr{L}}$ be as in Setting 2.3.4 page 49. Does the symbol*

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{V},\mathscr{C})_U\right] \mathbf{L}_k^{-\delta\cdot\omega_V^{-1}} \in \mathscr{M}_k$$

converge to $\tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V})$ in $\widehat{\mathscr{M}_k}^{w}$ (or even more optimistic, in $\widehat{\mathscr{M}_k}^{\dim}$), as $\delta \in \mathrm{Eff}(V)^{\vee}_{\mathbf{Z}}$ goes arbitrary far from the boundaries of the dual cone $\mathrm{Eff}(V)^{\vee}$?

Remark 3.1.8. We can refine the previous question as follows: given a choice β of vertical components of multiplicity one: does the symbol

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{V},\mathscr{C})_U^{\beta}\right] \mathbf{L}_k^{-\delta\cdot\omega_V^{-1}} \in \mathscr{M}_k$$

converge to $\tau(\mathscr{V})^{\beta}$ in $\widehat{\mathscr{M}_k}^{w}$, or even $\widehat{\mathscr{M}_k}^{\dim}$, as $\delta \in \mathrm{Eff}(V)^{\vee}_{\mathbf{Z}}$ goes arbitrary far from the boundaries of the dual cone $\mathrm{Eff}(V)^{\vee}$?

Example 3.1.9. Starting from previous works by Chambert-Loir-Loeser [**CLL16**] and Bilu [**Bil23**], we show in [**Fai22**] that the conjecture of Remark 3.1.8 is true when V is an equivariant compactification of a vector space and k algebraically closed with characteristic zero.

Example 3.1.10. Bilu and Browning show in [**BB23**] that the answer to Question 2 is positive whenever $\mathscr{C} = \mathbf{P}^1_{\mathbf{C}}$, $V \subset \mathbf{P}^{n-1}_{\mathbf{C}}$ is a hypersurface of degree $d \geqslant 3$ such that $n > 2^d(d-1)$, and $\mathscr{V} = \mathbf{P}^1_{\mathbf{C}} \times_{\mathbf{C}} V$.

3.1.4. Products of Fano-like varieties.

Proposition 3.1.11. *Let V_1 and V_2 be two Fano-like varities over F. Let $\mathscr{V}_1$ (respectively $\mathscr{V}_2$) be a model of V_1 above $\mathscr{C}$ and $\mathscr{L}_1$ be a model of $\omega_{V_1}^{-1}$ (resp. V_2, $\mathscr{L}_2$ and $\omega_{V_2}^{-1}$).*

Then $\mathscr{V}_1 \times_{\mathscr{C}} \mathscr{V}_2$ is a model of $V_1 \times_F V_2$ above $\mathscr{C}$, $(\mathrm{pr}_1^*\mathscr{L}_1) \otimes (\mathrm{pr}_2^*\mathscr{L}_2)$ is a model of $(\mathrm{pr}_1^*\omega_{V_1}^{-1}) \otimes (\mathrm{pr}_2^*\omega_{V_2}^{-1})$ and

$$\tau_{(\mathrm{pr}_1^*\mathscr{L}_1)\otimes(\mathrm{pr}_2^*\mathscr{L}_2)}(\mathscr{V}_1 \times_{\mathscr{C}} \mathscr{V}_2) = \tau_{\mathscr{L}_1}(\mathscr{V}_1)\tau_{\mathscr{L}_2}(\mathscr{V}_2).$$

PROOF. In order to apply Proposition 2.4.17, we have to check that for all closed point $p \in \mathscr{C}$,

$$\mu^*_{(\mathrm{pr}_1^*\mathscr{L}_1)\otimes(\mathrm{pr}_2^*\mathscr{L}_2)|(\mathscr{V}_1\times_{\mathscr{C}}\mathscr{V}_2)_{R_p}}(\mathrm{Gr}((\mathscr{V}_1 \times_{\mathscr{C}} \mathscr{V}_2)^{\circ}_{R_p})) = \mu^*_{\mathscr{L}_1|(\mathscr{V}_1)_{R_p}}(\mathrm{Gr}(\mathscr{V}^{\circ}_{1 R_p}))\mu^*_{\mathscr{L}_2|(\mathscr{V}_2)_{R_p}}(\mathrm{Gr}(\mathscr{V}^{\circ}_{2 R_p})).$$

Going back to Definition 2.3.21, and by functoriality of Greenberg schemes, it is enough to check that on $\mathrm{Gr}((\mathscr{V}_1 \times_{\mathscr{C}} \mathscr{V}_2)^{\circ}_{R_p})$ one has

$$\varepsilon_{(\mathrm{pr}_1^*\mathscr{L}_1)\otimes(\mathrm{pr}_2^*\mathscr{L}_2)-(\Lambda^{n_1+n_2}\Omega^1_{\mathscr{V}_1\times\mathscr{V}_2/R_p})^{\vee}} = \varepsilon_{\mathscr{L}_1-(\Lambda^{n_1}\Omega^1_{\mathscr{V}_1/R_p})^{\vee}} + \varepsilon_{\mathscr{L}_2-(\Lambda^{n_2}\Omega^1_{\mathscr{V}_2/R_p})^{\vee}}. \tag{3.1.4.15}$$

We can assume that $\mathscr{V}_1$ and $\mathscr{V}_2$ are R_p-smooth. Let R'_p be an unramified extension of R_p, $\tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in (\mathscr{V}_1 \times_{R_p} \mathscr{V}_2)(R'_p) \simeq \mathscr{V}_1(R'_p) \times_{\kappa(p)'} \mathscr{V}_2(R'_p)$ and take y_1, y_2, ω_1 and ω_2 to be generators respectively of $\tilde{x}_1^*\mathscr{L}_1$, $\tilde{x}_2^*\mathscr{L}_2$, $\tilde{x}_1^*(\Lambda^{n_1}\Omega^1_{\mathscr{V}_1/R_p})^{\vee}$ and $\tilde{x}_2^*(\Lambda^{n_2}\Omega^1_{\mathscr{V}_2/R_p})^{\vee}$. Then y_1y_2 is a generator of $\tilde{x}^*(\mathrm{pr}_1^* \mathscr{L}_1) \otimes (\mathrm{pr}_2^* \mathscr{L}_2)$ and $\omega_1\omega_2$ a generator of $\tilde{x}^*(\Lambda^{n_1+n_2}\Omega^1_{\mathscr{V}_1\times\mathscr{V}_2/R_p})^\vee$. Now (3.1.4.15) applied to $\tilde{x}$ is the identity $v_{R_p'}\left(\frac{\omega_1\omega_2}{y_1y_2}\right) = v_{R_p'}\left(\frac{\omega_1}{y_1}\right) + v_{R_p'}\left(\frac{\omega_2}{y_2}\right)$. □

3.2. Equidistribution of curves

In this section we assume that

— $\mathscr{V} \to \mathscr{C}$ is a proper model over $\mathscr{C}$ of a Fano-like varieties V together with a model $\underline{\mathscr{L}} = (\mathscr{L}_i)$ of a family of invertible sheaves (L_i) on V whose classes form a $\mathbf{Z}$-basis of $\mathrm{Pic}(V)$ (see Définition A and Setting 2.3.4);

— U is a dense open subset of V;

— the motivic Tamagawa number $\tau_{\mathscr{L}_\mathscr{V}}(\mathscr{V})$, see Definition 3.1.2 and (3.1.3.14), is well-defined in either $\widehat{\mathscr{M}_k}^{\dim}$ or $\widehat{\mathscr{M}_k}^{w}$. The following discussion will not depend on the choice of the filtration.

3.2.1. First approach. Let $\mathscr{S}$ be a zero-dimensional subscheme of the smooth projective curve $\mathscr{C}$, $|\mathscr{S}|$ its set of closed points and $\mathscr{C}'$ the complement of $|\mathscr{S}|$. This subscheme $\mathscr{S}$ is given by a disjoint union of spectrums of the form

$$\mathrm{Spec}\left(\mathscr{O}_{\mathscr{C},p}/(\mathfrak{m}_p^{m_p+1})\right) \simeq \mathrm{Spec}(\kappa(p)[[t]]/t^{m_p+1})$$

for $p \in |\mathscr{S}|$. Its length is

$$\ell(\mathscr{S}) = \sum_{p\in|\mathscr{S}|} (m_p+1)[\kappa(p):k].$$

Then for every $\mathscr{C}$-morphism $\varphi : \mathscr{S} \to \mathscr{V}$ and every $\delta \in \mathrm{Pic}(V)^\vee$ we define

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V} \mid \varphi)_U$$

as being the schematic fibre above φ of the restriction morphism

$$\mathrm{res}^{\mathscr{V}}_{\mathscr{S}} \, \mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V})_U \longrightarrow \mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V}).$$

We assume temporally that $\mathscr{L}_\mathscr{V}$ is isomorphic to $(\Lambda^n\Omega^1_{\mathscr{V}/\mathscr{C}})^\vee$ and that $\mathscr{V} \to \mathscr{C}$ is smooth above an open subset containing the closed points of $\mathscr{S}$. Then we say that there is weak equidistribution for $\mathscr{S}$ if for every $\mathscr{C}$-morphism $\varphi : \mathscr{S} \to \mathscr{V}$, the normalized class

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V} \mid \varphi)_U\right] \mathbf{L}_k^{-\delta\cdot\omega_V^{-1}} \in \mathscr{M}_k$$

converges to

$$\tau(\mathscr{V})_{|\mathscr{C}'} \times \prod_{p\in|\mathscr{S}|} \mathbf{L}_p^{-(m_p+1)\dim(V)} \in \widehat{\mathscr{M}_k}$$

when the multidegree δ tends to infinity – again, by this we will always mean $\delta \in \mathrm{Eff}(V)^\vee_{\mathbf{Z}}$ and $d(\delta, \partial\,\mathrm{Eff}(V)^\vee) \to \infty$. This definition may be seen as a first extension of Peyre's definition [**Pey21**, 5.8] to non-constant families $\mathscr{V} \to \mathscr{C}$.

Remark 3.2.1. $\mathrm{Hom}_{\mathscr{S}}(\mathscr{S},\mathscr{V}_\mathscr{S})$ can be interpreted as the product of spaces of jets

$$\prod_{p\in|\mathscr{S}|} \mathrm{Gr}_{m_p}(\mathscr{V}_{R_p}).$$

Since

$$\mathrm{Gr}_{m_i}(\mathscr{V}_{R_p}) \to \mathrm{Gr}_0(\mathscr{V}_{R_p}) \simeq \mathscr{V}_p$$

is a Zariski locally trivial fibration over $\mathscr{V}_p$ with fibre an affine space of dimension $m_p \dim(V)$, the class of the space of m_p-jets of $\mathscr{V}_{R_p}$ in $K_0\mathbf{Var}_{\mathscr{V}_p}$ is $\mathbf{L}_p^{m_p \dim(V)}[\mathscr{V}_p]$. Finally the class

$$[\mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})] \in K_0\mathbf{Var}_{\prod_{p\in|\mathscr{S}|}\mathscr{V}_p}$$

is send to the finite product

$$\prod_{p\in|\mathscr{S}|} \frac{[\mathscr{V}_p]}{\mathbf{L}_p^{\dim(V)}} \mathbf{L}_p^{(m_p+1)\dim(V)} \in K_0\mathbf{Var}_k.$$

Thus weak equidistribution for $\mathscr{S}$ implies that

$$\mathbf{L}_k^{-\delta\cdot\omega_V^{-1}}\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid\varphi)_U\right][\mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})]$$

tends to $\tau(\mathscr{V})$ when $\delta\to\infty$.

3.2.2. Equidistribution and arcs. Actually, the equidistribution hypothesis can be reformulated in terms of constructible sets of arcs. This reformulation is natural since we already interpreted the local factors of the motivic Tamagawa number as motivic densities of spaces of arcs. In this paragraph S is any finite set of closed points of $\mathscr{C}$. We drop as well the previous assumption on $\mathscr{L}_{\mathscr{V}}$.

The restriction to $\mathrm{Spec}(\widehat{\mathscr{O}_{\mathscr{C},p}})$ provides a morphism

$$\mathrm{res}_{\mathsf{S}}^{\mathscr{V}} : \mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}) \to \prod_{p\in\mathsf{S}} \mathrm{Gr}_{\infty}(\mathscr{V}_{R_p})$$

for every multidegree $\delta\in\mathrm{Eff}(V)^{\vee}_{\mathbf{Z}}$. If $\varphi=(\varphi_p)_{p\in\mathsf{S}}$ is a finite collection of jets such that $\varphi_p\in\mathrm{Gr}_{m_p}(\mathscr{V}_{R_p})$ for every p in S, the schematic fibre of

$$\prod_{p\in\mathsf{S}}\theta^{\infty}_{m_p}\circ\mathrm{res}_{\mathsf{S}}^{\mathscr{V}}$$

above φ is written

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid\varphi)_U.$$

Definition 3.2.2. We say that there is weak esquidistribution above S at level $(m_p)_{p\in\mathsf{S}}$ if for every collection $\varphi=(\varphi_p)_{p\in\mathsf{S}}\in\prod_{p\in\mathsf{S}}\mathrm{Gr}_{m_p}(\mathscr{V}_{R_p})$ of jets above S, the class

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid\varphi)_U\right]\mathbf{L}_k^{-\delta\cdot\omega_V^{-1}}$$

tends to the non-zero effective element

$$\tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V}\mid\varphi)=\tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V})_{|\mathscr{C}\setminus\mathsf{S}}\times\prod_{p\in\mathsf{S}}\mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}((\theta^{\infty}_{m_p})^{-1}(\varphi_p)\cap\mathrm{Gr}(\mathscr{V}^{\circ}_{R_p}))$$

of $\widehat{\mathscr{M}_k}$, when δ becomes arbitrarily large.

This definition is consistant with the previous one since we have the factorisation

$$\mathrm{res}_{\mathscr{S}}^{\mathscr{V}} : \mathrm{Hom}_{\mathscr{C}}^{\mathbf{d}}(\mathscr{C},\mathscr{V}) \xrightarrow{r^{\mathsf{S}}_{\mathscr{V}/\mathscr{C}}} \prod_{p\in\mathsf{S}}\mathrm{Gr}_{\infty}(\mathscr{V}_{R_p}) \longrightarrow \prod_{p\in\mathsf{S}}\mathrm{Gr}_{m_p}(\mathscr{V}_{R_p}) \simeq \mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})$$

for every S-tuple $(m_p)\in\mathbf{N}^{\mathsf{S}}$ and corresponding zero-dimensional subscheme $\mathscr{S}\subset\mathscr{C}$ with support $|\mathscr{S}|=\mathsf{S}$.

More generally, if W is a product $\prod_{p\in \mathsf{S}} W_p$ of constructible subsets W_p of $\mathrm{Gr}_\infty(\mathscr{V}_{R_p})$,

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid W)_U$$

is defined as the schematic fibre of $\mathrm{res}_{\mathsf{S}}^{\mathscr{V}}$ over W. Recall that each constructible set W_p of arcs is nothing else but the preimage by a projection morphism of a certain constructible subset of jets.

Definition 3.2.3. We will say that there is *equidistribution with respect to* $W=\prod_{p\in\mathsf{S}} W_p$ *and the multidegree* $\mathbf{deg}_{\underline{\mathscr{L}}}$, where each W_p is a constructible subset of arcs, if

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid W)_U\right]\mathbf{L}_k^{-\delta\cdot\omega_V^{-1}}$$

tends to

$$\tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V}\mid W)=\tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V})_{|\mathscr{C}\setminus S}\times\prod_{p\in\mathsf{S}}\mu^{*}_{\mathscr{L}_{|\mathscr{V}_{R_p}}}(W_p\cap\mathrm{Gr}(\mathscr{V}^{\circ}_{R_p}))$$

when the multidegree becomes arbitrary large.

We will say that there is *equidistribution of curves, with respect to the multidegree* $\mathbf{deg}_{\underline{\mathscr{L}}}$ if the previous statement holds for every such W.

Remark 3.2.4. Note that the notion of equidistribution of curves is stronger that the motivic Batyrev-Manin-Peyre principle for curves we formulate in Question 2.

Remark 3.2.5. In Definition 3.2.3 one may ask if it would be possible to replace *contructible* by *measurable* to get a more general notion of equidistribution, or consider constructible subsets which are not products over S of constructible sets, as in Theorem 5.1.7, but this higher level of generality would be mostly useless in the present work.

3.2.3. Checking equidistribution pointwise. Let $\mathscr{S}$ be a zero-dimensional subscheme of $\mathscr{C}$. Assume that for every $\delta\in\mathrm{Pic}(V)^\vee$, there exists a k-scheme F_δ (which depends on $\mathscr{S}$) such that

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid\varphi)_U\simeq(F_\delta\otimes\kappa(x))_{\mathrm{red}}$$

for every point $x\in\mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})$ corresponding to a $\mathscr{C}$-morphism $\varphi:\mathscr{S}\to\mathscr{V}$. Then by Proposition 2.3.31, the reduction map

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid\varphi)_U\to\mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})$$

is a piecewise trivial fibration with fibre F_δ and by Proposition 2.3.30, for every constructible subset W of $\mathrm{Hom}_{\mathscr{C}}(\mathscr{S},\mathscr{V})$,

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}}=\delta}(\mathscr{C},\mathscr{V}\mid W)_U\right]=[F_\delta][W]$$

in $K_0\mathbf{Var}_k$. Hence, if one whishes to show that there is equidistribution of curves on $\mathscr{V}$ above $\mathscr{S}$, a strategy is to prove the existence of such an F_δ and then study the convergence of the normalised class $[F_\delta]\mathbf{L}_k^{-\delta\cdot\omega_V^{-1}}$. In general the situation is not as simple, but we use a similar argument in Section 5.1.5 in order to prove Théorème B.

3.2.4. Equidistribution and models. Our goal for the end of this section is to prove the following main result, which does not depend on the choice of the filtration (dimensional or by the weight) on $\mathscr{M}_{\mathscr{C}}$.

THEOREM 3.2.6 (Change of model). *Let $\mathscr{V}$ and $\mathscr{V}'$ be two proper models over $\mathscr{C}$ of the same Fano-like F-variety V, together with models $\underline{\mathscr{L}} = (\mathscr{L}_i)$ and $\underline{\mathscr{L}'} = (\mathscr{L}'_i)$, respectively on $\mathscr{V}$ and $\mathscr{V}'$, of a family (L_i) of invertible sheaves forming a $\mathbf{Z}$-basis of $\mathrm{Pic}(V)$, as in Setting 2.3.4.*

Then there is equidistribution of curves for $(\mathscr{V}, \underline{\mathscr{L}})$, in the sense of Definition 3.2.3, if and only if there is equidistribution of curves for $(\mathscr{V}', \underline{\mathscr{L}'})$.

The remainder of this section is devoted to the proof of Theorem 3.2.6. We take $\mathscr{V}$, $\underline{\mathscr{L}}$, $\mathscr{V}'$ and $\underline{\mathscr{L}'}$ as in Setting 2.3.4. As before, we know the existence of a non-empty open subset $\mathscr{C}' \subset \mathscr{C}$ above which we have an isomorphism of $\mathscr{C}'$-schemes. By Corollary 2.3.28, we can find a proper model $\widetilde{\mathscr{V}} \to \mathscr{C}$ of V whose $\mathscr{C}$-smooth locus is a Néron smoothening of both $\mathscr{V}'$ and $\mathscr{V}$. Above $\mathscr{C}'$, the three models are isomorphic.

$$\begin{tikzcd} & \widetilde{\mathscr{V}} \arrow[dl, "f"'] \arrow[dr, "f'"] \arrow[dd, "\widetilde{\pi}"] & \\ \mathscr{V} \arrow[dr, "\pi"'] & & \mathscr{V}' \arrow[dl, "\pi'"] \\ & \mathscr{C} & \end{tikzcd}$$

This diagram induces morphisms

$$\begin{tikzcd} & \mathrm{Hom}_{\mathscr{C}}\left(\mathscr{C}, \widetilde{\mathscr{V}}\right)_U \arrow[dl, "f_*"'] \arrow[dr, "f'_*"] & \\ \mathrm{Hom}_{\mathscr{C}}(\mathscr{C}, \mathscr{V})_U & & \mathrm{Hom}_{\mathscr{C}}(\mathscr{C}, \mathscr{V}')_U \end{tikzcd}$$

between moduli spaces of sections.

Let $\widetilde{\mathscr{L}_i}$ and $\widetilde{\mathscr{L}'_i}$ be respectively the pull-backs of the sheaves $\mathscr{L}_i$ and $\mathscr{L}'_i$ to $\widetilde{\mathscr{V}}$ for all i. Then both $\widetilde{\mathscr{L}_i}$ and $\widetilde{\mathscr{L}'_i}$ are models of L_i on $\widetilde{\mathscr{V}}$. Up to shrinking $\mathscr{C}'$, we can assume that they are isomorphic above $\mathscr{C}'$. If $\widetilde{\sigma}$ is a section of $\widetilde{\mathscr{V}}$ and $\sigma = f \circ \widetilde{\sigma}$, one has the relation

$$\deg\left((\widetilde{\sigma})^* \widetilde{\mathscr{L}_i}\right) = \deg\left((\widetilde{\sigma})^* f^* \mathscr{L}_i\right) = \deg((f \circ \widetilde{\sigma})^* \mathscr{L}_i) = \deg(\sigma^* \mathscr{L}_i)$$

for all $i \in \{1, ..., r\}$, so that f_* bijectively sends points of

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\widetilde{\mathscr{L}}}} = \delta}\left(\mathscr{C}, \widetilde{\mathscr{V}}\right)_U$$

to points of

$$\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}} = \delta}(\mathscr{C}, \mathscr{V})_U,$$

and similarly for f'_* when considering the multidegrees given by $\underline{\widetilde{\mathscr{L}'}}$ and $\underline{\mathscr{L}'}$. By Proposition 2.3.31 this implies equality of the corresponding classes in $K_0\mathbf{Var}_k$.

We are going to compare the multidegrees $\mathbf{deg}_{\underline{\widetilde{\mathscr{L}}}}$ and $\mathbf{deg}_{\underline{\widetilde{\mathscr{L}'}}}$.

3.2.4.1. *Lifting equidistribution.* As an application of the change-of-variable formula Proposition 2.3.12, we show that equidistribution of curves holds for $(\mathscr{V}, \underline{\mathscr{L}})$ if and only if it holds for $(\widetilde{\mathscr{V}}, \widetilde{\underline{\mathscr{L}}})$.

Let S be the complement of $\mathscr{C}'$ in $\mathscr{C}$.

Lemma 3.2.7. *Let $\widetilde{W}$ be a finite product of constructible subsets $\widetilde{W}_p \subset \mathrm{Gr}(\widetilde{\mathscr{V}}_{R_p})$ for $p \in \mathsf{S}$, and let W be its image by f. Then*

$$\tau_{\widetilde{\mathscr{L}_{\mathscr{V}}}}(\widetilde{\mathscr{V}} \mid \widetilde{W}) = \tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V} \mid W)$$

and

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\widetilde{\underline{\mathscr{L}}}} = \delta}\left(\mathscr{C}, \widetilde{\mathscr{V}} \mid \widetilde{W}\right)_U\right] = \left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}} = \delta}(\mathscr{C}, \mathscr{V} \mid W)_U\right]$$

for every $\delta \in \mathrm{Pic}(V)^\vee$.

In particular, for every $m \in \mathbf{Z}$,

$$\tau_{\widetilde{\mathscr{L}_{\mathscr{V}}}}\left(\widetilde{\mathscr{V}} \mid \widetilde{W}\right) - \left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\widetilde{\underline{\mathscr{L}}}} = \delta}\left(\mathscr{C}, \widetilde{\mathscr{V}} \mid \widetilde{W}\right)_U\right] \mathbf{L}^{-\delta \cdot \omega_V^{-1}} \in \mathcal{F}^m \mathscr{M}_k$$

if and only if

$$\tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V} \mid W) - \left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}} = \delta}(\mathscr{C}, \mathscr{V} \mid W)_U\right] \mathbf{L}^{-\delta \cdot \omega_V^{-1}} \in \mathcal{F}^m \mathscr{M}_k.$$

Proof. Up to shrinking $\mathscr{C}'$ and adding trivial conditions, one can assume that S is contained in the complement of $\mathscr{C}'$. By Theorem 3.2.2 of [**CLNS18**, Chap. 5], the image of $\widetilde{W}_p$ in $\mathrm{Gr}(\mathscr{V}_{R_p})$ is a constructible subset W_p. Then f_* bijectively sends points of $\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\widetilde{\underline{\mathscr{L}}}} = \delta}\left(\mathscr{C}, \widetilde{\mathscr{V}} \mid \widetilde{W}\right)_U$ to points of $\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}} = \delta}(\mathscr{C}, \mathscr{V} \mid W)_U$ and by Proposition 2.3.31,

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\widetilde{\underline{\mathscr{L}}}} = \delta}\left(\mathscr{C}, \widetilde{\mathscr{V}} \mid \widetilde{W}\right)_U\right] = \left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}}} = \delta}(\mathscr{C}, \mathscr{V} \mid W)_U\right]$$

so that the only thing to show is the equality of the motivic Tamagawa numbers.

Up to shrinking $\mathscr{C}'$ again, we only have to show the equality of local factors

$$\mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}\left(W_p \cap \mathrm{Gr}\left(\mathscr{V}^\circ_{R_p}\right)\right) = \mu^*_{\widetilde{\mathscr{L}}_{|\widetilde{\mathscr{V}}_{R_p}}}\left(\widetilde{W}_p \cap \mathrm{Gr}\left(\widetilde{\mathscr{V}}^\circ_{R_p}\right)\right)$$

above closed points $p \in \mathsf{S}$. By assumption V is smooth, both $\widetilde{\mathscr{V}}$ and $\mathscr{V}$ are models of V, thus by Corollary 3.2.4 of [**CLNS18**, Chap. 5] $\mathrm{ordjac}_{f_{R_p}}$ only takes a finite number of values. By the change of variable formula, Proposition 2.3.12, applied to the constructible function

$$\varepsilon_{\mathscr{L}_{\mathscr{V}} - \left(\Lambda^n \Omega^1_{\mathscr{V}/R_p}\right)^\vee} \qquad \text{(see Definition 2.3.21)}$$

one has the following relation between local factors

$$\begin{aligned}
&\mu^*_{\mathscr{L}_{|\mathscr{V}_{R_p}}}\left(W_p \cap \mathrm{Gr}\left(\mathscr{V}^\circ_{R_p}\right)\right)\\
&= \int_{W_p\cap \mathrm{Gr}\left(\mathscr{V}^\circ_{R_p}\right)} \mathbf{L}^{-\varepsilon_{\mathscr{L}_{\mathscr{V}}-(\Lambda^n\Omega^1_{\mathscr{V}/R_p})^\vee}} \mathrm{d}\mu_{\mathscr{V}_{R_p}}\\
&= \int_{W_p\cap \mathrm{Gr}\left(\mathscr{V}^\circ_{R_p}\right)} \mathbf{L}^{-\operatorname{ord}_{\mathscr{L}_{\mathscr{V}}}+\operatorname{ord}_{(\Lambda^n\Omega^1_{\mathscr{V}/R_p})^\vee}} \mathrm{d}\mu_{\mathscr{V}_{R_p}}\\
&= \int_{\widetilde{W}_p\cap \mathrm{Gr}\left(\widetilde{\mathscr{V}}^\circ_{R_p}\right)} \mathbf{L}^{-f^*\operatorname{ord}_{\mathscr{L}_{\mathscr{V}}}+f^*\operatorname{ord}_{(\Lambda^n\Omega^1_{\mathscr{V}/R_p})^\vee}-\operatorname{ordjac}_f} \mathrm{d}\mu_{\widetilde{\mathscr{V}}_{R_p}}\\
&= \int_{\widetilde{W}_p\cap \mathrm{Gr}\left(\widetilde{\mathscr{V}}^\circ_{R_p}\right)} \mathbf{L}^{-\operatorname{ord}_{\widetilde{\mathscr{L}_{\mathscr{V}}}}+\operatorname{ord}_{(\Lambda^n\Omega^1_{\widetilde{\mathscr{V}}/R_p})^\vee}} \mathrm{d}\mu_{\widetilde{\mathscr{V}}_{R_p}}\\
&= \mu^*_{\widetilde{\mathscr{L}}_{|\widetilde{\mathscr{V}}_{R_p}}}\left(\widetilde{W}_p \cap \mathrm{Gr}\left(\widetilde{\mathscr{V}}^\circ_{R_p}\right)\right)
\end{aligned}$$

in $\mathscr{M}_{\mathscr{V}_{R_p}}$, where we used the relations

$$\operatorname{ord}_{\widetilde{\mathscr{L}_{\mathscr{V}}}} - f^*\operatorname{ord}_{\mathscr{L}_{\mathscr{V}}} = 0$$

and

$$f^*\operatorname{ord}_{\left(\Lambda^n\Omega^1_{\mathscr{V}/R_p}\right)^\vee} - \operatorname{ord}_{\left(\Lambda^n\Omega^1_{\widetilde{\mathscr{V}}/R_p}\right)^\vee} = \operatorname{ordjac}_f \qquad \text{(by Proposition 2.3.26)}$$

above the smooth R_p-locus. Taking the product over S, one gets

$$\tau_{\mathscr{L}_{\mathscr{V}}}(\mathscr{V} \mid W) = \tau_{\widetilde{\mathscr{L}_{\mathscr{V}}}}(\widetilde{\mathscr{V}} \mid \widetilde{W})$$

hence the lemma. □

3.2.4.2. *Switching the degree.* The difference of degrees on $\mathrm{Gr}\left(\widetilde{\mathscr{V}}_{R_p}\right)$ is given by the map

$$\varepsilon_{\underline{\widetilde{\mathscr{L}'}}-\underline{\widetilde{\mathscr{L}}}} : \left\{\begin{array}{ccc} \mathrm{Gr}\left(\widetilde{\mathscr{V}}_{R_p}\right) & \longrightarrow & \mathrm{Pic}(V)^\vee \\ x & \mapsto & \left(\otimes_i L_i^{\otimes d_i} \longmapsto \sum_{i=1}^r d_i \varepsilon_{\widetilde{\mathscr{L}'_i}-\widetilde{\mathscr{L}_i}}(x)\right)\end{array}\right.$$

which is trivial for all $p \notin \mathsf{S}$. For any $\varepsilon_p \in \mathrm{Pic}(V)^\vee$, let

$$\widetilde{\mathscr{W}}_p(\varepsilon_p) = \varepsilon^{-1}_{\underline{\widetilde{\mathscr{L}'}}-\underline{\widetilde{\mathscr{L}}}}(\{\varepsilon_p\}).$$

As a direct consequence of Lemma 2.3.19, we have the following.

Lemma 3.2.8. *The map $\varepsilon_{\underline{\widetilde{\mathscr{L}'}}-\underline{\widetilde{\mathscr{L}}}}$ is constructible and there is only a finite number of values of $\varepsilon_p \in \mathrm{Pic}(V)^\vee$ for which $\widetilde{\mathscr{W}}_p(\varepsilon_p)$ is non-empty.*

Now, for every $\varepsilon = (\varepsilon_s) \in (\mathrm{Pic}(V)^\vee)^{\mathsf{S}}$, let

$$\widetilde{\mathscr{W}}(\varepsilon) = \prod_{s\in\mathsf{S}} \widetilde{\mathscr{W}}_s(\varepsilon_s) \subset \prod_{s\in\mathsf{S}} \mathrm{Gr}_\infty\left(\widetilde{\mathscr{V}}_{R_s}\right)$$

and let $\widetilde{W}$ be any finite product $\prod_{s\in\mathsf{S}} \widetilde{W}_s$ of constructible subsets $\widetilde{W}_s \subset \mathrm{Gr}\left(\widetilde{\mathscr{V}}_{R_s}\right)$. Let W, W', W_s and W'_s be the corresponding images by f and f'. By the previous lifting

Lemma 3.2.7,

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\widetilde{\mathscr{L}}}}=\delta}(\mathscr{C},\widetilde{\mathscr{V}}\mid\widetilde{W}\cap\widetilde{\mathscr{W}}(\varepsilon))_U\right]\mathbf{L}^{-\,\delta\,\cdot\,\omega_V^{-1}}\longrightarrow\tau_{\widetilde{\mathscr{L}_{\mathscr{V}}}}\left(\widetilde{\mathscr{V}}\mid\widetilde{W}\cap\widetilde{\mathscr{W}}(\varepsilon)\right)$$

when $\delta\in\mathrm{Eff}(V)^{\vee}_{\mathbf{Z}}$ becomes arbitrary large.

Let $\mathscr{W}'(\varepsilon)$ be the image of $\widetilde{\mathscr{W}}(\varepsilon)$ in $\prod_{s\in\mathsf{S}}\mathrm{Gr}(\mathscr{V}'_{R_s})$. We decompose our classes as follows:

$$\begin{aligned}\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}'}}=\delta'}(\mathscr{C},\mathscr{V}'\mid W')\right]&=\sum_{\varepsilon\in(\mathrm{Pic}(V)^{\vee})^{\mathsf{S}}}\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}'}}=\delta'}(\mathscr{C},\mathscr{V}'\mid W'\cap\mathscr{W}'(\varepsilon))\right]\\&=\sum_{\varepsilon\in(\mathrm{Pic}(V)^{\vee})^{\mathsf{S}}}\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\widetilde{\mathscr{L}'}}}=\delta'}\left(\mathscr{C},\widetilde{\mathscr{V}}\mid\widetilde{W}\cap\widetilde{\mathscr{W}}(\varepsilon)\right)\right]\end{aligned}$$

these sums being finite by Lemma 3.2.8. Normalising, we get

$$\begin{aligned}&\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}'}}=\delta'}(\mathscr{C},\mathscr{V}'\mid W')\right]\mathbf{L}^{-\,\delta'\,\cdot\,\omega_V^{-1}}\\&=\sum_{\varepsilon\in(\mathrm{Pic}(V)^{\vee})^{\mathsf{S}}}\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\widetilde{\mathscr{L}'}}}=\delta'}\left(\mathscr{C},\widetilde{\mathscr{V}}\mid\widetilde{W}\cap\widetilde{\mathscr{W}}(\varepsilon)\right)\right]\mathbf{L}^{-\,\delta'\,\cdot\,\omega_V^{-1}}\\&=\sum_{\varepsilon\in(\mathrm{Pic}(V)^{\vee})^{\mathsf{S}}}\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\widetilde{\mathscr{L}}}}=\delta'-|\varepsilon|}\left(\mathscr{C},\widetilde{\mathscr{V}}\mid\widetilde{W}\cap\widetilde{\mathscr{W}}(\varepsilon)\right)\right]\mathbf{L}^{-\,(\delta'-|\varepsilon|)\,\cdot\,\omega_V^{-1}}\mathbf{L}^{-|\varepsilon|\,\cdot\,\omega_V^{-1}}\end{aligned}$$

where $|\varepsilon|$ stands for $\sum_{s\in\mathsf{S}}\varepsilon_s\in\mathrm{Pic}(V)^{\vee}$. Then

$$\left[\mathrm{Hom}_{\mathscr{C}}^{\mathbf{deg}_{\underline{\mathscr{L}'}}=\delta'}(\mathscr{C},\mathscr{V}'\mid W')\right]\mathbf{L}^{-\,\delta'\,\cdot\,\omega_V^{-1}}$$

$$\xrightarrow[d(\delta',\partial\,\mathrm{Eff}(V)^{\vee})\to\infty]{}\sum_{\varepsilon\in(\mathrm{Pic}(V)^{\vee})^{\mathsf{S}}}\tau_{\widetilde{\mathscr{L}_{\mathscr{V}}}}\left(\widetilde{\mathscr{V}}\mid\widetilde{W}\cap\widetilde{\mathscr{W}}(\varepsilon)\right)\mathbf{L}^{-|\varepsilon|\,\cdot\,\omega_V^{-1}}=\tau_{\widetilde{\mathscr{L}'_{\mathscr{V}}}}\left(\widetilde{\mathscr{V}}\mid\widetilde{W}\right)=\tau_{\mathscr{L}'_{\mathscr{V}}}(\mathscr{V}'\mid W')$$

and Theorem 3.2.6 is proved.

CHAPTER 4

Equivariant compactifications of vectors spaces

Abstract. We study the asymptotic behaviour of the moduli space of sections of a family over a projective curve whose generic fibre is a compactification of a power of the additive group. This is an extension of [Fai22] to prove equidistribution of curves in this setting, as well as a analogue of the motivic Batyrev-Manin-Peyre principle for Campana curves.

4.1. Geometric setting

We recall in this paragraph a few facts and notation coming from the geometric setting of [Bil23, CLL16, Fai22].

We assume the base field k to be algebraically closed of characteristic zero. In the whole chapter, $\mathscr{C}_0$ is a dense open subset of the smooth projective curve $\mathscr{C}$, whose genus is g and field of rational functions $F = k(\mathscr{C})$. The complement of $\mathscr{C}_0$ in $\mathscr{C}$ is a finite scheme S.

We take G to be the additive group scheme $\mathbf{G}_a^n$, for a given positive integer n, and X to be a smooth equivariant compactification of G_F, meaning that X is a smooth projective scheme over F containing a dense open subset isomorphic to G_F, such that the group law of G_F extends to an action of G_F on X. Let $\pi : \mathscr{X} \to \mathscr{C}$ be a *proper model of* X. We assume $\mathscr{X}$ to be projective over k.

Let $\mathscr{U}$ be a Zariski open subset of $\mathscr{X}$. Similarly, $U = \mathscr{U}_F$ is assumed to be stable under the action of G_F. We denote by D the complement of U in X.

By [HT99, Theorem 2.7], the boundary

$$X \setminus G_F$$

is a divisor whose irreducible components

$$(D_\alpha)_{\alpha\in\mathscr{A}}$$

freely generate the Picard group of X, as well as its effective cone. There exist integers $\rho_\alpha \geqslant 2$ such that an anticanonical divisor is given by

$$\sum_\alpha \rho_\alpha D_\alpha.$$

In particular, it is big. Then a log-anticanonical divisor with respect to $D = X \setminus U$ is

$$\sum_{\alpha\in\mathscr{A}} \rho'_\alpha D_\alpha$$

where $\rho'_\alpha = \rho_\alpha - 1$ if D_α is an irreducible component of D and $\rho'_\alpha = \rho_\alpha$ otherwise. The boundary D is a divisor which can be written $D = \sum_{\alpha\in\mathscr{A}_D} D_\alpha$ for a certain subset $\mathscr{A}_D$ of $\mathscr{A}$. We set

$$\mathscr{A}_U = \mathscr{A} \setminus \mathscr{A}_D.$$

For every α in $\mathscr{A}$, there is a Cartier divisor $\mathscr{L}_\alpha$ on $\mathscr{X}$ extending D_α. Given a tuple of integers $\mathbf{n} = (n_\alpha)_{\alpha\in\mathscr{A}}$, the moduli space

$$\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{U})_G$$

is the subspace of $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C}, \mathscr{X})_G$ parametrising sections $\sigma : \mathscr{C} \to \mathscr{X}$ such that:
— σ maps the generic point $\eta_{\mathscr{C}}$ of $\mathscr{C}$ to a point of G_F;
— the image of $\mathscr{C}_0$ by σ is contained in $\mathscr{U}$;
— for all α in $\mathscr{A}$, $\deg(\sigma^*\mathscr{L}_\alpha) = n_\alpha$.
The existence of these moduli spaces follows from [**CLL16**, Proposition 2.2.2]. In order to avoid any local obstruction to the existence of such sections, we assume that local sections exists: we suppose that for every closed point v of $\mathscr{C}_0$ the intersection of $G(F_v)$ with $\mathscr{U}(\mathcal{O}_v)$ is non-empty, where F_v is the completion of F at v and $\mathcal{O}_v$ is its ring of integers.

Let $\mathscr{D}_\alpha$ be the Zariski closure of D_α in $\mathscr{X}$, for all $\alpha \in \mathscr{A}$. Using resolution of singularities in characteristic zero, one can assume that $\mathscr{X}$ is smooth over k and that the sum of the non-smooth fibres of $\mathscr{X}$ and of the $\mathscr{D}_\alpha$ is a divisor with strict normal crossings. We may assume that $\mathscr{X} \setminus \mathscr{U}$ is a divisor with strict normal crossings as well. These assumptions will not change our counting problem (see §3.2 and Lemma 3.4.1 of [**CLL16**]).

4.1.1. S-integral curves. For any k-point v of $\mathscr{C}$, let $\mathscr{B}_v$ be the set of irreducible components of $\pi^{-1}(v)$. Given an irreducible component $\beta \in \mathscr{B}_v$, let us denote by E_β the corresponding component and μ_β its multiplicity in the special fibre of $\mathscr{X}$ at v (that is, the length of the $\mathcal{O}_v$-module of regular functions on E_β in $\mathscr{X}_v = \mathscr{X} \times_{\mathcal{O}_v} \mathrm{Spec}(k)$). Let $\mathscr{B} = \cup_{v\in\mathscr{C}(k)}\mathscr{B}_v$ be the union of all the $\mathscr{B}_v$'s and $\mathscr{B}_1$ the subset of components of multiplicity one. Over any place v this subset restricts to a subset $\mathscr{B}_{1,v} = \mathscr{B}_1 \cap \mathscr{B}_v$ of $\mathscr{B}_v$. If $v \in \mathscr{C}_0(k)$, we write $\mathscr{B}_v^{\mathscr{U}}$ for the subset of $\mathscr{B}_{1,v}$ of vertical components intersecting $\mathscr{U}$. This definition makes sense since we assumed that $\mathscr{X} \setminus \mathscr{U}$ is a divisor: it is the union of the $\mathscr{D}_\alpha$ for $\alpha \in \mathscr{A}_D$ together with a finite number of vertical divisors. It is convenient to set $\mathscr{B}_s^{\mathscr{U}} = \mathscr{B}_{1,s}$ if $s \in S$.

Recall that for every α in $\mathscr{A}$, $\mathscr{L}_\alpha$ is a Cartier divisor on $\mathscr{X}$ extending D_α. There is a finitely supported family of integers (e_α^β) indexed by $\mathscr{A}$ and $\mathscr{B}$ such that

$$\mathscr{L}_\alpha = \mathscr{D}_\alpha + \sum_{\beta\in\mathscr{B}} e_\alpha^\beta E_\beta$$

for any α in $\mathscr{A}$, as well as a finitely supported family of integers (ρ^β) indexed by $\mathscr{B}$ such that

$$-\operatorname{div}(\omega_X) = \sum_{\alpha\in\mathscr{A}} \rho_\alpha \mathscr{D}_\alpha + \sum_{\beta\in\mathscr{B}} \rho^\beta E_\beta$$

with ω_X the unique (up to a multiplicative constant) G_F-invariant rational differential form on X, which is understood here as a rational section of the relative canonical bundle of $\mathscr{X}$ over $\mathscr{C}$. By Lemmas 3.3.3 and 3.3.4 in [**CLL16**], there exists an open dense subset $\mathscr{C}_1 \subset \mathscr{C}_0$ such that $|\mathscr{B}_v| = 1$ for every $v \in \mathscr{C}_1(k)$ (so that $\mathscr{B}_v = \mathscr{B}_{1,v} = \mathscr{B}_v^{\mathscr{U}}$ above $\mathscr{C}_1$, since we assumed $\mathscr{U}(\mathcal{O}_v) \neq \varnothing$). Furthermore one can assume $e_\alpha^{\beta_v} = \rho^{\beta_v} = 0$ for all $v \in \mathscr{C}_1(k)$ and $\mathscr{D}_\alpha \times_{\mathscr{C}} \mathscr{C}_1 \to \mathscr{C}_1$ smooth for all $\alpha \in \mathscr{A}_U$. This open subset $\mathscr{C}_1$ can be understood as the set of places of *good reduction*.

4.1.2. Decomposition of arcs. Given any closed point $v \in \mathscr{C}(k)$, for every subset A of $\mathscr{A}$ and every irreducible component β of multiplicity one above v, let $\Delta_v(A,\beta)$ be the set of points of the special fibre $\mathscr{X}_v$ of $\mathscr{X} \to \mathscr{C}$ over v belonging exclusively to $\mathscr{D}_\alpha$, for every $\alpha \in A$, and to E_β:

$$\Delta_v(A,\beta) = \left(\bigcap_{\alpha\in A} \mathscr{D}_{\alpha,v} \cap E_\beta\right) \setminus \left(\bigcup_{\substack{\alpha\notin A\\ \beta'\neq\beta}} \mathscr{D}_{\alpha,v} \cup E_{\beta'}\right).$$

We denote by $\Omega_v(A,\beta)$ the preimage of $\Delta_v(A,\beta)$ in the arc space $\mathscr{L}(\mathscr{X}_{\mathcal{O}_v})$ through the projection $\mathscr{L}(\mathscr{X}_{\mathcal{O}_v}) \to \mathscr{L}_0(\mathscr{X}_{\mathcal{O}_v}) = \mathscr{X}_v$. Lemma 5.2.6 in [**CLL16**] (recalled as Lemma 6.3.3.2 in [**Bil23**]) gives the existence of an isomorphism

$$\begin{array}{rcl} \Theta : \Delta_v(A,\beta) \times \mathscr{L}(\mathbf{A}^1,0)^A \times \mathscr{L}(\mathbf{A}^1,0)^{n-|A|} & \longrightarrow & \Omega_v(A,\beta) \\ x = (x_\Delta, (x_\alpha)_{\alpha\in A}, (y_\alpha)) & \longmapsto & \Theta(x) \end{array}$$

which preserves the motivic measures and such that $\mathrm{ord}_0(x_\alpha) = \mathrm{ord}_{\mathscr{D}_\alpha}(\Theta(x))$ whenever $\alpha \in A$ and $\mathrm{ord}_{\mathscr{D}_\alpha}(\Theta(x)) = 0$ otherwise. For further details, we refer to [**CLL16**, §2.4 & §6.2].

To conclude this section, one may note that our problem is defined by a finite number of polynomial equations over k. Fixing an embedding of k into $\mathbf{C}$, we can assume that everything is defined over the field of complex numbers (using the fact that the definition of the moduli spaces of sections is functorial). Moreover, the assumption $\mathscr{U}(\mathscr{O}_v) \neq \varnothing$ for all $v \in \mathscr{C}_0$ implies that for such v, at least one of the $\Delta_v(A,\beta)$ has a k-point, thus has a $\mathbf{C}$-point [**Bil23**, §6.4.4].

4.1.3. Campana curves. Let $(m_\alpha)_{\alpha\in\mathscr{A}_U}$ be a finite family of positive integers and $(\epsilon_\alpha)_{\alpha\in\mathscr{A}}$ the associated finite family of rational numbers in $[0,1]$ given by

$$\epsilon_\alpha = \begin{cases} 1 - \frac{1}{m_\alpha} & \text{if } \alpha \in \mathscr{A}_U \\ 1 & \text{if } \alpha \in \mathscr{A}_D. \end{cases}$$

It is convenient to set $m_\alpha = \infty$ whenever $\alpha \in \mathscr{A}_D$. Let D_ϵ be the $\mathbf{Q}$-divisor given by

$$D_\epsilon = \sum_{\alpha\in\mathscr{A}} \epsilon_\alpha D_\alpha$$

as well as its model

$$\mathscr{D}_\epsilon = \sum_{\alpha\in\mathscr{A}} \epsilon_\alpha \mathscr{D}_\alpha.$$

Let $\mathscr{L}_{\rho-\epsilon}$ be the model of $-(K_X + D_\epsilon)$ given by

$$\mathscr{L}_{\rho-\epsilon} = \sum_{\alpha\in\mathscr{A}} (\rho_\alpha - \epsilon_\alpha)\mathscr{L}_\alpha.$$

The pair (X,D) is a Campana orbifold over the function field F. In the arithmetical setting, M. Pieropan, A. Smeets, S. Tanimoto and A. Várilly-Alvarado [**PSTVA21**] formulated a variant of the Batyrev-Manin-Peyre principle for Campana points, and proved it for equivariant compactification of vector spaces, adapting previous works by Chambert-Loir and Tschinkel [**CLT12**]. In this chapter we prove a motivic analogue of their result.

Definition 4.1.1. An (S,ϵ)-Campana section of $\mathscr{X}$ is a section $\sigma : \mathscr{C} \to \mathscr{X}$ such that
$$(\sigma, \mathscr{D}_\alpha)_v = 0 \quad \text{or} \quad (\sigma, \mathscr{D}_\alpha)_v \geqslant m_\alpha$$
for all $\alpha \in \mathscr{A}$ and all places $v \in \mathscr{C}_0(k)$.

4.1.4. Arc schemes and integrals of motivic residual functions. In this paragraph we recall and collect a few additional facts about jet schemes, arc schemes and a particular case of motivic measure on such spaces.

4.1.4.1. *Jet schemes, arc schemes and integration.* The field k is an algebraically closed field of characteristic zero and $\mathscr{Y}$ is a flat scheme of finite type over $k[[t]]$, which we assume to be equidimensional of relative dimension n. A simple example is given by $\mathscr{Y} = Y \times_{\operatorname{Spec} k} \operatorname{Spec}(k[[t]])$ for some k-variety Y. For any non-negative integer m, the jet-scheme $\mathscr{L}_m(\mathscr{Y})$ of order m of $\mathscr{Y}$ is the k-variety representing the functor
$$A \mapsto \operatorname{Hom}_k(\operatorname{Spec}(A[[t]]/(t^{m+1})), \mathscr{Y})$$
on the category of k-algebras. There are canonical affine projection morphisms
$$\theta_m^{m+1} : \mathscr{L}_{m+1}(\mathscr{Y}) \to \mathscr{L}_m(\mathscr{Y})$$
and one can consider the proscheme over k
$$\mathscr{L}(\mathscr{Y}) = \lim_{\substack{\leftarrow \\ m}} \mathscr{L}_m(\mathscr{Y})$$
of arcs on $\mathscr{Y}$. This projective limit carries canonical projections $\theta_m^\infty : \mathscr{L}(\mathscr{Y}) \to \mathscr{L}_m(\mathscr{Y})$.

The projections θ_m^{m+1} induce ring morphisms
$$\left(\theta_m^{m+1}\right)^* : \mathscr{E}xp\mathscr{M}_{\mathscr{L}_m(\mathscr{Y})} \to \mathscr{E}xp\mathscr{M}_{\mathscr{L}_{m+1}(\mathscr{Y})}$$
between the corresponding localised Grothendieck rings with exponential.

Definition 4.1.2. The ring of motivic residual functions on $\mathscr{L}(\mathscr{Y})$ is defined as the direct limit $\lim\limits_{\substack{\rightarrow \\ m}} \mathscr{E}xp\mathscr{M}_{\mathscr{L}_m(\mathscr{Y})}$. Let h be a motivic residual function, which we assume to be of the form $[H,f]_{\mathscr{L}_m(\mathscr{Y})}$ with H a variety over $\mathscr{L}_m(\mathscr{Y})$ for some m and $f : H \to \mathbf{A}^1$ a morphism. The integral of h over $\mathscr{L}(\mathscr{Y})$ is the element of $\mathscr{E}xp\mathscr{M}_k$
$$\int_{\mathscr{L}(\mathscr{Y})} h(x)\mathbf{d}x = \mathbf{L}^{-(m+1)n}[H,f]_k.$$

Since the projection $\mathscr{L}_{m'}(\mathscr{Y}) \to \mathscr{L}_m(\mathscr{Y})$ is a locally trivial fibration with fibres isomorphic to $\mathbf{A}^{(m'-m)n}$ for any $m' \geqslant m$ [**CLNS18**, Proposition 3.7.5], this definition does not depend on the choice of m.

4.1.4.2. *Motivic volumes.* A useful particular case of such integrals is given by the characteristic function of a constructible subset $W = p_m^{-1}(W_m)$ of $\mathscr{L}(\mathscr{Y})$. The integral of $\mathbf{1}_W$ over $\mathscr{L}(\mathscr{Y})$ is by definition the *volume* $\mu(W)$ of W. There are three particular volumes of interest for our purpose. The first one is the volume of the whole arc space $\mathscr{L}(\mathscr{Y}) = p_0^{-1}(\mathscr{Y}_k)$, which is
$$\mu(\mathscr{L}(\mathscr{Y})) = \mathbf{L}^{-n}[\mathscr{Y}_k, 0].$$
Then the volume of the subspace $W = p_0^{-1}(\{0\}) = \mathscr{L}(\mathbf{A}^1, 0)$ of arcs in $\mathbf{A}^1$ with origin at 0 is
$$\mu(\mathscr{L}(\mathbf{A}^1, 0)) = \mathbf{L}^{-1}[\{0\}, 0] = \mathbf{L}^{-1}.$$
Finally, an arc on $\mathbf{A}^1$ of order zero at $\{0\}$ is an arc whose image in $\mathbf{A}^1$ do not belongs to $\{0\}$, thus the corresponding volume is $1 - \mathbf{L}^{-1}$. More generaly, the set of arcs of order $m \in \mathbf{N}$ at $\{0\}$ has volume $(1 - \mathbf{L}^{-1})\mathbf{L}^{-m}$.

4.1.4.3. *Weights and volumes.* The weight function satisfies the following property [**Bil23**, Remark 6.3.1.3]: for any constructible subset W of $\mathscr{L}(\mathscr{Y})$ and motivic residual function h we have the inequality

$$w\left(\int_{\mathscr{L}(\mathscr{Y})} \mathbf{1}_W(x)h(x)dx\right) \leqslant w(\mu(W)).$$

4.2. Motivic multivariate zeta series

For the sake of completeness, we start by recalling the method leading to the decomposition of the motivic Zeta function, as it is developed in [**CLL16**] and [**Bil23**]. A motivation for doing that is giving the reader a precise meaning of the summation symbols involved here: our asymptotic study will require us to permute a sum over rational points and a limit in the completed ring $\widehat{\mathscr{E}xp\mathscr{M}_k}$.

4.2.1. Local intersection degrees. Given a k-point $v \in \mathscr{C}(k)$, it is possible to define local intersection degrees $(g, D_\alpha)_v$ and $(g, E_\beta)_v$ for all $g \in G(F_v)$, $\alpha \in \mathscr{A}$ and $\beta \in \mathscr{B}_v$, in the following manner [**CLL16**, §3.3]. Let $g : \mathrm{Spec}(F_v) \to G_F$ be such a point. Since $\mathscr{X} \to \mathscr{C}$ is proper, by the valuative criterion of properness this map extends to a map $\tilde{g} : \mathrm{Spec}(\mathcal{O}_v) \to \mathscr{X}$.

$$\begin{tikzcd}
\mathrm{Spec}(F_v) \arrow[r, "g"] \arrow[d] & G_F \arrow[r, hook] & \mathscr{X} \arrow[d, "\pi"] \\
\mathrm{Spec}(\mathcal{O}_v) \arrow[urr, dashed, "\tilde{g}"] \arrow[rr] & & \mathscr{C}
\end{tikzcd}$$

The non-negative local intersection number $(g, \mathscr{D}_\alpha)_v$ is given by the effective Cartier divisor on $\mathrm{Spec}(\mathcal{O}_v)$ obtained by pulling-back $\mathscr{D}_\alpha$:

$$\tilde{g}^*\mathscr{D}_\alpha = (g, \mathscr{D}_\alpha)_v[v].$$

The number $(g, E_\beta)_v$ is defined by pulling-back E_β on $\mathrm{Spec}(\mathcal{O}_v)$. Such invariants satisfy the following two properties:

— they are compatible with the global degree of the section $\sigma_g : \mathscr{C} \to \mathscr{X}$ canonically extending $g \in G(F)$, with respect to $\mathscr{D}_\alpha$ for any α:

$$\deg_{\mathscr{C}}(\sigma_g^*(\mathscr{D}_\alpha)) = \sum_{v \in \mathscr{C}(k)} (g, \mathscr{D}_\alpha)_v\,;$$

— the F_v-point g intersects exactly one vertical component of multiplicity one: there exists a unique $\beta \in \mathscr{B}_v$ such that

$$(g, E_\beta)_v = 1 \text{ with } \mu_\beta = 1$$

and

$$\beta' \neq \beta \text{ implies } (g, E_{\beta'})_v = 0.$$

4.2.2. Adelic sets and moduli spaces. Then one defines the sets

$$G(\mathsf{m}_v, \beta_v)_v = \{g \in G(F_v) \mid (g, E_\beta)_v = 1 \text{ and } (g, \mathscr{D}_\alpha)_v = \mathsf{m}_{v,\alpha} \text{ for all } \alpha \in \mathscr{A}\}$$

for all $\mathsf{m}_v \in \mathbf{N}^{\mathscr{A}}$ and $\beta_v \in \mathscr{B}_v$. These sets provide a decomposition of $G(F_v)$ into disjoint bounded definable subsets [**CLL16**, Lemma 3.3.2].

Definition 4.2.1. A pair $(\mathsf{m}_v, \beta_v) \in \mathbf{N}^{\mathscr{A}} \times \mathscr{B}_v$ is said to be v-integral if

— either $v \in \mathscr{C}_0$, $\beta \in \mathscr{B}_v^{\mathscr{U}}$ and $\mathsf{m}_{v,\alpha} = 0$ for every $\alpha \in \mathscr{A}_D$;

— or $v \in \mathscr{C} \setminus \mathscr{C}_0$.

A pair (m_v, β_v) is said to be v-Campana if

— either $v \in \mathscr{C}_0$, $\beta \in \mathscr{B}_v^{\mathscr{U}}$ and $\mathsf{m}_{v,\alpha} = 0$ or $\mathsf{m}_{v,\alpha} \geqslant \frac{1}{1-\epsilon_\alpha}$ for every $\alpha \in \mathscr{A}$;

— or $v \in \mathscr{C} \setminus \mathscr{C}_0$.

In particular, if $\mathscr{A}_D \neq \varnothing$ then v-Campana pairs are v-integral pairs.

One then introduces the corresponding sets

$$H(\mathsf{m}_v, \beta_v)_v = G(\mathsf{m}_v, \beta_v)_v \text{ if and only if } (\mathsf{m}_v, \beta_v) \text{ is } v\text{-integral, and } \varnothing \text{ otherwise,}$$
$$H^\epsilon(\mathsf{m}_v, \beta_v)_v = G(\mathsf{m}_v, \beta_v)_v \text{ if and only if } (\mathsf{m}_v, \beta_v) \text{ is } v\text{-Campana, and } \varnothing \text{ otherwise,}$$

for any place $v \in \mathscr{C}(k)$ and any pair (m_v, β_v). These definitions provide adelic sets

$$H^\epsilon(\mathsf{m}, \beta) = \prod_{v \in \mathscr{C}(k)} H^\epsilon(\mathsf{m}_v, \beta_v)_v \subset G(\mathsf{m}, \beta) = \prod_{v \in \mathscr{C}(k)} G(\mathsf{m}_v, \beta_v)_v \subset G(\mathbb{A}_{k(\mathscr{C})})$$

for any $\mathsf{m} = (\mathsf{m}_v) \in \mathrm{Sym}^{\mathbf{n}}_{/k} \mathscr{C}$ and $\beta = (\beta_v) \in \prod_v \mathscr{B}_v$. By Propositions 6.2.3.3 and 6.2.4.2 of [**Bil23**], there exist

— almost zero functions $s, s' : \mathscr{C} \to \mathbf{Z}$ (playing the roles of a and b in Section 2.2.5);

— an unbounded family $N = (N_{\mathbf{m}})_{\mathbf{m} \in \mathbf{N}^{\mathscr{A}}}$ such that $N_{\mathbf{0}} = 0$;

— for all $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ and $\beta \in \prod_v \mathscr{B}_v$, a constructible subset $H^\epsilon_{\mathbf{m},\beta} \subset \mathbf{A}_{\mathscr{C}}^{n(s'-N_{\mathbf{m}},s)}$;

such that for every $v \in \mathscr{C}(k)$, the fibre of $H^\epsilon_{\mathbf{m},\beta}$ at v is $H^\epsilon(\mathbf{m}, \beta_v)_v$.

Let $W = \prod_{v \in \mathscr{C}(k)} W_v$ be a product of non-empty constructible subsets of the $G(F_v)$'s such that $W_v = G(F_v)$ for all but finitely many $v \in (\mathscr{C}_0 \setminus \mathscr{C}_1)(k)$ (up to shrinking $\mathscr{C}_1$ a little bit). For all $\mathsf{m} = (\mathsf{m}_v) \in \mathrm{Sym}^{\mathbf{n}}_{/k} \mathscr{C}$ and $\beta = (\beta_v) \in \prod_v \mathscr{B}_v$ we set

$$W^\epsilon(\mathsf{m}, \beta) = \prod_{v \in \mathscr{C}(k)} H^\epsilon(\mathsf{m}_v, \beta_v)_v \cap W_v.$$

Since we modify $H^\epsilon_{\mathbf{m},\beta}$ only above a finite number closed points of $\mathscr{C}_0$, this defines for every $\mathbf{m} \in \mathbf{N}^{\mathscr{A}}$ a constructible subset $W^\epsilon_{\mathbf{m},\beta} \subseteq H^\epsilon_{\mathbf{m},\beta}$ whose fibre at every $v \in \mathscr{C}(k)$ is $H^\epsilon(\mathbf{m}, \beta_v)_v \cap W_v$.

By taking symmetric products, this data provides constructible subsets

$$\mathrm{Sym}^{\mathbf{n}}_{\mathscr{C}/k}((W^\epsilon_{\mathbf{m},\beta})_{\mathbf{m} \in \mathbf{N}^{\mathscr{A}}}) \subset \mathscr{A}_{\mathbf{n}}(s', s, N, 0)$$

for any $\mathbf{n} \in \mathbf{N}^{\mathscr{A}}$, which themselves define uniformly smooth constructible families of Schwartz-Bruhat functions (of any level $\mathbf{n}$), denoted by

$$\left(((\mathbf{1}_{H^\epsilon(\mathsf{m},\beta)\cap W}))_{\mathsf{m} \in \mathrm{Sym}^{\mathbf{n}}_{/k} \mathscr{C}}\right) \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(s',s,N,0)}.$$

Such functions correspond to the characteristic functions of $H^\epsilon(\mathsf{m}, \beta) \cap W$, for fixed β and varying $\mathsf{m} \in \mathrm{Sym}^{\mathbf{n}}_{/k} \mathscr{C}$. Therefore, by taking Fourier transforms, one gets a uniformly compactly supported family

$$(\mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m},\beta)\cap W}))_{\mathsf{m} \in \mathrm{Sym}^{\mathbf{n}}_{/k} \mathscr{C}} \in \mathscr{E}xp\mathscr{M}_{\mathscr{A}_{\mathbf{n}}(\nu-s,\nu-s',0,N)}$$

where ν is the conductor of a rational differential form $\omega \in \Omega_{k(\mathscr{C})/k}$.

The moduli space $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C}, \mathscr{X} \mid \mathscr{U})_G$ of sections $\sigma : \mathscr{C} \to \mathscr{X}$ such that:

— σ maps the generic point $\eta_{\mathscr{C}}$ of $\mathscr{C}$ to a point of G_F;

— the image of $\mathscr{C}_0$ by σ is contained in $\mathscr{U}$;

— for all α in $\mathscr{A}$, $\deg(\sigma^* \mathscr{L}_\alpha) = n_\alpha$, can be rewritten as a disjoint union of constructible subsets $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\mathscr{U})^{\beta}_G$, where for any $\beta\in\prod_{v\in\mathscr{C}(k)}\mathscr{B}^{\mathscr{U}}_v$, the subset $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\mathscr{U})^{\beta}_G$ is the set of sections $\sigma\in M_{\mathbf{n}}$ such that $(\sigma(\eta_{\mathscr{C}}),E_{\beta_v})_v=1$ for all $v\in\mathscr{C}(k)$. In what follows,

$$\mathbf{e}^{\beta_v}=(e^{\beta_v}_{\alpha})_{\alpha\in\mathscr{A}}\qquad \mathbf{e}^{\beta}=\sum_v\mathbf{e}^{\beta_v}\qquad \mathbf{n}^{\beta}=\mathbf{n}-\mathbf{e}^{\beta}\in\mathbf{N}^{\mathscr{A}}$$

hence elements of $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\mathscr{U})^{\beta}_G$ are sections $\sigma\in\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\mathscr{U})_G$ such that $(\deg(\sigma^*\mathscr{D}_\alpha))_{\alpha\in\mathscr{A}}=\mathbf{n}^{\beta}$. Since $\mathscr{B}^{\mathscr{U}}$ is the set of vertical components of multiplicity one lying above S or contained in $\mathscr{U}$, one has $\mathbf{e}^{\beta_v}\in\mathbf{N}^{\mathscr{A}_U}$ whenever $v\in\mathscr{C}_0(k)$. Moreover, if $v\in\mathscr{C}_1(k)$ then the condition $(\sigma(\eta_{\mathscr{C}}),E_{\beta_v})_v=1$ is automatically satisfied since $|\mathscr{B}_v|=1$. Thus the partition of $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\mathscr{U})_G$ we are describing here is actually finite, since it only depends on the local intersection numbers with respect to a finite number of vertical divisors.

The adaptation of the previous decomposition to the submoduli spaces

$$\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)_G$$

of constrained $(S,\epsilon)-$Campana curves is straightforward.

4.2.3. Motivic height zeta functions. By [Bil23, Lemma 6.2.6.1], if

$$\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_G$$

is non-empty there is a morphism

$$\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_G\to\mathrm{Sym}^{\mathbf{n}^{\beta}}_{/k}\mathscr{C}$$

of constructible sets sending a section σ_g to the tuple of zero-cycles

$$\sum_{v\in\mathscr{C}(k)}((g,\mathscr{D}_\alpha)_v)_{\alpha\in\mathscr{A}}[v].$$

The *coarse motivic height zeta function* $Z(\mathbf{t})$ can be rewritten

$$Z^{\epsilon}_{|W}(\mathbf{t})=\sum_{\beta\in\prod_v\mathscr{B}^{\mathscr{U}}_v}\ \sum_{\mathbf{n}^{\beta}\in\mathbf{N}^{\mathscr{A}}}\left[\mathrm{Hom}^{\mathbf{n}^{\beta}+\mathbf{e}^{\beta}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_G\right]\mathbf{t}^{\mathbf{n}^{\beta}+\mathbf{e}^{\beta}}=\sum_{\beta\in\prod_v\mathscr{B}^{\mathscr{U}}_v}\mathbf{t}^{\mathbf{e}^{\beta}}Z^{\epsilon,\beta}_{|W}(\mathbf{t}).\tag{4.2.3.16}$$

where

$$Z^{\epsilon,\beta}_{|W}(\mathbf{t})=\sum_{\mathbf{n}^{\beta}\in\mathbf{N}^{\mathscr{A}}}\left[\mathrm{Hom}^{\mathbf{n}^{\beta}+\mathbf{e}^{\beta}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_G\right]\mathbf{t}^{\mathbf{n}^{\beta}}$$

for every $\beta\in\prod_v\mathscr{B}^{\mathscr{U}}_v$ (note again that this last product is finite).

When one considers S-integral points and uses this function, it appears that it is not precise enough: poles of multiple order appear, which means that we miss relevant invariants which correspond to distinct components of the moduli space of sections. In order to deal with it, remark that a section $\sigma:\mathscr{C}\to\mathscr{X}$ intersects the divisors $\mathscr{D}_\alpha$ for $\alpha\in\mathscr{A}_D$ only above the finite set of closed points $S=\mathscr{C}\setminus\mathscr{C}_0$ and *each point* $s\in S$ gives an *invariant*

$$((\sigma(\eta_{\mathscr{C}}),\mathscr{D}_\alpha)_s)_{\alpha\in\mathscr{A}_D}.$$

This leads us to define for any $(\mathbf{n},\beta)$ and $\mathsf{m}_{\mathsf{S}}=(\mathsf{m}_s)_{s\in\mathsf{S}}\in\mathbf{N}^{\mathscr{A}_D\times S}$ the subset

$$\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid\mathsf{m}_S)^{\beta}$$

of sections $\sigma\in\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_G$ such that

$$((\sigma(\eta_{\mathscr{C}}),\mathscr{D}_\alpha))_s)_{s\in S}=\mathsf{m}_S.$$

This gives a decomposition of $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_{G}$ into a finite disjoint union of definable subsets. If $\mathbf{n}^{\beta}=(\mathbf{n}^{\beta}_{U},\mathbf{n}^{\beta}_{D})$ is an element of $\mathbf{N}^{\mathscr{A}}=\mathbf{N}^{\mathscr{A}_U}\times\mathbf{N}^{\mathscr{A}_D}$, this subset can be identified with the fibre of

$$\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_{G}\to\mathrm{Sym}^{\mathbf{n}^{\beta}}_{/k}\mathscr{C}$$

over the points

$$\mathsf{m}\in\mathrm{Sym}^{\mathbf{n}^{\beta}}_{/k}\mathscr{C}=\mathrm{Sym}^{\mathbf{n}^{\beta}_{U}}_{/k}\mathscr{C}\times_k\mathrm{Sym}^{\mathbf{n}^{\beta}_{D}}_{/k}\mathscr{C}$$

whose image by the projection $\mathrm{Sym}^{\mathbf{n}^{\beta}}_{/k}\mathscr{C}\to\mathrm{Sym}^{\mathbf{n}^{\beta}_{D}}_{/k}\mathscr{C}$ has support in $\mathsf{S}=\mathscr{C}\setminus\mathscr{C}_0$ and is given by m_{S}.

Definition 4.2.2. Let $\mathbf{u}=(\mathbf{u}_0,(\mathbf{u}_s)_{s\in S})$ be a family of indeterminates indexed by the set

$$\mathscr{A}_U\sqcup(\mathscr{A}_D\times S)$$

and let

$$\mathbf{e}^{\beta}=\left(\sum_{v\in\mathscr{C}_0}\mathbf{e}^{\beta_v}+\sum_{s\in S}\mathbf{e}^{\beta_s}_{\mathscr{A}_U},\left(\mathbf{e}^{\beta_s}_{\mathscr{A}_D}\right)_{s\in S}\right)$$

for all $\beta\in\prod_v\mathscr{B}^{\mathscr{U}}_v$.

We define the *refined motivic height zeta function with respect to ϵ and W* by

$$\mathscr{Z}^{\epsilon}_{|W}(\mathbf{u})=\sum_{\beta\in\prod_v\mathscr{B}^{\mathscr{U}}_v}\mathbf{u}^{\mathbf{e}^{\beta}}\mathscr{Z}^{\epsilon,\beta}_{|W}(\mathbf{u})\tag{4.2.3.17}$$

where

$$\mathscr{Z}^{\epsilon,\beta}_{|W}(\mathbf{u})=\sum_{\substack{\mathbf{n}^{\beta}_{\mathscr{A}_U}\in\mathbf{N}^{\mathscr{A}_U}\\ \mathsf{m}_S\in\mathbf{N}^{\mathscr{A}_D\times S}}}\left[\mathrm{Hom}^{\mathbf{n}^{\beta}+\mathbf{e}^{\beta}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid\mathsf{m}_{\mathsf{S}})^{\beta}\right]\mathbf{u}^{\mathbf{n}_U+\mathsf{m}_S}$$

for every $\beta\in\prod_v\mathscr{B}^{\mathscr{U}}_v$.

Remark 4.2.3. By definition, the specialisation of $\mathscr{Z}(\mathbf{u})$ obtained by replacing respectively $\mathbf{u}_0=(U_\alpha)_{\alpha\in\mathscr{A}_U}$ by $(t_\alpha)_{\alpha\in\mathscr{A}_U}$ and $\mathbf{u}_s=(U_{\alpha,s})_{\alpha\in\mathscr{A}_D}$ by $(t_\alpha)_{\alpha\in\mathscr{A}_D}$, for every $s\in S$, is the coarse zeta function $Z(\mathbf{t})$. Furthermore, if $\mathscr{C}_0=\mathscr{C}$ and $\mathscr{U}=\mathscr{X}$ then this refined function $\mathscr{Z}$ coincides with the coarse one Z, since in that case $S=\varnothing$ and $\mathscr{A}=\mathscr{A}_U$. A similar remark can be done for $\mathscr{Z}^{\beta}$ and Z^{β}.

The map sending a section $\sigma:\mathscr{C}\to X$ to $\sigma(\eta_{\mathscr{C}})\in X(F)$ induces an exact correspondence between

$$\begin{gathered}\text{sections }\sigma\in\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid W^{\epsilon})_G\\ \text{such that }\sum_{v\in\mathscr{C}(k)}((g,\mathscr{D}_\alpha)_v)_{\alpha\in\mathscr{A}}[v]=\mathsf{m}\in S^{\mathbf{n}^{\beta}}\mathscr{C}\end{gathered}$$

and

$$\text{elements of }G(F)\cap W^{\epsilon}(\mathsf{m},\beta).$$

Remark that if $W^{\epsilon}(\mathsf{m},\beta)$ is non-empty then the image of m by the projection

$$\mathrm{Sym}^{\mathbf{n}^{\beta}}_{/k}\mathscr{C}\to\mathrm{Sym}^{\mathbf{n}^{\beta}_{D}}_{/k}\mathscr{C}$$

has support in S.

By definition of the summation over rational points (2.2.4.11), one has the equalities in $\mathscr{M}_{\kappa(\mathsf{m})}$

$$\sum_{x\in\kappa(\mathsf{m})(\mathscr{C})} \mathbf{1}_{G(F)\cap W^{\epsilon}(\mathsf{m},\beta)}(x) = [G(F)\cap H^{\epsilon}(\mathsf{m},\beta)\cap W] = \left(\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_{G}\right)_{\mathsf{m}}$$

where $\left(\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_{G}\right)_{\mathsf{m}}$ is the class of the fibre of

$$\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_{G} \longrightarrow \mathrm{Sym}^{\mathbf{n}^{\beta}}_{/k}(\mathscr{C})$$

over the schematic point $\mathsf{m}\in S^{\mathbf{n}^{\beta}}\mathscr{C}$. Applying the motivic Poisson formula for families (see Section 2.2.5) and adapting Bilu's computations [**Bil23**, p.161-162], we obtain for the coarse subspaces

$$\begin{aligned}\left[\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W)^{\beta}_{G}\right] &= \sum_{\mathsf{m}\in S^{\mathbf{n}^{\beta}}(C)} [G(F)\cap H^{\epsilon}(\mathsf{m},\beta)\cap W]\\ &= \mathbf{L}^{(1-g)n}\sum_{\xi\in k(\mathscr{C})^{n}}\sum_{\mathsf{m}\in S^{\mathbf{n}^{\beta}}(C)} \mathscr{F}(\mathbf{1}_{H^{\epsilon}(\mathsf{m},\beta)\cap W})(\xi)\end{aligned}$$

and for the refined subspaces

$$\begin{aligned}\left[\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid \mathsf{m}_S)^{\beta}\right] &= \sum_{\mathsf{m}\in S^{\mathbf{n}^{\beta}}_{U}(C)} [G(F)\cap H^{\epsilon}(\mathsf{m}+\mathsf{m}_S,\beta)\cap W]\\ &= \mathbf{L}^{(1-g)n}\sum_{\xi\in k(\mathscr{C})^{n}}\sum_{\mathsf{m}\in S^{\mathbf{n}^{\beta}}_{U}(C)} \mathscr{F}(\mathbf{1}_{H^{\epsilon}(\mathsf{m}+\mathsf{m}_S,\beta)\cap W})(\xi).\end{aligned}$$

Since $\left(\mathscr{F}(\mathbf{1}_{H^{\epsilon}(\mathsf{m},\beta)\cap W})\right)_{\mathsf{m}\in\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} = S^{\mathbf{n}}((\mathscr{F}\mathbf{1}_{H^{\epsilon}(\mathbf{m},\beta)\cap W})_{\mathbf{m}\in\mathbf{N}^{\mathscr{A}}})$ (see [**Bil23**, Proposition 5.4.4.2]), by definition of the motivic Euler product notation one can write

$$\begin{aligned}Z^{\epsilon,\beta}_{|W}(\mathbf{t}) &= \mathbf{L}^{(1-g)n}\sum_{\xi\in k(\mathscr{C})^{n}}\sum_{\mathbf{n}\in\mathbf{N}^{\mathscr{A}}}\sum_{\mathsf{m}\in\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \mathscr{F}(\mathbf{1}_{H^{\epsilon}(\mathsf{m},\beta)\cap W})(\xi)\mathbf{t}^{\mathbf{n}}\\ &= \mathbf{L}^{(1-g)n}\sum_{\xi\in k(\mathscr{C})^{n}}\prod_{v\in\mathscr{C}}\left(\sum_{\mathsf{m}_v\in\mathbf{N}^{\mathscr{A}}} \mathscr{F}(\mathbf{1}_{H^{\epsilon}(\mathsf{m}_v,\beta_v)_v\cap W_v})(\xi_v)\mathbf{t}^{\mathsf{m}_v}\right)\\ &= \mathbf{L}^{(1-g)n}\sum_{\xi\in k(\mathscr{C})^{n}}\prod_{v\in\mathscr{C}} Z^{\epsilon,\beta}_{|W_v}(\mathbf{t},\xi)\end{aligned}$$

and since the Euler product is compatible with finite products, we have

$$\begin{aligned}\mathscr{Z}^{\epsilon,\beta}_{|W}(\mathbf{u}) =& \mathbf{L}^{(1-g)n} \sum_{\xi\in k(\mathscr{C})^n} \sum_{\substack{\mathbf{n}_U\in\mathbf{N}^{\mathscr{A}_U}\\ \mathbf{n}_D\in\mathbf{N}^{\mathscr{A}_D}}} \sum_{\substack{\mathsf{m}\in S^{\mathbf{n}_U}(\mathscr{C})\\ \mathsf{m}_S\in S^{\mathbf{n}_D}(\mathscr{C}\setminus\mathscr{C}_0)}} \mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m}+\mathsf{m}_S,\beta)\cap W})(\xi)\mathbf{u}^{\mathbf{n}_U+\mathsf{m}_S}\\ &= \mathbf{L}^{(1-g)n} \sum_{\xi\in k(\mathscr{C})^n} \left(\prod_{v\in\mathscr{C}_0}\left(\sum_{\mathsf{m}_v\in\mathbf{N}^{\mathscr{A}_U}} \mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m}_v,\beta_v)_v\cap W_v})(\xi_v)\mathbf{u}^{\mathsf{m}_v}\right)\right.\\ &\qquad \left.\times\prod_{s\in S}\left(\sum_{\mathsf{m}_s\in\mathbf{N}^{\mathscr{A}_U}\times\mathbf{N}^{\{s\}\times\mathscr{A}_D}} \mathscr{F}(\mathbf{1}_{H(\mathsf{m}_s,\beta_s)_s})(\xi_s)\mathbf{u}^{\mathsf{m}_s}\right)\right)\\ &= \mathbf{L}^{(1-g)n} \sum_{\xi\in k(\mathscr{C})^n}\prod_{v\in\mathscr{C}_0} \mathscr{Z}^{\epsilon,\beta_v}_{|W_v}(\mathbf{u}_0,\xi)\prod_{s\in S}\mathscr{Z}^{\beta_s}\left((\mathbf{u}_0,\mathbf{u}_s),\xi\right).\end{aligned}$$

This last decomposition is consistent with the one of $Z^{\epsilon,\beta}_{|W}(\mathbf{t})$ if one applies the specialisation of Remark 4.2.3. Combined with Equation (4.2.3.17) it finally gives

$$\mathscr{Z}^{\epsilon}_{|W}(\mathbf{u}) = \mathbf{L}^{(1-g)n}\sum_{\xi\in k(C)^n}\prod_{v\in\mathscr{C}}\left(\sum_{\beta_v\in\mathscr{B}^{\mathscr{U}}_v}\mathbf{u}^{\mathbf{e}^{\beta_v}}\mathscr{Z}^{\beta_v}_v(\mathbf{u},\xi_v)\right).$$

In this expression, the summation over $k(\mathscr{C})^n$ is actually a summation over $L(\tilde{E})^n$, where $L(\tilde{E})$ is the Riemann-Roch space of the k-divisor

$$\tilde{E} = -\sum_v(\nu_v - s_v)[v], \tag{4.2.3.18}$$

$s:\mathscr{C}\to\mathbf{Z}$ being the almost zero function of [**Bil23**, Proposition 6.2.3.3] and ν being the order function of $\omega\in\Omega_{k(\mathscr{C})/k}$ defined at the beginning of Section 2.2.3.

Hence the multivariate zeta functions $\mathscr{Z}^{\epsilon}_{|W}(\mathbf{u})$ and $Z^{\epsilon}_{|W}(\mathbf{t})$ can be written as

$$\mathscr{Z}^{\epsilon}_{|W}(\mathbf{u}) = \mathbf{L}^{(1-g)n}\sum_{\xi\in V}\mathscr{Z}^{\epsilon}_{|W}(\mathbf{u},\xi)$$
$$Z^{\epsilon}_{|W}(\mathbf{t}) = \mathbf{L}^{(1-g)n}\sum_{\xi\in V}Z^{\epsilon}_{|W}(\mathbf{t},\xi)$$

where V is the finite dimensional k-vector space

$$V = L(\tilde{E})^n$$

and each $\mathscr{Z}^{\epsilon}_{|W}(\mathbf{u},\xi)$ (respectively $Z^{\epsilon}_{|W}(\mathbf{t},\xi)$) can be expressed as a motivic Euler product with local factors

$$\mathscr{Z}^{\epsilon}_{|W_v}(\mathbf{u},\xi) = \sum_{\beta_v\in\mathscr{B}_{\mathscr{U},v}}\mathbf{u}^{\mathbf{e}^{\beta_v}}\mathscr{Z}^{\epsilon,\beta_v}_{|W_v}(\mathbf{u},\xi)$$

(resp. $Z^{\epsilon}_{|W_v}(\mathbf{t},\xi)$) for any place $v\in\mathscr{C}$. For any ξ in V, we will first study the asymptotic behaviour of the $\mathbf{n}$-th coefficient of the Euler product

$$\prod_{v\in\mathscr{C}}\mathscr{Z}^{\epsilon}_{|W_v}(\mathbf{u},\xi)$$

when $\min_{(\alpha,s)\in\mathscr{A}\times(\{0\}\cup S)}(n_{\alpha,s})$ tends to infinity. When we restrict this product to $\mathscr{C}_0$, its local factors coincide with the ones of the coarse version $Z(\mathbf{t},\xi)$. Therefore it is natural for us to identify them and keep the notations of [**Bil23**, **CLL16**] for places of $\mathscr{C}_0$.

Note that this notation is slightly abusive: it actually means that we restrict a certain relative motivic Euler product to a constructible subset of V containing ξ. Indeed, recall that we use the commutativity of the diagram

$$\begin{array}{ccc} \mathscr{E}xp\mathscr{M}_{V\times \mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} & \longrightarrow & \mathscr{E}xp\mathscr{M}_{\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \\ \downarrow & & \downarrow \\ \mathscr{E}xp\mathscr{M}_V & \longrightarrow & \mathscr{E}xp\mathscr{M}_k \end{array}$$

while summing over ξ and m the families $\left(\theta^*_{\mathbf{n}}\mathscr{F}(\mathbf{1}_{H(\mathsf{m},\beta)})\right)_{\mathsf{m}\in\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}} \in \mathscr{E}xp\mathscr{M}_{V\times\mathrm{Sym}^{\mathbf{n}}_{/k}\mathscr{C}}$. In particular, one can understand

$$\prod_{v\in\mathscr{C}}\left(\sum_{\mathsf{m}_v\in\mathbf{N}^{\mathscr{A}}}\mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m}_v,\beta_v)_v\cap W_v})\mathbf{t}^{\mathsf{m}_v}\right) = \sum_{\mathbf{n}\in\mathbf{N}^{\mathscr{A}}}\sum_{\mathsf{m}\in S^{\mathbf{n}}(C)}\mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m},\beta)\cap W})\mathbf{t}^{\mathbf{n}}$$

as a relative motivic Euler product with coefficients in $\mathscr{E}xp\mathscr{M}_V$ (in which we omit $\theta^*_{\mathbf{n}}$). Its convergence will be studied with respect to a finite constructible partition of V (described in Section 4.4.3). Since the motivic sum

$$\sum_{\xi\in V} : \mathscr{E}xp\mathscr{M}_V \to \mathscr{E}xp\mathscr{M}_k$$

is a morphism, it is compatible with such a cutting process. In the end, the remaining task will be to justify that one can actually permute the sum over rational points $\sum_{\xi\in k(\mathscr{C})^n}$ and the limit of the coefficients of $\mathscr{Z}(\mathbf{u},\xi)$.

It is shown in [**CLL16**] that the local factor of $Z(\mathbf{t})$ can be rewritten as a motivic integral over the arc spaces $\Omega_v(A,\beta)$ defined in Section 4.1. When $v\in\mathscr{C}_0$ and $W_v = G(F_v)$, this procedure gives

$$Z^\epsilon_v(\mathbf{t},\xi) = \sum_{\beta_v\in\mathscr{B}^{\mathscr{U}}_v}\mathbf{t}^{e^{\beta_v}}Z^{\epsilon,\beta_v}_v(\mathbf{t},\xi) = \sum_{\substack{A\subset\mathscr{A}_U\\ \beta_v\in\mathscr{B}^{\mathscr{U}}_v}}\mathbf{t}^{e^{\beta_v}}\mathbf{L}^{\rho^{\beta_v}}\int_{\Omega^\epsilon_v(A,\beta_v)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\mathrm{ord}_v(x_\alpha)}\mathbf{e}(\langle x,\xi\rangle)\mathrm{d}x$$

(see [**CLL16**, §6.1-2] and [**Bil23**, §6.3.1]). The ρ-weight-linear convergence of the local factor at $t_\alpha = \mathbf{L}^{-\rho'_\alpha}$ is actually proved in [**Bil23**, Lemma 6.3.5.1]. Above places $s\in S$ such that $W_s = G(F_s)$, the expression looks similar, replacing $\mathscr{A}_U$ by $\mathscr{A}$

$$Z_s(\mathbf{t},\xi) = \sum_{\beta_s\in\mathscr{B}^{\mathscr{U}}_s}\mathbf{t}^{e^{\beta_s}}Z^{\beta_s}_s(\mathbf{t},\xi) = \sum_{\substack{A\subset\mathscr{A}\\ \beta_s\in\mathscr{B}^{\mathscr{U}}_s}}\mathbf{t}^{e^{\beta_s}}\int_{\Omega_s(A,\beta)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\mathrm{ord}_s(x_\alpha)}\mathbf{e}(\langle x,\xi\rangle)\mathrm{d}x.$$

Actually Lemmas 6.1.1 and 6.2.6 in [**CLL16**] say that for every motivic residual function h on the arc space of $\mathscr{X}_{\mathscr{O}_v}$ and every character $\xi\in G(F_v)^\vee$

$$\int_{G(F_v)}\mathbf{t}^{(g,\underline{\mathscr{L}})_v}h(g)\mathbf{e}(\langle g,\xi\rangle)\mathrm{d}g = \sum_{\substack{A\subset\mathscr{A}\\ \beta_v\in\mathscr{B}^{\mathscr{U}}_v}}\mathbf{t}^{e^{\beta_v}}\int_{\Omega_v(A,\beta)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\mathrm{ord}_v(x_\alpha)}h(x)\mathbf{e}(\langle x,\xi\rangle)\mathrm{d}x.$$

One can take for h the characteristic function of $\Omega^\epsilon_v(A,\beta)\cap W_v$ and then can replace $\Omega_v(A,\beta)$ by $\Omega^\epsilon_v(A,\beta)\cap W_v$ in the expressions above. This operation does not modify the local factor above $\mathscr{C}_1$ and provides a unified expression

$$Z^\epsilon_v(\mathbf{t},\xi) = \sum_{\substack{A\subset\mathscr{A}\\ \beta_v\in\mathscr{B}^{\mathscr{U}}_v}}\mathbf{t}^{e^{\beta_v}}\mathbf{L}^{\rho^{\beta_v}}\int_{\Omega^\epsilon_v(A,\beta_v)\cap W_v}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\mathrm{ord}_v(x_\alpha)}\mathbf{e}(\langle x,\xi\rangle)\mathrm{d}x$$

valid for all $v \in \mathscr{C}(k)$.

Remark 4.2.4. Remark that for any character ξ and place $v \in \mathscr{C}_0(k)$ we have a decomposition

$$Z_v^{\epsilon,\beta_v}(\mathbf{t},\xi) = \sum_{A\subset \mathscr{A}_U} \mathbf{t}^{e^{\beta_v}} Z_{v,A}^{\epsilon,\beta_v}(\mathbf{t},\xi)$$

where $Z_{v,A}^{\epsilon,\beta_v}(\mathbf{t},\xi)$ only depends on the indeterminates $(t_\alpha)_{\alpha\in A}$.

This remark remains valid for places s of S, if one just replaces $\mathscr{A}_U$ by $\mathscr{A}$ in the decomposition above, and adapting this decomposition to $\mathscr{Z}(\mathbf{u},\xi)$ is straightforward.

4.3. Convergence of Euler products in our setting

As the reader familiar with Tauberian theorem in complex analysis might guess, the asymptotic behaviour of the coefficients of $Z(t)$ is closely linked to the poles of the series. The precise study of these poles has been done by Bilu in [**Bil23**, Chapter 6]. In this subsection we will both recall and adapt the relevant results for our purpose; in particular, we precisely check weight-linear convergence of these products, which is locally uniform with respect to a stratification of the space of characters. This information will be crucial both for the control of various error terms appearing through the last section of this paper and for the final step of our proof.

Since the motivic Euler product notation is compatible with finite products, it will be enough to prove convergence over $\mathscr{C}_0$ (and even $\mathscr{C}_1$). As we already pointed out, in this situation the local factors of the coarse zeta function and refined zeta function coincide. Therefore the content of this paragraph stays valid if one replaces everywhere $Z_v(\mathbf{t},\xi)$ by $\mathscr{Z}_v(\mathbf{u},\xi)$ and the inderterminates t_α by U_α, $\alpha \in \mathscr{A}_U$, in Notation 4.3.1 and Notation 4.3.4 below.

For any tuple $\mathbf{n} = (n_\alpha)_{\alpha\in\mathscr{A}}$ of integers and any $\rho' \in \mathbf{Q}^{\mathscr{A}}$, we will use freely the pairing notation

$$\langle \rho', \mathbf{n}\rangle = \sum_{\alpha\in\mathscr{A}} \rho'_\alpha n_\alpha$$

as well as its obvious restriction $\langle \rho', \mathbf{n}\rangle_A$ to any subset A of $\mathscr{A}$.

Furthermore, for any $\mathbf{x},\mathbf{y} \in \mathbf{Q}^{\mathscr{A}}$, $\mathbf{xy}$ is the coordinatewise product $(x_\alpha y_\alpha)_{\alpha\in\mathscr{A}}$, and if all the coordinates of $\mathbf{y}$ are non-zero, $\frac{\mathbf{x}}{\mathbf{y}}$ is the tuple $(x_\alpha/y_\alpha)_{\alpha\in\mathscr{A}}$.

4.3.1. Trivial character.

In this paragraph we study the main term of the motivic Zeta function.

Notation 4.3.1. For any place v of $\mathscr{C}$ and any choice of vertical components $\beta \in \prod_v \mathscr{B}_v^{\mathscr{U}}$ we set

$$F_v^{\epsilon,\beta}(\mathbf{t},0) = Z_v^{\epsilon,\beta}(\mathbf{t},0) \prod_{\alpha\in\mathscr{A}_U} \left(1 - (\mathbf{L}^{\rho_\alpha - 1} t_\alpha)^{m_\alpha}\right).$$

One can directly compute the local factors corresponding to the trivial character [**CLL16**, §6]. First assume that v is a place of $\mathscr{C}_0$. Then,

$$\begin{aligned}
Z_v^{\epsilon}(\mathbf{t},0) &= \sum_{\beta\in\mathscr{B}_v^{\mathscr{U}}} \mathbf{t}^{e^{\beta_v}} Z_v^{\epsilon,\beta_v}(\mathbf{t},0) \\
&= \sum_{\beta_v\in\mathscr{B}_v^{\mathscr{U}}} \mathbf{t}^{e^{\beta_v}} \mathbf{L}^{\rho^\beta} \sum_{A\subset\mathscr{A}_U} [\Delta_v(A,\beta)] \mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|} \prod_{\alpha\in A} \frac{(\mathbf{L}^{\rho_\alpha - 1} t_\alpha)^{m_\alpha}}{1 - \mathbf{L}^{\rho_\alpha-1} t_\alpha}
\end{aligned}$$

since for any $A \subset \mathscr{A}_U$ and β one has

$$\begin{aligned}
&\int_{\Omega_v^{\epsilon}(A,\beta_v)} \prod_{\alpha\in A} (\mathbf{L}^{\rho_\alpha} t_\alpha)^{\mathrm{ord}_v(x_\alpha)}\,\mathrm{d}x \\
&= \sum_{\substack{\mathbf{m}'\in\mathbf{Z}^A_{>0}\\ \mathbf{m}'\geqslant\mathbf{m}}} \int_{\substack{\Delta_v(A,\beta)\times\mathscr{L}(\mathbf{A}^1)^A\times\mathscr{L}(\mathbf{A}^1,0)^{n-|A|}\\ \mathrm{ord}_v\, x_\alpha = m'_\alpha}} \prod_{\alpha\in A} (\mathbf{L}^{\rho_\alpha} t_\alpha)^{m'_\alpha}\,\mathrm{d}x \\
&= \sum_{\substack{\mathbf{m}'\in\mathbf{Z}^A_{>0}\\ \mathbf{m}'\geqslant\mathbf{m}}} [\Delta_v(A,\beta)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|} \prod_{\alpha\in A} \mathbf{L}^{-m'_\alpha} (\mathbf{L}^{\rho_\alpha} t_\alpha)^{m'_\alpha} \\
&= [\Delta_v(A,\beta)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|} \prod_{\alpha\in A} \frac{(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}.
\end{aligned}$$

In case v is a place of the dense subset $\mathscr{C}_1 \subset \mathscr{C}_0$, since $\mathscr{B}_v = \{\beta_v\}$ and both e^β_α and ρ_β equal zero, this expression becomes slightly nicer:

$$Z_v(\mathbf{t},0) = Z_v^{\beta_v}(\mathbf{t},0) = \sum_{A\subset\mathscr{A}_U} [\Delta_v(A)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|} \prod_{\alpha\in A} \frac{(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}.$$

Concerning places of $S = \mathscr{C}\setminus\mathscr{C}_0$, the local factor becomes

$$Z_v(\mathbf{t},0) = \sum_{\substack{\beta\in\mathscr{B}_v^{\mathscr{U}}\\ A\subset\mathscr{A}}} \mathbf{t}^{e^{\beta_v}}\mathbf{L}^{\rho^\beta}[\Delta_v(A,\beta)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|} \prod_{\alpha\in A} \frac{\mathbf{L}^{\rho_\alpha-1}t_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}.$$

We will use these expressions to prove the following proposition.

Proposition 4.3.2. *Let*

$$M = \left\langle \rho-\epsilon, \frac{1+\epsilon}{1-\epsilon}\right\rangle_{\mathscr{A}_U}.$$

The $\mathbf{m}'$*-th coefficient of*

$$\prod_{v\in\mathscr{C}_0} F_v^{\epsilon,\beta}(\mathbf{t},0)$$

has weight bounded by

$$\left(2-\frac{1}{M}\right)\langle\rho-\epsilon,\mathbf{m}'\rangle + 2\rho^\beta_{\mathscr{C}_0}$$

where $\rho^\beta_{\mathscr{C}_0}$ *is the finite sum* $\sum_{v\in\mathscr{C}_0(k)}\rho^{\beta_v}$. *Thus this product is* $(\rho-\epsilon)$*-convergent for* $|\mathbf{t}| < \mathbf{L}^{-1+\frac{1}{2M}}$ *and is* $(\rho-\epsilon)$*-weight-linearly convergent at* $t_\alpha = \mathbf{L}_k^{-(\rho_\alpha-\epsilon_\alpha)}$ *with explicit control of the error term. The resulting sum is a non-zero effective element of* $\widehat{\mathscr{M}_{k,r}}^{w}$, *where* $r = \prod_{\alpha\in\mathscr{A}_U}\frac{1}{m_\alpha}$.

Note that the term $\rho^\beta_{\mathscr{C}_0}$ in the upper bound comes from the $\mathbf{L}^{\rho_v}$ appearing in the expression of the local factor given page 97, see also the definition of $Z^{\epsilon,\beta}_{|W}(\mathbf{t},0)$ given page 93

Proof. The proof is a variant of the one of Proposition 6.3.5.2 of [**Bil23**]. It is important to check that we are in the situation of Proposition 2.4.29, in order to obtain linear convergence with respect to $\rho-\epsilon$.

One can assume $\mathscr{A}_U = \mathscr{A}$. For a place $v\in\mathscr{C}_1$ the local factor

$$F_v^\epsilon(\mathbf{t},0) = \prod_{\alpha\in\mathscr{A}} (1-(\mathbf{L}^{\rho_\alpha-1}\mathbf{t})^{m_\alpha})Z_v(\mathbf{t},0)$$

is given by

$$F_v^\epsilon(\mathbf{t},0) = \sum_{A\subset\mathscr{A}} [\Delta_v(A)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|} \prod_{\alpha\in A}\sum_{\ell_\alpha=0}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha} \prod_{\alpha\in\mathscr{A}\setminus A}\left(1-\left(\mathbf{L}^{\rho_\alpha-1}t_\alpha\right)^{m_\alpha}\right)$$

If we expand the product over $\mathscr{A}\setminus A$ for any $A\subset\mathscr{A}$ we get

$$\prod_{\alpha'\in\mathscr{A}\setminus A}(1-(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_{\alpha'}}) = \sum_{B\subset\mathscr{A}\setminus A}(-1)^{|B|}\prod_{\alpha'\in B}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_\alpha}$$

and

$$F_v^\epsilon(\mathbf{t},0) = \sum_{A\subset\mathscr{A}}\sum_{B\subset\mathscr{A}\setminus A}(-1)^{|B|}[\Delta_v(A)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|}\prod_{\alpha\in A}\sum_{\ell_\alpha=0}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha}\prod_{\alpha'\in B}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_{\alpha'}}.$$

Remark that $[\Delta_v(\varnothing)] = [\mathbf{G}_a^n] = \mathbf{L}^n$. Then the previous sums can be decomposed with respect to the cardinalities of the sets A and B:

$$\begin{aligned}
F_v^\epsilon(\mathbf{t},0) = 1 &- \sum_{\alpha'\in\mathscr{A}}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_{\alpha'}} + \sum_{\substack{B\subset\mathscr{A}\\|B|\geqslant 2}}(-1)^{|B|}\prod_{\alpha'\in B}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_{\alpha'}} && (A=\varnothing)\\
&+\sum_{\alpha\in\mathscr{A}}[\Delta_v(\{\alpha\})]\mathbf{L}^{-n+1}(1-\mathbf{L}^{-1})\sum_{\ell_\alpha=0}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha} && (|A|=1,\ B=\varnothing)\\
&+\sum_{\alpha\in\mathscr{A}}\sum_{\substack{B\subset\mathscr{A}\setminus\{\alpha\}\\B\neq\varnothing}}(-1)^{|B|}[\Delta_v(\{\alpha\})]\mathbf{L}^{-n+1}(1-\mathbf{L}^{-1})\sum_{\ell_\alpha=0}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha}\prod_{\alpha'\in B}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m'_\alpha}\\
& && (|A|=1,\ B\neq\varnothing)\\
&+\sum_{\substack{A\subset\mathscr{A}\\|A|\geqslant 2}}\sum_{B\subset\mathscr{A}\setminus A}(-1)^{|B|}[\Delta_v(A)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|}\prod_{\alpha\in A}\sum_{\ell_\alpha=0}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha}\prod_{\alpha'\in B}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_{\alpha'}}.\\
& && (|A|\geqslant 2)
\end{aligned}$$

The definition of $\Delta_v(A)$ gives the equality of classes

$$[\mathscr{D}_{\alpha,v}] = [\Delta_v(\{\alpha\})] + \sum_{\substack{A'\subset\mathscr{A}\setminus\{\alpha\}\\A'\neq\varnothing}}[\Delta_v(A'\cup\{\alpha\})]$$

for every $\alpha\in\mathscr{A}$. Finally

$$F_v^\epsilon(\mathbf{t},0) = 1+\sum_{\alpha\in\mathscr{A}}\left([\mathscr{D}_{\alpha,v}]-\mathbf{L}^{n-1}\right)\mathbf{L}^{-(n-1)}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}+P_v^\epsilon(\mathbf{t})$$

where $P_v^\epsilon(\mathbf{t})$ is the polynomial

$$P_v^\epsilon(\mathbf{t}) =$$

$$\sum_{\substack{B\subset\mathscr{A}\\|B|\geqslant 2}}(-1)^{|B|}\prod_{\alpha'\in B}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_\alpha} \tag{4.3.1.19}$$

$$-\sum_{\alpha\in\mathscr{A}}\sum_{\substack{A'\subset\mathscr{A}\setminus\{\alpha\}\\A'\neq\varnothing}}[\Delta_v(A'\cup\{\alpha\})]\mathbf{L}^{-n+1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha} \tag{4.3.1.20}$$

$$+\sum_{\alpha\in\mathscr{A}}[\Delta_v(\{\alpha\})]\mathbf{L}^{-n+1}(1-\mathbf{L}^{-1})\sum_{\ell_\alpha=1}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha} \tag{4.3.1.21}$$

$$-\sum_{\alpha\in\mathscr{A}}[\Delta_v(\{\alpha\})]\mathbf{L}^{-n}\sum_{\ell_\alpha=0}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha} \tag{4.3.1.22}$$

$$+\sum_{\substack{A\subset\mathscr{A}\\A\neq\varnothing}}\sum_{\substack{B\subset\mathscr{A}\setminus A\\|A|=1\Rightarrow B\neq\varnothing}}(-1)^{|B|}[\Delta_v(A)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|}\prod_{\alpha\in A}\sum_{\ell_\alpha=0}^{m_\alpha-1}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha+\ell_\alpha}\prod_{\alpha'\in B}(\mathbf{L}^{\rho_{\alpha'}-1}t_{\alpha'})^{m_{\alpha'}}. \tag{4.3.1.23}$$

Let us analyse the dimensions of the coefficients of this polynomial term by term, and compare them to the multidegrees of the corresponding monomials.

— In the sum of the first line, since $|B|\geqslant 2$, the corresponding coefficient has dimension

$$\sum_{\alpha\in B}m_\alpha(\rho_\alpha-1)=\sum_{\alpha\in B}m_\alpha(\rho_\alpha-\epsilon_\alpha)-|B|\leqslant\sum_{\alpha\in B}m_\alpha(\rho_\alpha-\epsilon_\alpha)-2.$$

Here the corresponding monomial is $\mathbf{t}^{\mathbf{m}'}$ with $m'_\alpha=m_\alpha$ if $\alpha\in B$ and 0 otherwise; in this case $\sum_{\alpha\in B}m'_\alpha(\rho_\alpha-\epsilon_\alpha)=\langle\rho-\epsilon,\mathbf{m}'\rangle$.

— Concerning the sum of the second line, since we assumed that $\mathscr{X}$ is a good model, the dimension of $\Delta_v(A'\cup\{\alpha\})$ is at most $n-2$ and the whole coefficient has dimension at most

$$m_\alpha(\rho_\alpha-1)-1=m_\alpha(\rho_\alpha-\epsilon_\alpha)-2.$$

— The coefficient of $t_\alpha^{m_\alpha+\ell_\alpha}$ in the third line only appears if $m_\alpha\geqslant 2$ and has dimension

$$\begin{aligned}(\rho_\alpha-1)(m_\alpha+\ell_\alpha)&=(\rho_\alpha-\epsilon_\alpha)(m_\alpha+\ell_\alpha)+(\epsilon_\alpha-1)(m_\alpha+\ell_\alpha)\\&=(\rho_\alpha-\epsilon_\alpha)(m_\alpha+\ell_\alpha)-\frac{m_\alpha+\ell_\alpha}{m_\alpha}\\&\leqslant(m_\alpha+\ell_\alpha)(\rho_\alpha-\epsilon_\alpha)-\frac{3}{2}\end{aligned}$$

since $\ell_\alpha\geqslant 1$ in this case.

— The coefficient of $t_\alpha^{m_\alpha+\ell_\alpha}$ in the fourth line only appears if $m_\alpha\geqslant 2$ but ℓ_α can be zero this time. It has dimension

$$\begin{aligned}-1+(m_\alpha+\ell_\alpha)(\rho_\alpha-1)&=(m_\alpha+\ell_\alpha)(\rho_\alpha-\epsilon_\alpha)-1+(\epsilon_\alpha-1)(m_\alpha+\ell_\alpha)\\&\leqslant(m_\alpha+\ell_\alpha)(\rho_\alpha-\epsilon_\alpha)-2.\end{aligned}$$

— Concerning the coefficients in the fifth and last sum they have dimension

$$\sum_{\alpha\in A\cup B}(m_\alpha+\ell_\alpha)(\rho_\alpha-1)$$

(with the convention $\ell_\alpha=0$ whenever $\alpha\in B$) which can be rewritten

$$\sum_{\alpha\in A\cup B}(m_\alpha+\ell_\alpha)(\rho_\alpha-\epsilon)-\sum_{\alpha\in A\cup B}\frac{m_\alpha+\ell_\alpha}{m_\alpha}$$

Since $|A\cup B|\geqslant 2$ the second sum is at most -2 and an upper bound is this is bounded by

$$\langle\rho-\epsilon,m+\ell\rangle-2$$

It corresponds to the monomial $\mathbf{t}^{\mathbf{m}'}$ with $m'_\alpha=(m_\alpha+\ell_\alpha)$ if $\alpha\in A\cup B$ and 0 otherwise; in this case

$$\langle\rho-\epsilon,\mathbf{m}'\rangle=\sum_{\alpha\in A\cup B}(m_\alpha+\ell_\alpha)(\rho_\alpha-\epsilon_\alpha).$$

Remark that our computations do not depend on v in the sense that there exist polynomials $F^\epsilon(\mathbf{t},0)$ and $P^\epsilon(\mathbf{t})$ with coefficients in $\mathscr{E}xp\mathscr{M}_{\mathscr{C}_1}$, such that their pull-backs by v are respectively F_v^ϵ and P_v^ϵ. Therefore we will be able to use Proposition 2.4.29 with $X=\mathscr{C}_1$, $b=\frac{1}{2}$,

$$\begin{aligned}M&=\sum_{\alpha\in\mathscr{A}_U}\underbrace{(2m_\alpha-1)}_{=\frac{1+\epsilon_\alpha}{1-\epsilon_\alpha}}(\rho_\alpha-\epsilon_\alpha)\\&=\left\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\right\rangle_{\mathscr{A}_U}\end{aligned}$$

and $\beta=0$ so that the first condition of Proposition 2.4.29 becomes in our case

$$w_{\mathscr{C}_1}(\mathfrak{c}_{\mathbf{m}})\leqslant 2\left(\langle\rho-\epsilon,\mathbf{m}\rangle-1\right). \tag{4.3.1.24}$$

By the last property of the weight recalled in Proposition 2.4.3, we obtain the crucial argument of this proof, which is the inequality

$$\begin{aligned}w_{\mathscr{C}_1}\left(\left([\mathscr{D}_{\alpha,v}]-\mathbf{L}^{n-1}\right)\mathbf{L}^{-(n-1)}\mathbf{L}^{m_\alpha(\rho_\alpha-1)}\right)&\leqslant 2(n-1)-2(n-1)+2m_\alpha(\rho_\alpha-1)\\&=2m_\alpha(\rho_\alpha-1)\\&=2m_\alpha(\rho_\alpha-\epsilon_\alpha)-2\end{aligned}$$

As Bilu points out in [**Bil23**, Remarks 6.3.4.2-3], the bounds on dimensions obtained in the previous paragraph ensure that the coefficients of multidegree $\mathbf{m}$ of $P_v(\mathbf{t})$ satisfy (4.3.1.24). Thus we can apply Proposition 2.4.29 and $(\rho-\epsilon)$-weight-linear convergence over $\mathscr{C}_1$ at $t_\alpha=\mathbf{L}^{-\rho_\alpha}$ follows. Since the Euler product notation is compatible with finite products and since we already know that the local factors converge weight-linearly, applying Lemma 2.4.27 we deduce that the $(\rho-\epsilon)$-weight-linear convergence holds for the product over $\mathscr{C}_0$. □

Remark 4.3.3. Assume that $\epsilon_{\mathscr{A}_U}=\mathbf{0}$ and that W_v is trivial above every place. Since for any place $v\in\mathscr{C}_1$ the fibre $\mathscr{U}_v$ is the disjoint union of all the $\Delta_v(A)$ for $A\subset\mathscr{A}_U$, and in general $E_{\beta,v}$ is the disjoint union of all the $\Delta_v(A,\beta)$, it is straightforward to check that the value at $t_\alpha = \mathbf{L}^{-\rho'_\alpha}$ of the motivic Euler product $\prod_{v\in\mathscr{C}_0} \mathbf{t}^{\mathbf{e}^{\beta_v}} F_v^{\beta}(\mathbf{t},0)$ is

$$\begin{aligned}
&\prod_{v\in\mathscr{C}_0}\left(\mathbf{t}^{\mathbf{e}^{\beta_v}} Z_v^{\beta_v}(\mathbf{t},0)\prod_{\alpha\in\mathscr{A}_U}(1-\mathbf{L}^{\rho_\alpha-1}t^{\rho_\alpha})\right)\left((\mathbf{L}^{-\rho_\alpha})_{\alpha\in\mathscr{A}}\right)\\
&=\prod_{v\in\mathscr{C}_1}\left((1-\mathbf{L}^{-1})^{\mathrm{rk}(\mathrm{Pic}(U))}\frac{[\mathscr{U}_v]}{\mathbf{L}^n}\right)\times\prod_{v\in\mathscr{C}_0\setminus\mathscr{C}_1}\left((1-\mathbf{L}^{-1})^{\mathrm{rk}(\mathrm{Pic}(U))}\mathbf{L}^{\rho^{\beta_v}-\langle\rho,\mathbf{e}^{\beta}\rangle}\frac{[E^\circ_{\beta_v}]}{\mathbf{L}^n}\right).
\end{aligned}$$

The local term $\mathbf{L}^{\rho^{\beta_v}-\langle\rho,\mathbf{e}^{\beta}\rangle}[E^\circ_{\beta_v}]\mathbf{L}^{-n}$ (which is just $[\mathscr{U}_v]\mathbf{L}^{-n}$ if $v\in\mathscr{C}_1$) can be interpreted as the motivic integral

$$\int_{H(F_v,\beta_v)}\mathbf{L}^{-(g,\mathscr{L}_{\rho'})}|\omega_X|$$

over the local space $H(F_v,\beta_v)=\{g\in H(F_v)\mid (g,E_{\beta_v})=1\}$. By [**CLL16**, Lemma 6.1.1] this integral can we rewritten as the motivic integral over arc spaces

$$\begin{aligned}
\int_{\mathscr{L}(\mathscr{U}_{\mathcal{O}_v},E_{\beta_v})}\mathbf{L}^{-\mathrm{ord}_{\mathscr{L}_{\rho'}}(x)}\mathbf{L}^{-\mathrm{ord}_{\omega_X}(x)} &= \int_{\mathscr{L}(\mathscr{U}_{\mathcal{O}_v},E_{\beta_v})}\mathbf{L}^{-\sum_{\alpha\in\mathscr{A}_U}\rho_\alpha\left(\mathrm{ord}_v(x_\alpha)+e_\alpha^{\beta_v}\right)}\mathbf{L}^{\rho^{\beta_v}+\sum_{\alpha\in\mathscr{A}_U}\rho_\alpha\,\mathrm{ord}_v(x_\alpha)}\\
&=\int_{\mathscr{L}(\mathscr{U}_{\mathcal{O}_v},E_{\beta_v})}\mathbf{L}^{\rho^{\beta_v}-\langle\rho,\mathbf{e}^{\beta}\rangle}\\
&=\mathbf{L}^{\rho^{\beta_v}-\langle\rho,\mathbf{e}^{\beta}\rangle}\,\mu(\mathscr{L}(\mathscr{U}_{\mathcal{O}_v},E_{\beta_v}))\\
&=\mathbf{L}^{\rho^{\beta_v}-\langle\rho,\mathbf{e}^{\beta_v}\rangle}\frac{[E^\circ_{\beta_v}]}{\mathbf{L}^n}.
\end{aligned}$$

This comes from the fact that $\mathscr{X}(\mathcal{O}_v)=X(F_v)$ so that one can view $G(F)$ as a subset of $\mathscr{X}(\mathcal{O}_v)$ and any Schwartz-Bruhat function on $G(F_v)$ as a motivic function on $\mathscr{L}(\mathscr{X})$, see [**CLL16**, §6.1].

In general, looking at the expression of $Z^{\epsilon,\beta}_{|W,v}(\mathbf{t})$ given page 97, we see that the value at $t_\alpha=\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)}$ of the motivic Euler product $\prod_{v\in\mathscr{C}_0}\mathbf{t}^{\mathbf{e}^{\beta_v}}F_v^{\epsilon,\beta}(\mathbf{t},0)$ is given by the convergent effective motivic Euler product

$$\prod_{v\in\mathscr{C}_0}\left(1-\mathbf{L}^{-1}\right)^{|\mathscr{A}_U|}\int_{\mathscr{L}^\epsilon(\mathscr{X}_v,E_{\beta_v}|W_v)}\mathbf{L}^{-\mathrm{ord}_{\mathscr{L}_{\rho-\epsilon}}(x)}\mathbf{L}^{-\mathrm{ord}_{\omega_X}(x)}\in\widehat{\mathscr{M}_{k,r}}$$

where $\mathscr{L}^\epsilon(\mathscr{X}_v,E^\circ_{\beta_v}\mid W_v)$ is the subspace of arcs with origin in $E^\circ_{\beta_v}$, satisfying the Campana conditions above v and lying in W_v. By the valuative criterion of properness, such arcs correspond to elements of $W^\epsilon(F_v,\beta)=\{g\in H^\epsilon(F_v)\cap W_v\mid (g,E_{\beta_v})=1\}$

4.3.2. Non-trivial characters. Given a place $v\in\mathscr{C}$ and a non-trivial character $\xi\in G(F)^\vee$, the linear form $x\in G(F)\mapsto\langle x,\xi\rangle$ can be seen as a rational function f_ξ on X whose divisor of poles has support contained in the union of the D_α. We denote by $d_\alpha(\xi)$ the order of the pole of f_ξ with respect to D_α and define a subset of $\mathscr{A}_U$ by setting

$$\mathscr{A}_U^0(\xi)=\{\alpha\in\mathscr{A}_U\mid d_\alpha(\xi)=0\}.$$

If $U\neq X$, this is automatically a proper subset of $\mathscr{A}$. Otherwise if $U=X$, since X is projective and ξ is non-trivial, this is a proper subset of $\mathscr{A}$.

Notation 4.3.4. For any place v, any $\beta \in \prod_v \mathscr{B}_v^{\mathscr{U}}$ and any non-trivial character ξ, we write

$$F_v^{\epsilon,\beta}(\mathbf{t},\xi_v) = Z_v^{\epsilon,\beta}(\mathbf{t},\xi_v) \prod_{\alpha\in\mathscr{A}_U^0(\xi)} \left(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right).$$

Proposition 4.3.5. *Let*

$$c_\epsilon = 1 - \frac{1}{2}\min_{\alpha\in\mathscr{A}_U} \frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}.$$

The product

$$\prod_{v\in\mathscr{C}_0} F_v^{\epsilon,\beta}(\mathbf{t},\xi_v)$$

has its $\mathbf{m}'$*-th coefficient of weight bounded by*

$$2c_\epsilon\langle\rho-\epsilon,\mathbf{m}'\rangle + 2\rho^\beta_{\mathscr{C}_0}$$

where $\rho^\beta_{\mathscr{C}_0}$ *is the finite sum* $\sum_{v\in\mathscr{C}_0(k)}\rho^{\beta_v}$*. In particular, it* $(\rho-\epsilon)$*-converges for* $|\mathbf{t}| < \mathbf{L}^{-1+\frac{1}{2}\min_{\alpha\in\mathscr{A}_U}\frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}}$*. It is* $(\rho-\epsilon)$*-weight-linearly convergent at* $t_\alpha = \mathbf{L}^{-\rho_\alpha+\epsilon_\alpha}$ *and the resulting sum is a non-zero effective element of* $\widehat{\mathscr{E}xp\mathscr{M}}_{k,r}$*.*

The proof of this proposition we give here consists of a summary of the arguments of the proof of Proposition 6.3.5.3 in Bilu's thesis [**Bil23**], since our main interest lies in obtaining weight-linear convergence of $\prod_{v\in\mathscr{C}_0} F_v^\epsilon(\mathbf{t},\xi)$ at $t_\alpha = \mathbf{L}^{-\rho_\alpha+\epsilon_\alpha}$.

Note again that the term $2\rho^\beta$ in the upper bound comes from the contribution of the places v not in $\mathscr{C}_1$, for which there is a factor $\mathbf{L}^{\rho_v}$ appearing in our definition of $Z_v^{\epsilon,\beta}(\mathbf{t},\xi_v)$ given page 93.

Proof of Proposition 4.3.5. Once again it is enough to check convergence on the dense subset $\mathscr{C}_1 \subset \mathscr{C}$. In order to apply Proposition 2.4.29, one has to bound the weight of the coefficients of $F_v(\mathbf{t},\xi)$. We start by doing this for $Z_v(\mathbf{t},\xi)$.

First, the divisor of f_ξ can be written

$$\operatorname{div}(f_\xi) = \Xi_\xi - \sum_{\alpha\in\mathscr{A}} d_\alpha(\xi) D_\alpha$$

where Ξ_ξ is the Zariski-closure of $\{\langle x,\xi\rangle = 0\}$ in X. Over a place $v\in\mathscr{C}_1$, one has $\mathscr{B}_v^{\mathscr{U}} = \{\beta_v\}$ and the local factor $Z_v^\epsilon(\mathbf{t},\xi)$ can be written $Z_v^\epsilon(\mathbf{t},\xi) = \sum_{A\subset\mathscr{A}_U} Z_{v,A}^\epsilon(\mathbf{t},\xi)$ with

$$Z_{v,A}^\epsilon(\mathbf{t},\xi) = \int_{\Omega_v^\epsilon(A,\beta_v)} \prod_{\alpha\in A} (\mathbf{L}^{\rho_\alpha}t_\alpha)^{\operatorname{ord} x_\alpha}\mathbf{e}(f_\xi(x))\mathbf{d}x$$

as we already pointed out in Remark 4.2.4. It is possible to compute or at least bound the weight of this integral. We refer to [**Bil23**, §6.3.5.2] for the details of such computations; in this proof we will restrict ourselves to giving a list of results we need to apply Proposition 2.4.29. In particular we obtain weight-linear convergence of the Euler product $\prod_{v\in\mathscr{C}_1} F_v^\epsilon(\mathbf{t},\xi)$. In what follows $\mathscr{X}_{\mathcal{O}_v}$ is written $\mathscr{X}$ for conciseness and the index v may be dropped.

If $A=\varnothing$ then the integral above equals one. If $A=\{\alpha\}$ the intermediate step is to cut the integral into two pieces: one part corresponding to arcs with origin in the zero divisor Ξ_ξ of f_ξ and another part corresponding to arcs with origin outside Ξ_ξ.

$$\begin{aligned}Z^{\epsilon}_{v,\{\alpha\}}(\mathbf{t},\xi) = &\int_{\mathscr{L}(\mathscr{X},D^\circ_\alpha\setminus\Xi_\xi)\cap\Omega^\epsilon_v(\{\alpha\})}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\operatorname{ord}x_\alpha}\,\mathbf{e}(f_\xi(x))\mathbf{d}x\\ &+\int_{\mathscr{L}(\mathscr{X},D^\circ_\alpha\cap\Xi_\xi)\cap\Omega^\epsilon_v(\{\alpha\})}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\operatorname{ord}x_\alpha}\,\mathbf{e}(f_\xi(x))\mathbf{d}x\end{aligned}$$

Then there are several cases to distinguish, according to the order of the pole of f_ξ at D_α. On one hand, concerning arcs with origins outside Ξ_ξ, one has the following results.

— If $d_\alpha(\xi)=0$ then

$$\begin{aligned}&\int_{\mathscr{L}(\mathscr{X},D^\circ_\alpha\setminus\Xi_\xi)\cap\Omega^\epsilon_v(\{\alpha\})}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\operatorname{ord}x_\alpha}\,\mathbf{e}(f_\xi(x))\mathbf{d}x\\ &=(1-\mathbf{L}^{-1})\frac{(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}[D^\circ_\alpha\setminus\Xi_\xi]\mathbf{L}^{-(n-1)}.\end{aligned}$$

— If $d_\alpha(\xi)=1$ then

$$\begin{aligned}&\int_{\mathscr{L}(\mathscr{X},D^\circ_\alpha\setminus\Xi_\xi)\cap\Omega^\epsilon_v(\{\alpha\})}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\operatorname{ord}x_\alpha}\,\mathbf{e}(f_\xi(x))\mathbf{d}x\\ &=\begin{cases}-\mathbf{L}^{-2}[D^\circ_\alpha\setminus\Xi_\xi]\mathbf{L}^{-(n-1)}\mathbf{L}^{\rho_\alpha}t_\alpha & \text{if } m_\alpha=1\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Since $\dim(D^\circ_\alpha\setminus\Xi_\xi)=n-1$, one has the following upper bound on the weight

$$w_{\mathscr{C}_1}\left(-\mathbf{L}^2[D^\circ_\alpha\setminus\Xi_\xi]\mathbf{L}^{1-n}\mathbf{L}^{\rho_\alpha}\right)\leqslant 2(\rho_\alpha-2)+1\leqslant 2\rho_\alpha-1.$$

— If $d_\alpha(\xi)>1$ then this integral equals zero.

On the other hand, concerning arcs with origin in Ξ_ξ, the corresponding integral can be rewritten

$$\begin{aligned}&\int_{\mathscr{L}(\mathscr{X},D^\circ_\alpha\cap\Xi_\xi)\cap\Omega^\epsilon_v(\{\alpha\})}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\operatorname{ord}x_\alpha}\,\mathbf{e}(f_\xi(x))\mathbf{d}x\\ &=\sum_{m'\geqslant m_\alpha}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{m'}\int_{\substack{(D^\circ_\alpha\cap\Xi_\xi)\times\mathscr{L}(\mathbf{A}^1)\times\mathscr{L}(\mathbf{A}^1,0)^{n-1}\\ \operatorname{ord}x=m'}}\mathbf{e}(f_\xi(t^{m'}x,\mathbf{y}))\mathrm{d}x\mathbf{d}\mathbf{y}.\end{aligned}$$

By (4.1.4.3), the weight of the coefficient of order m is smaller than the weight of the motivic volume

$$\begin{aligned}&\mu\left(\{(w,x,\mathbf{y})\in(D^\circ_\alpha\cap\Xi_\xi)\times\mathscr{L}(\mathbf{A}^1)\times\mathscr{L}(\mathbf{A}^1,0)^{n-1}\mid\operatorname{ord}x=m'\}\right)\\ &=[D^\circ_\alpha\cap\Xi_\xi]\,\mathbf{L}^{-m'}\left(1-\mathbf{L}^{-1}\right)\mathbf{L}^{-n+1}.\end{aligned}$$

One has the upper bound on dimensions

$$\begin{aligned}\dim\left(\mathbf{L}^{\rho_\alpha m'}[D^\circ_\alpha\cap\Xi_\xi]\mathbf{L}^{-m'}\left(1-\mathbf{L}^{-1}\right)\mathbf{L}^{-n+1}\right)&\leqslant m'\rho_\alpha+(n-2)-m'-(n-1)\\ &=m'(\rho_\alpha-1)-1\\ &=m'(\rho_\alpha-\epsilon_\alpha)-\frac{m'}{m_\alpha}-1\end{aligned}$$

and thus on weights

$$\begin{aligned} w_{\mathscr{C}_1}\left(\mathbf{L}^{\rho_\alpha m'}[D_\alpha^\circ \cap \Xi_\xi]\mathbf{L}^{-m'}\left(1-\mathbf{L}^{-1}\right)\mathbf{L}^{-n+1}\right) &\leqslant 2(m'(\rho_\alpha - 1) - 1) + 1 \\ &= 2\left(m'(\rho_\alpha - \epsilon_\alpha) - \frac{m'}{m_\alpha} - 1\right) + 1. \end{aligned}$$

Setting

$$c = \max_{\alpha\in\mathscr{A}_U}\left(1 - \frac{1}{2\rho_\alpha}\right) \qquad \text{(as in [Bil23])}$$

$$c_\epsilon = \max_{\alpha\in\mathscr{A}_U}\left(1 - \frac{1}{2m_\alpha(\rho_\alpha - \epsilon_\alpha)}\right) \qquad \text{(for Campana curves)}$$

one gets respectively

$$2\rho_\alpha - 1 \leqslant 2\rho_\alpha c$$

and

$$2(\rho_\alpha - \epsilon_\alpha) - \frac{1}{m_\alpha} \leqslant 2c_\epsilon(\rho_\alpha - \epsilon_\alpha)$$

for every $\alpha \in \mathscr{A}_U$, so that

$$w_{\mathscr{C}_1}\left(\mathbf{L}^{\rho_\alpha m'}[D_\alpha^\circ \cap \Xi_\xi]\mathbf{L}^{-m'}\left(1-\mathbf{L}^{-1}\right)\mathbf{L}^{-n+1}\right) \leqslant 2cm'\rho_\alpha - 1$$

$$w_{\mathscr{C}_1}\left(\mathbf{L}^{\rho_\alpha m'}[D_\alpha^\circ \cap \Xi_\xi]\mathbf{L}^{-m'}\left(1-\mathbf{L}^{-1}\right)\mathbf{L}^{-n+1}\right) \leqslant 2c_\epsilon m'(\rho_\alpha - \epsilon_\alpha) - \frac{m'}{m_\alpha} - 1.$$

Now if $|A| \geqslant 2$ the idea is similar. The corresponding local term is

$$Z_{v,A} = \sum_{\substack{\mathbf{m}'\in\mathbf{N}_{>0}^A \\ m'_\alpha \geqslant m_\alpha}} \prod_{\alpha\in A} (\mathbf{L}^{\rho_\alpha} t_\alpha)^{m'_\alpha} \int_{\operatorname{ord} x_\alpha = m'_\alpha} \mathbf{e}(f_\xi(x,y))\mathbf{d}x\mathbf{d}y.$$

The volume of the constructible subsets of $\Omega_v(A)$ over which integration is done is actually

$$\begin{aligned} &\mu\left(\left\{(w,(x_\alpha)_{\alpha\in A},y) \in D_A^\circ \times \mathscr{L}(\mathbf{A}^1)^A \times \mathscr{L}(\mathbf{A}^1,0)^{n-|A|} \mid \operatorname{ord} x_\alpha = m'_\alpha,\ \alpha \in A\right\}\right) \\ &= [D_A^\circ] \prod_{\alpha\in A}\left(\mathbf{L}^{-m'_\alpha}\left(1-\mathbf{L}^{-1}\right)\right)\mathbf{L}^{-n+|A|}. \end{aligned}$$

By definition of D_A°, it has dimension at most $n-|A|$. Thus the dimension of the integral is at most $-\sum_{\alpha\in A} m'_\alpha$ and, using again the inequality of Section 4.1.4.3, the weight of the $\mathbf{m}$-th coefficient of $Z_{v,A}(\mathbf{t},\xi)$ is bounded by

$$2\sum_{\alpha\in A} m'_\alpha(\rho_\alpha - 1) + \dim\mathscr{C}_1 \leqslant 2c\sum_{\alpha\in A} m'_\alpha\rho_\alpha - 1.$$

and for Campana curves

$$\begin{aligned} 2\sum_{\alpha\in A}\left(m'_\alpha(\rho_\alpha - \epsilon_\alpha) - \frac{m'_\alpha}{m_\alpha}\right) + \dim\mathscr{C}_1 &\leqslant 2c_\epsilon\sum_{\alpha\in A} m'_\alpha(\rho_\alpha - \epsilon_\alpha) - \sum_{\alpha\in A}\frac{m'_\alpha}{m_\alpha} + 1 \\ &\leqslant 2c_\epsilon\sum_{\alpha\in A} m'_\alpha(\rho_\alpha - \epsilon_\alpha) - 1 \end{aligned}$$

To summarize, we obtained

$$Z_v^\epsilon(\mathbf{t},\xi) = 1 + \sum_{\substack{\alpha\in\mathscr{A}_U\\ d_\alpha(\xi)\leqslant 1}} Z^\epsilon_{v,\alpha}(\mathbf{t},\xi) + \sum_{|A|\geqslant 2} Z^\epsilon_{v,A}(\mathbf{t},\xi)$$

$$Z_v^\epsilon(\mathbf{t},\xi) = 1 + \left(1-\mathbf{L}^{-1}\right)\sum_{\alpha\in\mathscr{A}_U^0(\xi)} \frac{[D_\alpha^\circ\setminus \Xi_\xi]}{\mathbf{L}^{n-1}}\frac{(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha} - \mathbf{L}^{-n-1}\sum_{\substack{\alpha\in\mathscr{A}_U(\xi)\\ d_\alpha(\xi)=1\\ m_\alpha=1}}[D_\alpha^\circ\setminus\Xi_\xi]\mathbf{L}^{\rho_\alpha}t_\alpha$$

$$+ \text{ terms of weights bounded as in Proposition 2.4.29, locally uniformly in } \xi$$

The last step of the proof consists in multiplying by $\prod_{\alpha\in\mathscr{A}_U^0(\xi)}(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})$. This operation does not affect the bounds we need in order to apply Proposition 2.4.29, since $\rho_\alpha - 1\leqslant c\rho_\alpha$ and

$$\begin{aligned}
\left(1-\frac{1}{2m_\alpha(\rho_\alpha-\epsilon_\alpha)}\right)m'_\alpha(\rho_\alpha-\epsilon_\alpha) &= m'_\alpha(\rho_\alpha-\epsilon_\alpha) - \frac{1}{2}\frac{m'_\alpha}{m_\alpha}\\
&= m'_\alpha(\rho_\alpha-1) + \frac{m'_\alpha}{m_\alpha}\underbrace{(1-\epsilon_\alpha)m_\alpha}_{=1} - \frac{1}{2}\frac{m'_\alpha}{m_\alpha}\\
&= m'_\alpha(\rho_\alpha-1) + \frac{1}{2}\frac{m'_\alpha}{m_\alpha}\\
&\geqslant m'_\alpha(\rho_\alpha-1).
\end{aligned}$$

hence $m'_\alpha(\rho_\alpha-1)\leqslant c_\epsilon m'_\alpha(\rho_\alpha-\epsilon_\alpha)$ for all $m'_\alpha\geqslant m_\alpha$ and $\alpha\in\mathscr{A}_U$. The only thing left to do is controlling the diverging term

$$\sum_{\alpha\in\mathscr{A}_U^0(\xi)}\frac{[D_\alpha^\circ\setminus\Xi_\xi]}{\mathbf{L}^{n-1}}\frac{(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}$$

in the expression of $Z_v^\epsilon(\mathbf{t},\xi)$. One gets

$$\begin{aligned}
F_v^\epsilon(\mathbf{t},\xi) &= 1 - \sum_{\alpha\in\mathscr{A}_U^0(\xi)}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha} + \mathbf{L}^{1-n}\sum_{\alpha\in\mathscr{A}_U^0(\xi)}[D_\alpha^\circ\setminus\Xi_\xi](\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha} + P_v^\epsilon(\mathbf{t},\xi)\\
&= 1 + \sum_{\alpha\in\mathscr{A}_U^0(\xi)}\left([D_\alpha^\circ\setminus\Xi_\xi]-\mathbf{L}^{n-1}\right)\mathbf{L}^{-(n-1)}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha} + P_v^\epsilon(\mathbf{t},\xi)
\end{aligned}$$

where $P_v^\epsilon(\mathbf{t},\xi)$ is a Laurent series consisting of terms satisfying the bounds of Proposition 2.4.29 for

$$c_\epsilon = \max_{\alpha\in\mathscr{A}_U}\left(1-\frac{1}{2m_\alpha(\rho_\alpha-\epsilon_\alpha)}\right).$$

By Proposition 2.4.3, for every $\alpha\in\mathscr{A}_U^0(\xi)$

$$\begin{aligned}
& w_{\mathscr{C}_1}\left(\left([D_\alpha^\circ\setminus\Xi_\xi]-\mathbf{L}^{n-1}\right)\mathbf{L}^{-(n-1)}\mathbf{L}^{m_\alpha(\rho_\alpha-1)}\right)\\
&\leqslant 2(n-1)-2(n-1)+2m_\alpha(\rho_\alpha-1)\\
&= 2m_\alpha(\rho_\alpha-1)\\
&= 2m_\alpha(\rho_\alpha-\epsilon_\alpha)-2
\end{aligned}$$

Our analysis shows that we can apply Proposition 2.4.29 with $X=\mathscr{C}_1$, $b=\frac{1}{2}$, M arbitrary and $\beta=0$. The result follows. $\square$

Remark 4.3.6. The previous proof shows that the weight-linear convergence of $\prod_{v\in\mathscr{C}_1} F_v(\mathbf{t},\xi)$ with respect to ρ is uniform on each set of a finite partition of $V\setminus\{0\}$ given by

$$V_{A_0^D,A_1^D,A_{\geqslant 2}^D} = \left(\bigcap_{\alpha\in A_0^D} d_\alpha^{-1}(\{0\})\right)\cap\left(\bigcap_{\alpha\in A_1^D} d_\alpha^{-1}(\{1\})\right)\cap\left(\bigcap_{\alpha\in A_{\geqslant 2}^D} d_\alpha^{-1}(\mathbf{N}_{\geqslant 2})\right)$$
$$A_0^D\sqcup A_1^D\sqcup A_{\geqslant 2}^D = \mathscr{A}_U$$

(recall that V is the n-th power of the Riemann-Roch space of the divisor $\tilde{E}$ (4.2.3.18)).

4.3.3. Places $v\in\mathscr{C}_0(k)$ above which $W_v\neq\mathscr{L}(\mathscr{X}_{\mathcal{O}_v})$. For any character ξ, we have to bound the weight over $\mathbf{C}$ of the coefficient of $F_v(\mathbf{t},\xi)$. First we bound the weight of the coefficients of

$$Z_v^\epsilon(\mathbf{t},\xi) = \sum_{\substack{A\subset\mathscr{A}\\ \beta_v\in\mathscr{B}_v^{\mathscr{U}}}} \mathbf{t}^{e^{\beta_v}}\mathbf{L}^{\rho^{\beta_v}} \sum_{\mathbf{m}'\in\mathbf{N}_{>0}^A} \mathbf{t}^{\mathbf{m}'} \int_{\substack{\Omega_v^\epsilon(A,\beta_v)\cap W_v\\ \mathrm{ord}_v(x_\alpha)=m'_\alpha}} \prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha})^{m'_\alpha}\mathbf{e}(\langle x,\xi\rangle)\mathrm{d}x.$$

The weight of each integral is bounded by the weight of the motivic volume of

$$\Omega_v^\epsilon(A,\beta_v)\cap_\alpha \mathrm{ord}(x_\alpha)^{-1}(m'_\alpha),$$

hence by $2\langle\rho-1,\mathbf{m}'\rangle_A$. Again multiplication by $\prod_{\alpha\in\mathscr{A}_U^0(\xi)}(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})$ does not change this bound. Conclusion: the $\mathbf{m}'$-th coefficient of $\mathbf{L}^{-\rho^{\beta_v}}F_v^{\epsilon,\beta}(\mathbf{t},\xi)$ has weight bounded by

$$\left(2-\min_{\alpha\in\mathscr{A}_U}\left(\frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}\right)\right)\langle\rho-\epsilon,\mathbf{m}'\rangle$$

if $\xi\neq 0$ and by

$$\left(2-\frac{1}{\left\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\right\rangle_{\mathscr{A}_U}}\right)\langle\rho-\epsilon,\mathbf{m}'\rangle$$

if $\xi=0$.

4.4. Moduli spaces of curves: asymptotic behaviour

We fix a *choice of vertical components* $\beta\in\prod_v\mathscr{B}_v^{\mathscr{U}}$. By the previous paragraph, we can drop the mention of W in the local terms of the motivic height zeta functions.

4.4.1. Simplified case: rational curves. For the sake of simplicity we start the proof of our result assuming that $\mathscr{C}$ is the projective line over k. For any $\mathbf{a}$ and $\mathbf{b}$ in $\mathbf{Z}^{\mathscr{A}}$ we write $\mathbf{a}\leqslant\mathbf{b}$ if and only if $\mathbf{b}-\mathbf{a}$ lies in $\mathbf{N}^{\mathscr{A}}$.

4.4.1.1. *Main term.* First we study the contribution of the trivial character $\xi=0$, which is expected to be the only one to contribute asymptotically after normalization by $\mathbf{L}^{\langle\rho-\epsilon,\mathbf{m}\rangle}$. For sake of simplicity we begin assuming $\mathscr{U}=\mathscr{X}$ so that $\mathscr{A}_U=\mathscr{A}$ and the coarse and refined height zeta functions coincide.

For every place v of $\mathbf{P}_k^1$ recall that we already defined a polynomial

$$F_v^{\epsilon,\beta}(\mathbf{t},0) = Z_v^{\epsilon,\beta}(\mathbf{t},0)\prod_{\alpha\in\mathscr{A}}\left(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)$$

with Notation 4.3.1. Thanks to Proposition 2.4.12 we are able to permute the products $\prod_{\alpha\in\mathscr{A}}$ and $\prod_{v\in\mathbf{P}^1_k}$ when considering $Z^{\epsilon,\beta}(\mathbf{t},0)$. It means that

$$Z^{\epsilon,\beta}(\mathbf{t},0)=\prod_{v\in\mathbf{P}^1_k}Z_v^{\epsilon,\beta}(\mathbf{t},0)=\left(\prod_{v\in\mathbf{P}^1_k}F_v^{\epsilon,\beta}(\mathbf{t},0)\right)\left(\prod_{\alpha\in\mathscr{A}}\prod_{v\in\mathbf{P}^1_k}\left(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)^{-1}\right).$$

By definition, the Kapranov Zeta function of $\mathbf{P}^1_k$ is the power series

$$Z_{\mathbf{P}^1_k}(t)=\prod_{v\in\mathbf{P}^1_k}(1-t)^{-1}$$

which is given explicitly (see e.g. [**Kap00**, Theorem 1.1.9] or [**CLNS18**, Chapter 7, Theorem 1.3.1]) by

$$Z_{\mathbf{P}^1_k}(t)=\frac{1}{(1-t)(1-\mathbf{L}t)}=\frac{1}{1-\mathbf{L}}\left(\frac{1}{1-t}-\frac{\mathbf{L}}{1-\mathbf{L}t}\right).$$

In our case, this gives for every index α

$$\begin{aligned}\prod_{v\in\mathbf{P}^1_k}\left(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)^{-1}&=Z_{\mathbf{P}^1_k}\left((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)\\&=\frac{1}{1-\mathbf{L}}\left(\frac{1}{1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}-\frac{\mathbf{L}}{1-\mathbf{L}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}\right)\\&=\frac{1}{1-\mathbf{L}}\left(\frac{1}{1-\mathbf{L}^{-1}(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha}}-\frac{\mathbf{L}}{1-(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha}}\right).\end{aligned}$$

Expanding the right side of this equality, one gets that the coefficient of $t_\alpha^{m_\alpha n_\alpha}$ for any non-negative integer n_α is given by

$$\frac{\mathbf{L}^{m_\alpha(\rho_\alpha-\epsilon_\alpha)n_\alpha}(\mathbf{L}^{-n_\alpha}-\mathbf{L})}{1-\mathbf{L}}=\frac{1}{1-\mathbf{L}^{-1}}\left(1-\mathbf{L}^{-n_\alpha-1}\right)\mathbf{L}^{m_\alpha(\rho_\alpha-\epsilon_\alpha)n_\alpha}.$$

Thus, loosely speaking, after normalisation by $\mathbf{L}^{m_\alpha(\rho_\alpha-\epsilon_\alpha)n_\alpha}$, the contribution of $(1-\mathbf{L}^{-1}(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha})^{-1}$ in this last formula tends to zero when n_α tends to infinity. In order to find the expected limit of $\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}[M_{U,\mathbf{m}'}]$ when $\min_{\alpha\in\mathscr{A}}(m'_\alpha)$ tends to infinity, it may be natural to consider for a while the coefficients of the series

$$\overline{Z}^{\epsilon,\beta}(\mathbf{t})=\left(\prod_{\alpha\in\mathscr{A}}\left(\left(1-\mathbf{L}^{-1}\right)(1-\mathbf{L}^{\rho_\alpha}t_\alpha)\right)^{-1}\right)\left(\prod_{v\in\mathbf{P}^1_k}F_v^{\epsilon,\beta}(\mathbf{t},0)\right)$$

instead of considering those of $Z(\mathbf{t},0)$.

For any r-tuple $\mathbf{m}'$ of integers, let us denote by $\bar{\mathfrak{a}}_{\mathbf{m}'}$ the coefficient of the monomial $\mathbf{t}^{\mathbf{m}'}$ in $\overline{Z}(\mathbf{t})$. Denoting by $\mathfrak{b}_{\mathbf{n}}$ the coefficient of $\mathbf{t}^{\mathbf{n}}$ in $\prod_{v\in\mathbf{P}^1_k}F_v^{\epsilon,\beta}(\mathbf{t},0)$ for any r-tuple $\mathbf{n}$ of integers, one gets the finite sum

$$\bar{\mathfrak{a}}_{\mathbf{m}'}=\frac{1}{(1-\mathbf{L}^{-1})^r}\sum_{\substack{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}\\ \mathbf{n}'\in\mathbf{N}^{\mathscr{A}}\\ \mathbf{n}+m\mathbf{n}'=\mathbf{m}'}}\mathfrak{b}_{\mathbf{n}}\times\mathbf{L}^{\langle m(\rho-\epsilon),\mathbf{n}'\rangle}.$$

After normalisation by $\mathbf{L}^{\langle \rho-\epsilon,\mathbf{m}'\rangle}$ in $\mathscr{M}_{k,r}$, this is actually the $\mathbf{m}'$-th partial sum of the Euler motivic product $\prod_{v\in\mathbf{P}^1_k} F_v^{\epsilon,\beta}\left(\left(\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)}\right)_{\alpha\in\mathscr{A}},0\right)$. Therefore by Proposition 4.3.2

$$\overline{\mathfrak{a}}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}$$

converges in $\widehat{\mathscr{M}_{k,r}}$ to

$$\frac{1}{(1-\mathbf{L}^{-1})^r}\prod_{v\in\mathbf{P}^1_k} F_v^{\epsilon,\beta}\left(\left(\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)}\right)_{\alpha\in\mathscr{A}},0\right)$$

when $\min_{\alpha\in\mathscr{A}}(m'_\alpha)$ tends to infinity, with an error term of weight bounded by

$$-\frac{1}{M}\langle\rho-\epsilon,\mathbf{m}'\rangle+2\rho^\beta$$

where $M=\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle$. Here the term $\rho^\beta=\sum_v\rho^{\beta_v}$ comes from all the $\mathbf{L}^{\rho_v}$ for v not in $\mathscr{C}_1$.

This heuristic argument given, now we have to justify that this is indeed the limit of $\mathbf{L}^{-\langle\rho,\mathbf{m}'\rangle}[M_{U,\mathbf{m}'}]$ (up to a factor $\mathbf{L}^n$) in $\widehat{\mathscr{M}_k}$ when $\min_{\alpha\in\mathscr{A}}(m'_\alpha)$ becomes infinitely large. First we evaluate the error term we introduced, then we check that terms corresponding to non-trivial characters do not constribute to the limit. We postpone the summation over all characters to the last section of this paper, where it will be performed in full generality.

4.4.1.2. *Contribution of the error term.* We go back to the general case which includes Campana curves. In order to control the error term we implicitly introduced in the previous paragraph, we develop the denominator of $Z(\mathbf{t},0)$ as follows. We still assume $\mathscr{U}=\mathscr{X}$. Then

$$\begin{aligned}
G(\mathbf{t},0)&=\prod_{\alpha\in\mathscr{A}} Z_{\mathbf{P}^1_k}((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})\\
&=\prod_{\alpha\in\mathscr{A}}\prod_{v\in\mathbf{P}^1_k}\left(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)^{-1}\\
&=\prod_{\alpha\in\mathscr{A}}\frac{1}{1-\mathbf{L}}\left(\frac{1}{1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}-\frac{\mathbf{L}}{1-\mathbf{L}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}\right)\\
&=\prod_{\alpha\in\mathscr{A}}\frac{1}{1-\mathbf{L}}\left(\frac{1}{1-\mathbf{L}^{-1}(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha}}-\frac{\mathbf{L}}{1-(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha}}\right)\\
&=\frac{1}{(1-\mathbf{L})^r}\sum_{b\in\{0,1\}^{\mathscr{A}}}\frac{(-\mathbf{L})^{|\mathscr{A}|-|b|}}{\prod_{\alpha\in\mathscr{A}}(1-\mathbf{L}^{m_\alpha(\rho_\alpha-\epsilon_\alpha)-b_\alpha}t_\alpha^{m_\alpha})}.
\end{aligned}$$

where $|b|=\sum_\alpha b_\alpha$. Let us introduce a few convenient notations. For any $b\in\{0,1\}^{\mathscr{A}}$ we define

$$G_b(\mathbf{T})=\prod_{\alpha\in\mathscr{A}}(1-\mathbf{L}^{m_\alpha(\rho_\alpha-\epsilon_\alpha)-b_\alpha}t_\alpha^{m_\alpha})^{-1}=\sum_{\mathbf{n}\in\mathbf{N}^{\mathscr{A}}}\mathbf{L}^{\langle m(\rho-\epsilon)-b,\mathbf{n}\rangle}\mathbf{t}^{m\mathbf{n}}$$

so that $G(\mathbf{t},0)$ admits a decomposition of the form

$$G(\mathbf{T},0)=\sum_{b\in\{0,1\}^{\mathscr{A}}}\frac{(-\mathbf{L})^{r-|b|}}{(1-\mathbf{L})^r}G_b(\mathbf{T}).$$

Now one easily sees that the term of multidegree $\mathbf{m}' \in \mathbf{Z}^{\mathscr{A}}$ of $G_b(\mathbf{T}) \prod_{v\in\mathbf{P}^1_k} F_v(\mathbf{T},0)$ is the finite sum

$$\mathfrak{g}^b_{\mathbf{m}'} = \sum_{\substack{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}\\ \mathbf{n}'\in\mathbf{N}^{\mathscr{A}}\\ \mathbf{n}+m\mathbf{n}'=\mathbf{m}'}} \mathfrak{b}_{\mathbf{n}} \mathbf{L}^{\langle m(\rho-\epsilon)-b,\mathbf{n}'\rangle}$$

from which we deduce in $\mathscr{M}_{k,r}$

$$\mathfrak{g}^b_{\mathbf{m}'}\mathbf{L}^{-\langle \rho-\epsilon,\mathbf{m}'\rangle} = \sum_{\substack{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}\\ \mathbf{n}'\in\mathbf{N}^{\mathscr{A}}\\ \mathbf{n}+m\mathbf{n}'=\mathbf{m}'}} \mathfrak{b}_{\mathbf{n}} \mathbf{L}^{-\langle\rho-\epsilon,\mathbf{n}\rangle}\mathbf{L}^{-\langle b,\mathbf{n}'\rangle}.$$

since

$$\langle m(\rho-\epsilon),\mathbf{n}'\rangle - \langle \rho-\epsilon,\mathbf{m}'\rangle = \langle \rho-\epsilon, m\mathbf{n}'-\mathbf{m}'\rangle = -\langle(\rho-\epsilon),\mathbf{n}\rangle$$

for all $\mathbf{n},\mathbf{n}' \in \mathbf{N}^{\mathscr{A}}$ such that $\mathbf{n}+m\mathbf{n}'=\mathbf{m}'$. The case $b=\mathbf{0}$ has been studied in the previous paragraph and it suffices to apply Lemma 2.4.28 to $\sum \mathfrak{b}_{\mathbf{n}}\mathbf{L}^{-\langle\rho,\mathbf{n}\rangle}$ in case $b\neq\mathbf{0}$. We obtain the following proposition.

Proposition 4.4.1 ($\mathscr{U}=\mathscr{X}$, $\mathscr{C}=\mathscr{C}_0=\mathbf{P}^1_k$, $\xi=0$)**.** *Let $M=\langle \rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle$ and $\rho^\beta=\sum_v \rho^{\beta_v}$. There exists a decomposition*

$$Z^\beta(\mathbf{t},0) = \sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}} \mathfrak{a}_{\mathbf{m}'}\mathbf{t}^{\mathbf{m}'} = \sum_{B\subset\mathscr{A}}\sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}} \mathfrak{a}^B_{\mathbf{m}'}\mathbf{t}^{\mathbf{m}'}$$

such that for all $\varnothing\neq B\subset\mathscr{A}$

$$w\left(\mathfrak{a}^B_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}\right) \leqslant -\frac{1}{M}\langle\rho-\epsilon,\mathbf{m}'\rangle_B + 2\rho^\beta$$

while

$$\mathfrak{a}^\varnothing_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle} \underset{\rho\text{-weight-lin.}}{\longrightarrow} \frac{1}{(1-\mathbf{L}^{-1})^{\operatorname{rk}(\operatorname{Pic}(X))}} \prod_{v\in\mathbf{P}^1_k} F^\beta_v\left((\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)})_{\alpha\in\mathscr{A}},0\right) \in \widehat{\mathscr{M}_{k,r}}$$

when $\min_{\alpha\in\mathscr{A}}(m'_\alpha)$ tends to infinity, with an error term of weight bounded by

$$-\frac{1}{M}\langle\rho-\epsilon,\mathbf{m}'\rangle + 2\rho^\beta$$

where ρ^β is the finite sum $\sum_v\rho^{\beta_v}$.

As an immediate corollary of this proposition, the terms $\mathfrak{a}^b_{\mathbf{m}'}\mathbf{L}^{-\langle\rho,\mathbf{m}'\rangle}$, for $b\neq\mathbf{0}$, are negligible when $\min_{\alpha\in\mathscr{A}}(m'_\alpha)$ becomes arbitrarily large, in comparison with $\mathfrak{a}^{\mathbf{0}}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho,\mathbf{m}'\rangle}$.

4.4.1.3. *Non-trivial characters.* We now study the asymptotic contribution of the coefficients of $Z(\mathbf{t},\xi)$ for $\xi\in G(F)^\vee$ non-trivial. We still assume $\mathscr{C}=\mathbf{P}^1_k$.

Recall that given a place v of $\mathscr{C}$, the linear form $x\mapsto\langle x,\xi\rangle$ on G_F can be seen as a rational function f_ξ on X with poles contained in $\cup_{\alpha\in\mathscr{A}}D_\alpha$. If $d_\alpha(\xi)$ denotes the order of the pole of f_ξ with respect to D_α, one can define a subset of $\mathscr{A}_U$ by setting

$$\mathscr{A}^0_U(\xi) = \{\alpha\in\mathscr{A}_U \mid d_\alpha(\xi)=0\},$$

see page 103. The character ξ being non-trivial, the cardinality of this set is strictly smaller than $|\mathscr{A}|$. Using Notation 4.3.4, the term of the motivic height Zeta function corresponding to the character ξ and the choice of vertical components β is

$$Z^\beta(\mathbf{t},\xi) = \prod_{v\in\mathbf{P}^1_k}\left(F^\beta_v(\mathbf{t},\xi_v)\prod_{\alpha\in\mathscr{A}^0_U(\xi)}\left(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)^{-1}\right).$$

Again we restrict our analysis to the case $\mathscr{U} = \mathscr{X}$ so that $\mathscr{A} = \mathscr{A}_U$. The general case will be treated together with the case of a general curve. For any $b \in \{0,1\}^{\mathscr{A}_U^0(\xi)}$ we introduce

$$G_b(\mathbf{t},\xi) = \prod_{\alpha\in\mathscr{A}_U^0(\xi)} \left(1-\mathbf{L}^{-b_\alpha}(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha}\right)^{-1}.$$

Adapting the computation done in the previous paragraph, one gets

$$G(\mathbf{t},\xi) = \prod_{v\in\mathbf{P}^1_k}\left(\prod_{\alpha\in\mathscr{A}_U^0(\xi)}(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})^{-1}\right) = \sum_{b\in\{0,1\}^{\mathscr{A}_U^0(\xi)}} \frac{(-\mathbf{L})^{|\mathscr{A}_U^0(\xi)|-|b|}}{(1-\mathbf{L})^{|\mathscr{A}_U^0(\xi)|}} G_b(\mathbf{t},\xi)$$

and the coefficient of multidegree $\mathbf{m}' \in \mathbf{N}^{\mathscr{A}}$ of the product

$$G_b(\mathbf{t},\xi)\prod_{v\in\mathbf{P}^1_k} F_v^\beta(\mathbf{t},\xi_v)$$

is

$$\mathfrak{g}^\xi_{b,\mathbf{m}'} = \sum_{\substack{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}\\ \mathbf{n}'\in\mathbf{N}^{\mathscr{A}}\\ \mathbf{n}+m\mathbf{n}'=\mathbf{m}'}} \mathfrak{b}^\xi_{\mathbf{n}}\mathbf{L}^{\langle m(\rho-\epsilon)-b,\mathbf{n}'\rangle_{\mathscr{A}_U^0(\xi)}}$$

where $\mathfrak{b}^\xi_{\mathbf{n}}$ is the coefficient of multidegree $\mathbf{n}\in\mathbf{N}^{\mathscr{A}}$ of $\prod_{v\in\mathbf{P}^1_k} F_v^\beta(\mathbf{t},\xi)$. Then we consider the following normalisation:

$$\mathfrak{g}^\xi_{b,\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle} = \sum_{\substack{\mathbf{n}\in\mathbf{Z}^{\mathscr{A}}\\ \mathbf{n}'\in\mathbf{N}^{\mathscr{A}}\\ \mathbf{n}+m\mathbf{n}'=\mathbf{m}'}} \mathfrak{b}^\xi_{\mathbf{n}}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{n}\rangle}\mathbf{L}^{-\langle m(\rho-\epsilon),\mathbf{n}'\rangle_{\mathscr{A}\setminus\mathscr{A}_U^0(\xi)}-\langle b,\mathbf{n}'\rangle_{\mathscr{A}_U^0(\xi)}}.$$

Applying Lemma 2.4.28 to this sum for every b, we get the following proposition.

Proposition 4.4.2 ($\mathscr{U} = \mathscr{X}$, $\mathscr{C} = \mathscr{C}_0 = \mathbf{P}^1_k$, $\xi \neq 0$)**.** *There exists a decomposition*

$$Z^\beta(\mathbf{t},\xi) = \sum_{B\subset\mathscr{A}_U^0(\xi)}\sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}} \mathfrak{a}^{\xi,B}_{\mathbf{m}'}\mathbf{t}^{\mathbf{m}'}$$

such that for every $B\subset\mathscr{A}_0(\xi)$

$$w\left(\mathfrak{a}^{\xi,B}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}\right) < -\delta\langle\rho-\epsilon,\mathbf{m}'\rangle_{B\sqcup(\mathscr{A}_U\setminus\mathscr{A}_U^0(\xi))} + 2\rho^\beta$$

for all $\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}$, where $\delta = \min\limits_{\alpha\in\mathscr{A}_U}\frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}$ and $\rho^\beta = \sum_v \rho^{\beta_v}$.

Since $\mathscr{A}_U\setminus\mathscr{A}_U^0(\xi)$ is non-empty for every $\xi\neq 0$, the inequality of this last proposition means that the normalised terms $\mathfrak{a}^{\xi,b}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}$ coming form $Z(\mathbf{t},\xi)$ are negligible in comparison with the main term $\mathfrak{a}^{0}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}$ coming from $Z(\mathbf{t},0)$. Furthermore we have a linear bound on the weight which is uniform on a finite partition of the space of characters by Remark 4.3.6.

4.4.2. The case of a general curve. From now on we assume that $\mathscr{C}$ is a projective smooth irreducible curve of genus g over k having a k-rational point. In this case, for $m\geqslant 2g-1$ the class of $\mathrm{Sym}^m(\mathscr{C})$ in $K_0\mathbf{Var}_k$ is $[\mathrm{Pic}^0(\mathscr{C})][\mathbf{P}^{m-g}_k]$ (this follows from Riemann-Roch and Serre's duality, see [**CLNS18**, Chapter 7, Example 1.1.10]) and the Kapranov zeta function of $\mathscr{C}$ is still rational (see [**Kap00**, Theorem 1.1.9] or [**CLNS18**, Chapter 7, Theorem 1.3.1]). We have

$$Z_{\mathscr{C}}^{\mathrm{Kap}}(t)=\sum_{m\in\mathbf{N}}[\mathrm{Sym}^m(\mathscr{C})]t^m=\sum_{m=0}^{2(g-1)}[\mathrm{Sym}^m(C)]t^m+[\mathrm{Pic}^0(\mathscr{C})]\sum_{m\geqslant 2g-1}\frac{\mathbf{L}^{m-g+1}-1}{\mathbf{L}-1}t^m$$

(where the first sum is empty if $g=0$). Consider for any $\xi\in G(F)^\vee$

$$G(\mathbf{t},\xi)=\prod_{v\in\mathscr{C}}\prod_{\alpha\in\mathscr{A}_U^0(\xi)}(1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})^{-1}$$

(remark that if $\xi=0$ then $\mathscr{A}_U^0(\xi)=\mathscr{A}_U$). By Proposition 2.4.12 one has

$$G(\mathbf{t},\xi)=\prod_{\alpha\in\mathscr{A}_U^0(\xi)}Z_{\mathscr{C}}^{\mathrm{Kap}}\left((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)$$

In order to easily adapt the computations done before, we may assume $g\geqslant 1$, replace $Z_{\mathscr{C}}^{\mathrm{Kap}}$ by

$$\widetilde{Z_{\mathscr{C}}^{\mathrm{Kap}}}(t)=\sum_{m\geqslant 0}[\mathrm{Pic}^0(\mathscr{C})]\frac{\mathbf{L}^{m-g+1}-1}{\mathbf{L}-1}t^m$$

in the expression of $G(\mathbf{t},\xi)$ and then control the error term coming from this slight modification. If we do so, then we obtain

$$Z^\beta(\mathbf{t},\xi)=G(\mathbf{t},\xi)\prod_{v\in\mathscr{C}}F_v^\beta(\mathbf{t},\xi)=\widetilde{G}(\mathbf{t},\xi)\prod_{v\in\mathscr{C}}F_v^\beta(\mathbf{t},\xi)+H(\mathbf{t},\xi)\prod_{v\in\mathscr{C}}F_v^\beta(\mathbf{t},\xi)$$

where

$$\widetilde{G}(\mathbf{t},\xi)=\prod_{\alpha\in\mathscr{A}_U^0(\xi)}\widetilde{Z_{\mathscr{C}}^{\mathrm{Kap}}}\left((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)\tag{4.4.2.25}$$

and

$$H(\mathbf{t},\xi)=\prod_{\alpha\in\mathscr{A}_U^0(\xi)}Z_{\mathscr{C}}^{\mathrm{Kap}}\left((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right)-\prod_{\alpha\in\mathscr{A}_U^0(\xi)}\widetilde{Z_{\mathscr{C}}^{\mathrm{Kap}}}\left((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}\right).\tag{4.4.2.26}$$

We then put

$$\widetilde{Z}^\beta(\mathbf{t},\xi)=\widetilde{G}(\mathbf{t},\xi)\prod_{v\in\mathscr{C}}F_v^\beta(\mathbf{t},\xi).\tag{4.4.2.27}$$

4.4.2.1. *Main term.* Basically, the contribution of $\widetilde{Z}(\mathbf{t},\xi)$ has already been treated in Section 4.4.1.2 for the case $\xi=0$ and in Section 4.4.1.3 for non-trivial characters. The only difference with the particular case of the projective line is a factor $[\mathrm{Pic}^0(\mathscr{C})]\mathbf{L}^{(1-g)}$. Indeed, starting from (4.4.2.25) one gets

$$\begin{aligned}\widetilde{G}(\mathbf{t},\xi)&=\prod_{\alpha\in\mathscr{A}_U^0(\xi)}\frac{[\mathrm{Pic}^0(\mathscr{C})]}{1-\mathbf{L}}\left(\frac{1}{1-\mathbf{L}^{-1}(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha}}-\frac{\mathbf{L}^{1-g}}{1-(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha}}\right)\\&=\frac{[\mathrm{Pic}^0(\mathscr{C})]^{|\mathscr{A}_U^0(\xi)|}}{(1-\mathbf{L})^{|\mathscr{A}|}}\sum_{b\in\{0,1\}^{\mathscr{A}_U^0(\xi)}}\frac{(-\mathbf{L}^{1-g})^{|\mathscr{A}_U^0(\xi)|-|b|}}{\prod_{\alpha\in\mathscr{A}_U^0(\xi)}(1-\mathbf{L}^{-b_\alpha}(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}t_\alpha)^{m_\alpha})}\\&=\left(\frac{[\mathrm{Pic}^0(\mathscr{C})]\mathbf{L}^{1-g}}{\mathbf{L}-1}\right)^{|\mathscr{A}_U^0(\xi)|}\sum_{b\in\{0,1\}^{\mathscr{A}_U^0(\xi)}}(-1)^{|b|}\widetilde{G}_b(\mathbf{t},\xi)\mathbf{L}^{|b|(g-1)}\end{aligned}$$

where

$$\widetilde{G_b}(\mathbf{t},\xi) = \prod_{\alpha\in\mathscr{A}_U^0(\xi)} \left(1-\mathbf{L}^{-b_\alpha}\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}{t_\alpha}^{m_\alpha}\right)^{-1} = \sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}_U^0(\xi)}} \mathbf{L}^{\langle m(\rho-\epsilon),\mathbf{m}'\rangle_{\mathscr{A}_U^0(\xi)}}\mathbf{L}^{-\langle b,\mathbf{m}'\rangle}\mathbf{t}^{m\mathbf{m}'}.$$

The following proposition summarizes what we obtain if we replace $G_b(\mathbf{t},\xi)$ by $\widetilde{G_b}(\mathbf{t},\xi)$ in the previous paragraph. Again, it ensures the negligibility of the terms corresponding to non-trivial characters (up to the error term coming from $H(\mathbf{t},\xi)$, which is treated in the next paragraph).

Proposition 4.4.3 ($\mathscr{U}=\mathscr{X}$, $\mathscr{C}=\mathscr{C}_0$, $\mathscr{A}=\mathscr{A}_U$)**.** *Let $\rho^\beta=\sum_v\rho^{\beta_v}$. For any character ξ, there exists a decomposition*

$$\tilde{Z}^\beta(\mathbf{t},\xi) = \sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}} \tilde{\mathfrak{a}}^{\xi}_{\mathbf{m}'}\mathbf{t}^{\mathbf{m}'} = \sum_{B\subset\mathscr{A}_U^0(\xi)}\sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}} \tilde{\mathfrak{a}}^{\xi,B}_{\mathbf{m}'}\mathbf{t}^{\mathbf{m}'}$$

such that

— *if $\xi=0$,*

$$\tilde{\mathfrak{a}}^{0,\varnothing}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle} \underset{(\rho-\epsilon)\text{-weight-lin.}}{\longrightarrow} \left(\frac{[\mathrm{Pic}^0(\mathscr{C})]\mathbf{L}^{1-g}}{\mathbf{L}-1}\right)^{|\mathscr{A}_U|}\prod_{v\in\mathscr{C}_0}F_v^\epsilon((\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)})_{\alpha\in\mathscr{A}},0)$$

with an error term of weight bounded by

$$-\frac{1}{\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle}\langle\rho-\epsilon,\mathbf{m}'\rangle+\,2\rho^\beta$$

and

$$w\left(\tilde{\mathfrak{a}}^{0,B}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}\right)\leqslant -\frac{1}{\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle}\langle\rho-\epsilon,\mathbf{m}'\rangle_B+\,2\rho^\beta$$

for all $B\neq\varnothing$ and $\mathbf{m}'\in\mathbf{N}^{\mathscr{A}_U}$;

— *if $\xi\neq0$*

$$w\left(\tilde{\mathfrak{a}}^{\xi,B}_{\mathbf{m}'}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}\right)\leqslant -\delta\langle\rho-\epsilon,\mathbf{m}'\rangle_{B\sqcup(\mathscr{A}_U\setminus\mathscr{A}_U^0(\xi))}+\,2\rho^\beta$$

for all $B\subset\mathscr{A}_U^0(\xi)$ and all $\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}$, where $\delta=\min\limits_{\alpha\in\mathscr{A}_U}\frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}$.

4.4.2.2. *Error term.* Now we study the contribution of $H(\mathbf{t},\xi)$, still assuming $\mathscr{U}=\mathscr{X}$. We rewrite this error term as follows, using the convenient notations $\mathbf{1}=(1,...,1)$ and $\mathbf{g}=g\mathbf{1}$. In this paragraph ξ is any character, including the trivial one; in this case $\mathscr{A}_U^0(0)=\mathscr{A}_U$.

$$\begin{aligned}H^\epsilon(\mathbf{t},\xi) &= \sum_{\varnothing\neq A\subset\mathscr{A}_U^0(\xi)}\prod_{\alpha\in\mathscr{A}_U^0(\xi)\setminus A}\widetilde{Z^{\mathrm{Kap}}_{\mathscr{C}}}((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})\prod_{\alpha\in A}\left(Z^{\mathrm{Kap}}_{\mathscr{C}}((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})-\widetilde{Z^{\mathrm{Kap}}_{\mathscr{C}}}((\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha})\right)\\ &= \sum_{\substack{\varnothing\neq A\subset\mathscr{A}_U^0(\xi)\\ b\in\{0,1\}^{\mathscr{A}_U^0(\xi)\setminus A}\times\{1\}^A}}[\mathrm{Pic}^0(\mathscr{C})]^{|\mathscr{A}_U^0(\xi)\setminus A|}\frac{(-\mathbf{L}^{1-g})^{|\mathscr{A}_U^0(\xi)|-|b|}}{(1-\mathbf{L})^{|\mathscr{A}_U^0(\xi)\setminus A|}}H^\epsilon_{A,b}(\mathbf{t},\xi)\end{aligned}$$

with for any $A\subset\mathscr{A}_U^0(\xi)$ non-empty and $b\in\{0,1\}^{\mathscr{A}_U^0(\xi)\setminus A}\times\{1\}^A$

$$H^\epsilon_{A,b}(\mathbf{t},\xi)=\sum_{\mathbf{n}\in\mathbf{N}^{\mathscr{A}_U^0(\xi)}}\mathbf{L}^{\langle m(\rho-\epsilon),\mathbf{n}\rangle}\mathbf{L}^{\langle -b,\mathbf{n}\rangle}\mathbf{t}^{m\mathbf{n}}\prod_{\alpha\in A}\left([\mathrm{Sym}^{n_\alpha}\mathscr{C}]-[\mathrm{Pic}^0(\mathscr{C})]\frac{\mathbf{L}^{n'_\alpha-g+1}-1}{\mathbf{L}-1}\right).$$

Thus studying the contribution of $H(\mathbf{t},\xi)$ amounts to studying the $\mathbf{m}'$-th coefficient of the product

$$H^{\epsilon}_{A,b}(\mathbf{t},\xi)\prod_{v\in\mathscr{C}}F^{\epsilon}_{v}(\mathbf{t},\xi)$$

for every $A\subset\mathscr{A}^0_U(\xi)$ non-empty and $b\in\{0,1\}^{\mathscr{A}^0_U(\xi)\setminus A}\times\{1\}^A$. In what follows we fix such an A. We know that the term of multidegree $m\mathbf{n}$ of $H^{\epsilon}_{A,b}(\mathbf{t})$ is zero whenever there is $\alpha\in A$ such that $n_\alpha\geqslant 2g-1$. So the $\mathbf{m}'$-th term of the product $H^{\epsilon}_{A,b}(\mathbf{t},\xi)\prod_{v\in\mathscr{C}}F^{\epsilon}_{v}(\mathbf{t},\xi)$ is equal to

$$\begin{aligned}\mathfrak{c}^{A,b}_{\mathbf{m}'} &= \mathbf{L}^{\langle\rho-\epsilon-b,\mathbf{m}'\rangle_{\mathscr{A}^0_U(\xi)}} \qquad (4.4.2.28)\\ &\times\sum_{\substack{(\mathbf{n},\mathbf{n}')\in\mathbf{Z}^{\mathscr{A}}\times\mathbf{N}^{\mathscr{A}}\\ \mathbf{n}+m\mathbf{n}'=\mathbf{m}'\\ \mathbf{n}'_A\leqslant 2(\mathbf{g}_A-\mathbf{1})}}\mathfrak{b}_{\mathbf{n}}\mathbf{L}^{-\langle\rho-\epsilon-b,\mathbf{n}\rangle_{\mathscr{A}^0_U(\xi)}}\prod_{\alpha\in A}\left([\mathrm{Sym}^{n'_\alpha}\mathscr{C}]-[\mathrm{Pic}^0(\mathscr{C})]\frac{\mathbf{L}^{n'_\alpha-g+1}-1}{\mathbf{L}-1}\right)\end{aligned}$$

where $\mathfrak{b}_{\mathbf{n}}$ is the $\mathbf{n}$-th coefficient of $\prod_{v\in\mathscr{C}}F^{\epsilon}_{v}(\mathbf{t},\xi)$ and $\mathbf{n}'_A$ denotes the restriction to A of $\mathbf{n}'\in\mathbf{N}^{\mathscr{A}}$. Since $A\neq\varnothing$, by Proposition 4.3.2 (for the trivial character), Proposition 4.3.5 (for non-trivial characters) and Lemma 2.4.28 we get that for any $b\in\{0,1\}^{\mathscr{A}^0_U(\xi)\setminus A}\times\{1\}^A$

$$\begin{aligned}& w\left(\sum_{\substack{(\mathbf{n},\mathbf{n}')\in\mathbf{Z}^{\mathscr{A}}\times\mathbf{N}^{\mathscr{A}}\\ \mathbf{n}+m\mathbf{n}'=\mathbf{m}'\\ \mathbf{n}'_A\leqslant 2(\mathbf{g}_A-\mathbf{1})}}\mathfrak{b}_{\mathbf{n}}\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{n}\rangle}\mathbf{L}^{-\langle b,\mathbf{n}'\rangle_{\mathscr{A}^0_U(\xi)}-\langle\rho-\epsilon,m\mathbf{n}'\rangle_{\mathscr{A}\setminus\mathscr{A}^0_U(\xi)}}\right)\\ &< -\delta\langle\rho-\epsilon,\mathbf{m}'\rangle_{B\cup\mathscr{A}_U\setminus\mathscr{A}^0_U(\xi)}+2\rho^\beta \qquad (4.4.2.29)\end{aligned}$$

where $\delta=\min\limits_{\alpha\in\mathscr{A}_U}\frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}$ if $\xi\neq 0$ and $1/\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle$ otherwise, and $B\subset\mathscr{A}^0_U(\xi)$ is given by the support of b. Remark that the product over A can only take a finite number of values, with weight uniformly bounded by $2|A|(g-1)$. Combining (4.4.2.28) and (4.4.2.29), we conclude that

$$w\left(\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}\mathfrak{c}^{A,b}_{\mathbf{m}'}\right)<-\delta\langle\rho-\epsilon,\mathbf{m}'\rangle_{B\cup\mathscr{A}_U\setminus\mathscr{A}^0_U(\xi)}+2(\rho^\beta+|A|(g-1)_+)$$

for all $\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}$.

This analysis proves that the slight modification we performed on the Kapranov Zeta function of $\mathscr{C}$ is painless in the case $\mathscr{U}=\mathscr{X}$. It allows us to extend the result to any smooth projective irreducible curve of genus g. Therefore the control of the error terms coming from $H^{\epsilon}(\mathbf{t},\xi)$ is given by the following proposition.

Proposition 4.4.4 ($\mathscr{U}=\mathscr{X}$, $\mathscr{C}=\mathscr{C}_0$). *Let ξ be any character. There exists a decomposition*

$$H^{\epsilon}(\mathbf{t},\xi)\prod_{v\in\mathscr{C}_0}F^{\epsilon,\beta}_{v}(\mathbf{t},\xi)=\sum_{\substack{\varnothing\neq A\subset\mathscr{A}^0_U(\xi)\\ B\subset\mathscr{A}^0_U(\xi)\\ A\cap B=\varnothing}}\sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}}\mathfrak{h}^{A,B}_{\mathbf{m}'}\mathbf{t}^{\mathbf{m}'}$$

such that for all non empty subsets $A,B\subset\mathscr{A}^0_U(\xi)$ with $A\neq\varnothing$ and $A\cap B=\varnothing$,

$$w\left(\mathbf{L}^{-\langle\rho-\epsilon,\mathbf{m}'\rangle}\mathfrak{h}^{A,B}_{\mathbf{m}'}\right)\leqslant-\delta\langle\rho-\epsilon,\mathbf{m}'\rangle_{A\cup B\cup\mathscr{A}_U\setminus\mathscr{A}^0_U(\xi)}+2(\rho^\beta+|A|(g-1)_+)$$

for all $\mathbf{m}'\in\mathbf{N}^{\mathscr{A}}$, where $\delta=\min\limits_{\alpha\in\mathscr{A}_U}\frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}$ if $\xi\neq 0$ and $\delta=1/\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle$ otherwise.

4.4.2.3. *S-integral points and the trivial character.* We now start treating the general case $\mathscr{U} \subset \mathscr{X}$. For clarity's sake we will keep using the coarse height zeta function $Z(\mathbf{t})$ until it becomes necessary to switch to its refined version $\mathscr{Z}(\mathbf{u})$, that is, when we will integrate the Euler product over S to the global one over $\mathscr{C}$.

In order to treat the case of S-integral points, as in [**CLL16**, §2.3] we introduce a simplicial complex encoding the intersecting data of the boundary's components. Let Y be a smooth algebraic variety over a field L and Δ a divisor on Y with strict normal crossings. The Clemens complex $\mathrm{Cl}(Y, \Delta)$ is the simplicial complex whose vertices are the irreducible components $(\Delta_i)_{i\in I}$ of Δ. It has an edge between Δ_i and Δ_j for $i \neq j$ if and only if $\Delta_i \cap \Delta_j \neq \varnothing$. It has a two dimensional face, given by the vertices Δ_i, Δ_j and Δ_k, for i, j, k pairwise distinct, if and only if $\Delta_i \cap \Delta_j \cap \Delta_k \neq \varnothing$, and so on for higher dimensional faces: in general, a subset $J \subset I$ corresponds to a face of dimension $|J| - 1$ of $\mathrm{Cl}(Y, \Delta)$ if and only if $\Delta_J = \cap_{j\in J} \Delta_j \neq \varnothing$. Since we assumed that Y is smooth and Δ has strict normal crossings, the intersections we are considering are smooth as well.

Then a maximal face of $\mathrm{Cl}(Y, \Delta)$ is a simplex whose vertices are indexed by a subset $J \subset I$ such that $\Delta_J \neq \varnothing$ and $\Delta_J \cap \Delta_k = \varnothing$ for any $k \in I \setminus J$. In particular, the dimension of $\mathrm{Cl}(Y, \Delta)$ is the maximal number of components of Δ with non-empty intersection minus one.

The L-Clemens complex[1] $\mathrm{Cl}_L(Y, \Delta)$ is defined in a similar way, by restriction to L-points. More precisely, $\mathrm{Cl}_L(Y, \Delta)$ is the subcomplex of $\mathrm{Cl}(Y, \Delta)$ consisting of simplices $\Delta_J \in \mathrm{Cl}(Y, \Delta)$ such that $\Delta_J(L) \neq \varnothing$. The set of maximal faces of $\mathrm{Cl}_L(Y, \Delta)$ is $\mathrm{Cl}_L^{\max}(Y, \Delta)$.

Let $s \in S$. By [**CLL16**, §6] and [**Bil23**, §6.3.6] we have

$$Z_s^{\beta_s}(\mathbf{t}, 0) = \sum_{A\subset\mathscr{A}} \mathbf{L}^{\rho^\beta}[\Delta_s(A, \beta)]\mathbf{L}^{-n+|A|}(1 - \mathbf{L}^{-1})^{|A|} \prod_{\alpha\in A} \frac{\mathbf{L}^{\rho_\alpha-1}t_\alpha}{1 - \mathbf{L}^{\rho_\alpha-1}t_\alpha}.$$

Our goal here is to isolate the contribution to the poles of the boundary D. To do so, following [**Bil23**, **CLL16**] we associate to any pair (A, β) such that $\Delta_s(A \cap \mathscr{A}_D, \beta) \neq \varnothing$ a maximal subset M_s of $\mathscr{A}_D$ such that $A \cap \mathscr{A}_D \subset \mathsf{M}_s$ and $\Delta_s(\mathsf{M}_s, \beta) \neq \varnothing$. For a general $A \subset \mathscr{A}$, there is no canonical choice of such a maximal subset: we arbitrarily choose such a map μ. This will not be a key issue, as we will see in Remarks 4.4.6 and 4.4.7 below. We assume furthermore that $\Delta_s(\mathsf{M}_s, \beta)$ has a F_s-point; this assumption is natural when one considers sections intersecting, above s, the divisors $\mathscr{D}_\alpha$ given by $\mathsf{M}_s \subset \mathscr{A}_D$, see Remark 4.4.14.

Lemma 4.4.5. *Such an $\mathsf{M}_s \subset \mathscr{A}_D$ corresponds to a maximal face of the analytic Clemens complex $\mathrm{Cl}_s(X, D) = \mathrm{Cl}_{F_s}(X_{F_s}, D_{F_s})$.*

Proof. By definition, $\Delta_s(A, \beta)$ is the set of points of the fibre $\mathscr{X}_s$ belonging exclusively to $\mathscr{D}_\alpha$ and E_β, that is to say

$$\Delta_s(A, \beta) = \left(\bigcap_{\alpha\in A} \mathscr{D}_{\alpha,s} \cap E_\beta\right) \setminus \left(\bigcup_{\substack{\alpha\notin A\\ \beta'\neq\beta}} \mathscr{D}_{\alpha,s} \cup E_{\beta'}\right).$$

For simplicity we assume that $\mathscr{B}_{1,s} = \{\beta\}$.

1. This complex is called *analytic Clemens complex* in [**CLL16**].

We argue by contradiction: assume that M_s is not a maximal face of $\mathrm{Cl}_{F_s}(X_{F_s}, D_{F_s})$. It means that there exists a non-empty subset $\mathsf{M}'_s \subset \mathscr{A}_D$ which contains M_s as a proper subset and such that

$$\left(\bigcap_{\alpha\in\mathsf{M}'_s} (D_\alpha)_{F_s}\right)(F_s) \neq \varnothing.$$

We can assume furthermore that M'_s is maximal for this property. In other words M'_s is a maximal face of $\mathrm{Cl}_{F_s}(X_{F_s}, D_{F_s})$ containing M_s as a proper subface. Remark that since $X \setminus U = \cup_{\alpha\in\mathscr{A}_D} D_\alpha$ and $X_{F_s}(F_s) = X(F_s)$, together with the maximality of M'_s, then

$$\bigcap_{\alpha\in\mathsf{M}'_s} D_\alpha \cap D_{\alpha'}(F_s) = \varnothing \tag{4.4.2.30}$$

for every $\alpha' \in \mathscr{A}_D \setminus \mathsf{M}'_s$.

Our argument relies on the fact that $\mathscr{X} \to \mathscr{C}$ is proper. Composing with the projection $X_{F_s} \to \mathscr{X}$ and applying the valuative criterion of properness, this F_s-point x_s uniquely lifts to a $\mathcal{O}_s$-point $\widetilde{x_s}$ of $\mathscr{X}$.

$$\begin{tikzcd}
X_{F_s} \arrow[rr] & & \mathscr{X} \arrow[d, "\pi"] \\
\mathrm{Spec}(F_s) \arrow[u, "x_s"] \arrow[r] & \mathrm{Spec}(\mathcal{O}_s) \arrow[r] \arrow[ur, dashed, "\widetilde{x_s}"'] & \mathscr{C}
\end{tikzcd}$$

Since for every α, $\mathscr{D}_\alpha$ is the closure in $\mathscr{X}$ of D_α, $\widetilde{x_s}$ is actually a $\mathcal{O}_s$-point of $\bigcap_{\alpha\in\mathsf{M}'_s}\mathscr{D}_\alpha$. By reduction modulo the maximal ideal of $\mathcal{O}_s$, one gets a well-defined reduction $\overline{x_s}$ to the special fibre $\mathscr{X}_s$.

$$\begin{tikzcd}
\mathscr{X}_s \arrow[r] \arrow[d, shift right] & \mathscr{X}_{\mathcal{O}_s} \arrow[r] \arrow[d, shift right] & \mathscr{X} \arrow[d] \\
\mathrm{Spec}(k) \arrow[r] \arrow[u, dashed, shift right, "\overline{x_s}"'] & \mathrm{Spec}(\mathcal{O}_s) \arrow[r] \arrow[u, dashed, shift right] \arrow[ur, dashed, "\widetilde{x_s}"] & \mathscr{C}
\end{tikzcd}$$

This very last point $\overline{x_s}$ is actually a point of $\bigcap_{\alpha\in\mathsf{M}'_s}\mathscr{D}_{\alpha,s}$ which does not belong to any of the $\mathscr{D}_{\alpha'}$ for $\alpha' \in \mathscr{A} \setminus \mathsf{M}'_s$: since all the intersections we are considering are smooth, we use Hensel's lemma here. This contradicts the maximality of M_s with respect to the hypothesis $\Delta_s(\mathsf{M}_s, \beta) \neq \varnothing$, since we have found a M'_s strictly containing M_s and such that $\Delta_s(\mathsf{M}'_s, \beta)$ is non-empty. Thus M_s is a maximal face of $\mathrm{Cl}_s(X, D)$. In the general case where the fibre above s is not irreducible, then the above argument gives at least one irreducible component β for which $\Delta_s(\mathsf{M}'_s, \beta) \neq \varnothing$. The conclusion follows. $\square$

Remark 4.4.6. If M_s is a maximal face of $\mathrm{Cl}_s(X, D)$, then for every $A \subset \mathscr{A}_U$ and $\beta \in \mathscr{B}_{1,s}$, the map μ sends the pair $(A \cup \mathsf{M}_s, \beta)$ to M_s by .

Collecting the terms with respect to every such a maximal face M_s, one gets

$$\begin{aligned}Z_s^{\beta_s}(\mathbf{t},0) &= \sum_{\mathsf{M}_s\in \mathrm{Cl}_s^{\max}(X,D)} \sum_{\substack{A\subset\mathscr{A}\\ \mu:(A,\beta_s)\mapsto \mathsf{M}_s}} \mathbf{L}^{\rho^\beta}[\Delta_s(A,\beta)]\mathbf{L}^{-n+|A|}(1-\mathbf{L}^{-1})^{|A|}\\ &\quad\times \prod_{\alpha\in A\setminus \mathsf{M}_s}\frac{\mathbf{L}^{\rho_\alpha-1}t_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}\prod_{\alpha\in A\cap \mathsf{M}_s}\frac{\mathbf{L}^{\rho_\alpha-1}t_\alpha}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}\\ &= \sum_{\mathsf{M}_s\in \mathrm{Cl}_s^{\max}(X,D)}\frac{P^{\beta_s}_{\mathsf{M}_s}(\mathbf{t})}{\prod_{\alpha\in\mathscr{A}_U}(1-\mathbf{L}^{\rho_\alpha-1}t_\alpha)}\prod_{\alpha\in \mathsf{M}_s}\frac{1}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}\end{aligned}$$

where for all $\mathsf{M}_s \in \mathrm{Cl}_s^{\max}(X,D)$

$$\begin{aligned}P^{\beta_s}_{\mathsf{M}_s}(\mathbf{t}) &= \sum_{\substack{A\subset\mathscr{A}\\ \mu:(A,\beta_s)\mapsto \mathsf{M}_s}} \mathbf{L}^{\rho^\beta}\left(\prod_{\alpha\in A}\mathbf{L}^{\rho_\alpha-1}t_\alpha\right)\\ &\quad\times [\Delta_s(A,\beta_s)]\mathbf{L}^{-n}(\mathbf{L}-1)^{|A|}\prod_{\alpha\in(\mathscr{A}_U\setminus A)\cup(\mathsf{M}_s\setminus A)}(1-\mathbf{L}^{\rho_\alpha-1}t_\alpha).\end{aligned}$$

Remark 4.4.7. In the expression of $P^{\beta_s}_{\mathsf{M}_s}(\mathbf{t})$, the polynomials $(1-\mathbf{L}^{\rho_\alpha-1}t_\alpha)$ for $\alpha\in\mathsf{M}_s$ give vanishing factors at $t_\alpha=\mathbf{L}^{-\rho_\alpha+1}$. In particular, for any $A\subset\mathscr{A}$ such that $A\cap\mathscr{A}_D\subset\mathsf{M}_s$, we have

$$\prod_{\alpha\in(\mathscr{A}_U\setminus A)\cup(\mathsf{M}_s\setminus A)}(1-\mathbf{L}^{\rho_\alpha-1}t_\alpha)(\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)})\neq 0$$

if and only if $A\cap\mathscr{A}_D=\mathsf{M}_s$ (since $\epsilon_\alpha=1$ whenever $\alpha\in\mathscr{A}_D$), and thus using Remark 4.4.6

$$\begin{aligned}&\left(P^{\beta_s}_{\mathsf{M}_s}\prod_{\alpha\in\mathscr{A}_U}\frac{1-(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}t_\alpha}\right)\left(\left(\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)}\right)_{\alpha\in\mathscr{A}}\right)\\ &= \sum_{\substack{A\subset\mathscr{A}\\ A\cap\mathscr{A}_D=\mathsf{M}_s}}\mathbf{L}^{\rho^\beta}\mathbf{L}^{-|A\cap\mathscr{A}_U|+\sum_{\alpha\in A\cap\mathscr{A}_U}\epsilon_\alpha}[\Delta_s(A,\beta_s)]\mathbf{L}^{-n}(\mathbf{L}-1)^{|A|}\prod_{\alpha\in\mathscr{A}_U\setminus A}\left(1-\mathbf{L}^{\epsilon_\alpha-1}\right)\prod_{\alpha\in\mathscr{A}_U}\frac{1-\mathbf{L}^{-1}}{1-\mathbf{L}^{\epsilon_\alpha-1}}\\ &\qquad\qquad\text{(since } m_\alpha(\epsilon_\alpha-1)=-1 \text{ whenever } \alpha\in\mathscr{A}_U)\\ &= (\mathbf{L}-1)^{|\mathsf{M}_s|}(1-\mathbf{L}^{-1})^{|\mathscr{A}_U|}\mathbf{L}^{-n}\mathbf{L}^{\rho^{\beta_s}}\sum_{A\subset\mathscr{A}_U}[\Delta_s(\mathsf{M}_s\cup A,\beta_s)]\prod_{\alpha\in A}\mathbf{L}^{\epsilon_\alpha}\frac{1-\mathbf{L}^{-1}}{1-\mathbf{L}^{\epsilon_\alpha-1}}.\end{aligned}$$

Assume only for a while that $\epsilon_\alpha=0$ for all $\alpha\in\mathscr{A}_U$. By definition of $\Delta_s(\mathsf{M}_s\cup A,\beta)$ one has

$$\sum_{A\subset\mathscr{A}_U}[\Delta_s(\mathsf{M}_s\cup A,\beta)]=[E^\circ_{\beta_s}\cap\mathscr{D}_{\mathsf{M}_s}]$$

where E°_β stands for $E_\beta\setminus\cup_{\beta'\neq\beta}E_{\beta'}$ and $\mathscr{D}_{\mathsf{M}_s}$ is the intersection $\cap_{\alpha\in\mathsf{M}_s}\mathscr{D}_{\alpha,s}$. Finally the value of $P^{\beta_s}_{\mathsf{M}_s}$ at $\left(\mathbf{L}^{-\rho'_\alpha}\right)_{\alpha\in\mathscr{A}}$ is

$$P^{\beta_s}_{\mathsf{M}_s}\left(\left(\mathbf{L}^{-\rho'_\alpha}\right)_{\alpha\in\mathscr{A}}\right)=(\mathbf{L}-1)^{|\mathsf{M}_s|}(1-\mathbf{L}^{-1})^{|\mathscr{A}_U|}\mathbf{L}^{\rho^{\beta_s}}[E^\circ_{\beta_s}\cap\mathscr{D}_{\mathsf{M}_s}]\mathbf{L}^{-n}.$$

In general, one rewrites the sum

$$\sum_{A\subset\mathscr{A}_U}[\Delta_s(\mathsf{M}_s\cup A,\beta_s)]\prod_{\alpha\in A}\mathbf{L}^{\epsilon_\alpha}\frac{1-\mathbf{L}^{-1}}{1-\mathbf{L}^{\epsilon_\alpha-1}}\mathbf{L}^{-n}$$

as

$$\sum_{A\subset\mathscr{A}_U}\sum_{\mathbf{m}'\in\mathbf{N}^A_{>0}}\int_{\substack{\Delta_s(A\cup\mathsf{M}_s,\beta)\times\mathscr{L}(\mathbf{A}^1,0)^A\times\mathscr{L}(\mathbf{A}^1,0)^{n-|A|}\\ \mathrm{ord}_s(x_\alpha)=m'_\alpha}}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha-1}t_\alpha)^{m'_\alpha}$$

evaluated at $t_\alpha=\mathbf{L}^{-(\rho_\alpha-\epsilon_\alpha)}$. In the end, adding a factor $\mathbf{L}^{-\langle\rho,\mathbf{e}^{\beta_s}\rangle}$ (the tuples $\mathbf{e}^{\beta_s}$ being defined pages 88 and 93), this gives

$$\begin{aligned}&\mathbf{L}^{-\langle\rho,\mathbf{e}^{\beta_s}\rangle}(\mathbf{L}-1)^{|\mathsf{M}_s|}(1-\mathbf{L}^{-1})^{|\mathscr{A}_U|}\mathbf{L}^{-n}\mathbf{L}^{\rho^{\beta_s}}\sum_{A\subset\mathscr{A}_U}[\Delta_s(\mathsf{M}_s\cup A,\beta_s)]\prod_{\alpha\in A}\mathbf{L}^{\epsilon_\alpha}\frac{1-\mathbf{L}^{-1}}{1-\mathbf{L}^{\epsilon_\alpha-1}}\\ &=(1-\mathbf{L}^{-1})^{|\mathscr{A}_U|+|\mathsf{M}_s|}\int_{\mathscr{L}((\mathscr{D}_{\mathcal{O}_s})_{\mathsf{M}_s},E^\circ_{\beta_s})}\mathbf{L}^{-\mathrm{ord}\,\mathscr{L}_\epsilon}\mathbf{L}^{-\mathrm{ord}\,\omega_X}.\end{aligned}$$

See Remark 4.4.13 for more about this interpretation.

Now we are able to consider the whole Euler product for $\xi=0$ and evaluate its coefficients. For any *choice of vertical components* $\beta\in\prod_v\mathscr{B}^{\mathscr{U}}_v$ (recall that this product is actually finite, see Section 4.1 for the definition of $\mathscr{B}^{\mathscr{U}}_v$), let us refine Notation 4.3.1 by setting for every $v\in\mathscr{C}(k)$

$$\mathscr{F}^\beta_v(\mathbf{u},0)=\prod_{\alpha\in\mathscr{A}_U}(1-\mathbf{L}^{\rho_\alpha-1}u_\alpha)\mathscr{Z}^\beta_v(\mathbf{u},0)$$

where $\mathscr{Z}^\beta_v(\mathbf{u},0)$ is the local factor of the refined zeta function $\mathscr{Z}^\beta(\mathbf{u},0)$, see Definition 4.2.2. This local factor only depends on the indeterminates $\mathbf{u}_0$ when $v\in\mathscr{C}_0(k)$ and on $(\mathbf{u}_0,\mathbf{u}_s)$ when $v=s\in S$. Similarly we define $\mathscr{P}^\beta_{\mathsf{M}_s}(\mathbf{u}_0,\mathbf{u}_s)$ starting from $P^\beta_{\mathsf{M}_s}(\mathbf{t})$ for every $s\in S$ and we write $\mathrm{Cl}^{\max}_S(X,D)$ for the product $\prod_{s\in S}\mathrm{Cl}^{\max}_s(X,D)$. Then

$$\begin{aligned}\mathscr{Z}^\beta(\mathbf{u},0)&=\prod_{v\in\mathscr{C}}\left(\prod_{\alpha\in\mathscr{A}_U}(1-\mathbf{L}^{\rho_\alpha-1}u_\alpha)^{-1}\mathscr{F}^\beta_v(\mathbf{u},0)\right)\\ &=\prod_{\alpha\in\mathscr{A}_U}Z_{\mathscr{C}}((\mathbf{L}^{\rho_\alpha-1}u_\alpha)^{m_\alpha})\prod_{v\in\mathscr{C}}\mathscr{F}^\beta_v(\mathbf{u},0)\\ &=\left(\prod_{\alpha\in\mathscr{A}_U}Z_C\left((\mathbf{L}^{\rho_\alpha-1}u_\alpha)^{m_\alpha}\right)\prod_{v\in\mathscr{C}_0}\mathscr{F}^\beta_v(\mathbf{u}_0,0)\right)\\ &\quad\times\prod_{s\in S}\left(\sum_{\mathsf{M}_s\in\mathrm{Cl}^{\max}_s(X,D)}\mathscr{P}^\beta_{\mathsf{M}_s}(\mathbf{u}_0,\mathbf{u}_s)\prod_{\alpha\in\mathsf{M}_s}\frac{1}{1-\mathbf{L}^{\rho_\alpha-1}u_{\alpha,s}}\prod_{\alpha\in\mathscr{A}_U}\frac{1-(\mathbf{L}^{\rho_\alpha-1}u_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}u_\alpha}\right)\\ &=\sum_{\mathsf{M}\in\mathrm{Cl}^{\max}_S(X,D)}\mathscr{Z}^\beta_{\mathsf{M}}(\mathbf{u},0)\end{aligned}$$

where for every $\mathsf{M}\in\mathrm{Cl}^{\max}_S(X,D)\subset\mathcal{P}(\mathscr{A}_D)^S$

$$\begin{aligned}\mathscr{Z}^\beta_{\mathsf{M}}(\mathbf{u},0)&=\left(\prod_{s\in S}\prod_{\alpha\in\mathsf{M}_s}\frac{1}{1-\mathbf{L}^{\rho_\alpha-1}u_{\alpha,s}}\right)\left(\prod_{\alpha\in\mathscr{A}_U}Z_{\mathscr{C}}((\mathbf{L}^{\rho_\alpha-1}u_\alpha)^{m_\alpha})\right)\\ &\quad\times\left(\prod_{v\in\mathscr{C}_0}\mathscr{F}^\beta_v(\mathbf{u}_0,0)\right)\left(\prod_{s\in S}\mathscr{P}^\beta_{\mathsf{M}_s}(\mathbf{u}_0,\mathbf{u}_s)\prod_{\alpha\in\mathscr{A}_U}\frac{1-(\mathbf{L}_s^{\rho_\alpha-1}u_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}u_\alpha}\right).\end{aligned}$$

This series only depends on the inderminates indexed by $\mathscr{A}_U$ or $\mathsf{M} = (\mathsf{M}_s)_{s\in S} \subset \mathscr{A}_D^S$. Let us introduce the series

$$\mathscr{G}_{\mathsf{M}}(\mathbf{u},0) = \prod_{\alpha\in\mathscr{A}_U} Z_{\mathscr{C}}\left(\left(\mathbf{L}^{\rho'_\alpha-1}u_\alpha\right)^{m_\alpha}\right)\prod_{s\in S}\left(\prod_{\alpha\in\mathsf{M}_s}(1-\mathbf{L}_s^{\rho_\alpha-1}u_{\alpha,s})^{-1}\right)$$

for all choice of maximal faces $\mathsf{M} = (\mathsf{M}_s)_{s\in S} \in \mathrm{Cl}_S^{\max}(X,D)$. These notations allow us to rewrite $\mathscr{Z}_{\mathsf{M}}(\mathbf{t},0)$ as

$$\mathscr{Z}_{\mathsf{M}}^{\beta}(\mathbf{u},0) = \mathscr{G}_{\mathsf{M}}(\mathbf{u},0)\prod_{v\in\mathscr{C}_0}\left(\mathscr{F}_v^{\beta}(\mathbf{u}_0,0)\right)\prod_{s\in S}\left(\mathscr{P}_{\mathsf{M}_s}^{\beta}(\mathbf{u}_0,\mathbf{u}_s)\right).$$

We may decompose $\mathscr{G}_{\mathsf{M}}(\mathbf{u},0)$ as

$$\mathscr{G}_{\mathsf{M}}(\mathbf{u},0) = \widetilde{\mathscr{G}_{\mathsf{M}}}(\mathbf{u},0) + \mathscr{H}_{\mathsf{M}}(\mathbf{u},0)$$

where

$$\widetilde{\mathscr{G}_{\mathsf{M}}}(\mathbf{u},0) = \widetilde{G}(\mathbf{u}_0,0)\prod_{s\in S}\prod_{\alpha\in\mathsf{M}_s}(1-\mathbf{L}_s^{\rho'_\alpha}u_{\alpha,s})^{-1}$$
$$\mathscr{H}_{\mathsf{M}}(\mathbf{u},0) = H(\mathbf{u}_0,0)\prod_{s\in S}\prod_{\alpha\in\mathsf{M}_s}(1-\mathbf{L}_s^{\rho'_\alpha}u_{\alpha,s})^{-1}$$

with $\widetilde{G}(\mathbf{t},0)$ and $H(\mathbf{t},0)$ defined in (4.4.2.25) and (4.4.2.26) (this definition is licit since $\widetilde{G}(\mathbf{t},0)$ and $H(\mathbf{t},0)$ only depend on the indeterminates indexed by $\mathscr{A}_U$). This leads to a decomposition

$$\mathscr{Z}_{\mathsf{M}}^{\beta}(\mathbf{u},0) = \widetilde{\mathscr{Z}_{\mathsf{M}}^{\beta}}(\mathbf{u},0)+\mathscr{H}_{\mathsf{M}}(\mathbf{u},0)\left(\prod_{v\in\mathscr{C}_0}\mathscr{F}_v^{\beta}(\mathbf{u}_0,0)\right)\left(\prod_{s\in S}\mathscr{P}_{\mathsf{M}_s}^{\beta_s}(\mathbf{u}_0,\mathbf{u}_s)\prod_{\alpha\in\mathscr{A}_U}\frac{1-(\mathbf{L}_s^{\rho_\alpha-1}u_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}u_\alpha}\right)$$

where

$$\widetilde{\mathscr{Z}_{\mathsf{M}}^{\beta}}(\mathbf{u},0) = \widetilde{\mathscr{G}_{\mathsf{M}}}(\mathbf{u},0)\left(\prod_{v\in\mathscr{C}_0}\mathscr{F}_v^{\beta}(\mathbf{u}_0,0)\right)\left(\prod_{s\in S}\mathscr{P}_{\mathsf{M}_s}^{\beta_s}(\mathbf{u}_0,\mathbf{u}_s)\prod_{\alpha\in\mathscr{A}_U}\frac{1-(\mathbf{L}_s^{\rho_\alpha-1}u_\alpha)^{m_\alpha}}{1-\mathbf{L}^{\rho_\alpha-1}u_\alpha}\right).$$

Now we are ready to adapt the $\mathscr{U}=\mathscr{X}$ case to the study of the coefficients of $\mathscr{Z}_{\mathsf{M}}^{\beta}(\mathbf{u},0)$ and $\widetilde{\mathscr{Z}_{\mathsf{M}}^{\beta}}(\mathbf{u},0)$ for a fixed $\mathsf{M}\in\mathrm{Cl}_S^{\max}(X,D)$. We are going to show that asymptotically the main contribution in this decomposition of $\mathscr{Z}_{\mathsf{M}}^{\beta}(\mathbf{u},0)$ comes from $\widetilde{\mathscr{Z}_{\mathsf{M}}^{\beta}}(\mathbf{u},0)$ while the error term coming from $\widetilde{\mathscr{Z}_{\mathsf{M}}^{\beta}}(\mathbf{u},0)-\mathscr{Z}_{\mathsf{M}}^{\beta}(\mathbf{u},0)$ is negligible.

An expression of $\widetilde{\mathscr{G}_{\mathsf{M}}}(\mathbf{u},0)$ is given by

$$\widetilde{\mathscr{G}_{\mathsf{M}}}(\mathbf{u},0) = \left(\frac{[\mathrm{Pic}^0(\mathscr{C})]\mathbf{L}^{1-g}}{\mathbf{L}-1}\right)^{|\mathscr{A}_U|}\sum_{b\in\{0,1\}^{\mathscr{A}_U}}(-1)^{|b|}\widetilde{\mathscr{G}_{\mathsf{M},b}}(\mathbf{u},0)\mathbf{L}^{|b|(g-1)}$$

where for all $b\in\{0,1\}^{\mathscr{A}_U}$

$$\begin{aligned}\widetilde{\mathscr{G}_{\mathsf{M},b}}(\mathbf{u},0) &= \prod_{\alpha\in\mathscr{A}_U}\left(1-\mathbf{L}^{-b_\alpha}(\mathbf{L}^{\rho_\alpha-\epsilon_\alpha}u_\alpha)^{m_\alpha}\right)^{-1}\prod_{s\in S}\prod_{\alpha\in\mathsf{M}_s}(1-\mathbf{L}_s^{\rho_\alpha-1}U_{\alpha,s})^{-1}\\ &= \sum_{\substack{\mathsf{m}_0\in\mathbf{N}^{\mathscr{A}_U}\\ (\mathsf{m}_s)\in\prod_{s\in S}\mathbf{N}^{\mathsf{M}_s}}}\mathbf{L}^{\langle\rho-\epsilon-b,m\cdot\mathsf{m}_0\rangle_{\mathscr{A}_U}+\sum_{s\in S}\langle\rho-1,\mathsf{m}_s\rangle_{\mathsf{M}_s}}\mathbf{u}^{\mathsf{m}_0+(\mathsf{m}_s)_{s\in S}}.\end{aligned}$$

The coefficient of order $\mathbf{m}' \in \mathbf{N}^{\mathscr{A}_U} \times \mathbf{N}^{\mathsf{M}} \subset \mathbf{N}^{\mathscr{A}_U} \times \mathbf{N}^{\mathscr{A}_D \times S}$ of $\widetilde{\mathscr{Z}^{\beta}_{\mathsf{M}}}(\mathbf{u}, 0)$ is equal to

$$\sum_{\substack{\mathbf{n},\mathbf{n}' \in \mathbf{N}^{\mathscr{A}_U} \times \mathbf{N}^{\mathsf{M}} \\ \mathbf{n}_{\mathscr{A}_U} + m\mathbf{n}'_{\mathscr{A}_U} = \mathbf{m}'_{\mathscr{A}_U} \\ \mathbf{n}_{\mathsf{M}} + \mathbf{n}'_{\mathsf{M}} = \mathbf{m}'_{\mathsf{M}}}} \mathfrak{c}^{\beta}_{\mathsf{M}}(\mathbf{n}) \mathbf{L}^{\langle \rho - \epsilon - b, m\mathbf{n}' \rangle_{\mathscr{A}_U} + \langle \rho - 1, \mathbf{n}' \rangle_{\mathsf{M}}} \tag{4.4.2.31}$$

where $\mathfrak{c}^{\beta}_{\mathsf{M}}(\mathbf{n})$ is the $\mathbf{n}$-th coefficient (with $\mathbf{n} \in \mathbf{N}^{\mathscr{A}_U} \times \mathbf{N}^{\mathsf{M}}$) of

$$\prod_{v \in \mathscr{C}_0} \mathscr{F}^{\beta}_v(\mathbf{u}_0, 0) \prod_{s \in S} \left(P^{\beta_s}_{\mathsf{M}_s}(\mathbf{u}_0, \mathbf{u}_s) \prod_{\alpha \in \mathscr{A}_U} \frac{1 - (\mathbf{L}^{\rho_\alpha - 1} u_\alpha)^{m_\alpha}}{1 - \mathbf{L}^{\rho_\alpha - 1} u_\alpha} \right)$$

and $\mathscr{A}_U \sqcup \mathsf{M}$ is the disjoint union of $\mathscr{A}_U$ and M in $\mathscr{A}_U \sqcup \mathscr{A}_D^S$. After normalising by $\mathbf{L}^{\langle \rho - \epsilon, \mathbf{m}' \rangle_{\mathscr{A}_U \sqcup \mathsf{M}}}$, the coefficient of (4.4.2.31) for $b = \mathbf{0}$ is the $\mathbf{m}'$-th partial sum of the motivic Euler product

$$\left(\prod_{v \in \mathscr{C}_0} \mathscr{F}^{\beta}_v(\mathbf{u}_0, 0) \right) ((\mathbf{L}^{-(\rho_\alpha - \epsilon_\alpha)})_{\alpha \in \mathscr{A}_U})$$
$$\times \prod_{s \in S} \left(P^{\beta_s}_{\mathsf{M}_s}(\mathbf{u}_0, \mathbf{u}_s) \prod_{\alpha \in \mathscr{A}_U} \frac{1 - (\mathbf{L}^{\rho_\alpha - 1} u_\alpha)^{m_\alpha}}{1 - \mathbf{L}^{\rho_\alpha - 1} u_\alpha} ((\mathbf{L}^{-(\rho_\alpha - \epsilon_\alpha)})_{\alpha \in \mathscr{A}_U}, (\mathbf{L}^{-(\rho_\alpha - 1)})_{\alpha \in \mathscr{A}_D}) \right)$$

to which one can apply Proposition 4.3.2. If $b \neq \mathbf{0}$, the only difference is again the factor $\mathbf{L}^{-\langle b, m\mathbf{n} - \mathbf{m}' \rangle_{\mathscr{A}_U}}$, which is controlled with Lemma 2.4.28 as in the case $\mathscr{U} = \mathscr{X}$. Since by Lemma 2.4.27 we know that weight-linear convergence is stable by finite product, we deduce the following proposition.

Proposition 4.4.8. *For all choice of maximal faces $\mathsf{M} \in \mathrm{Cl}^{\max}_S(X, D)$ and vertical components $\beta \in \prod_v \mathscr{B}^{\mathscr{U}}_v$, there exists a decomposition*

$$\widetilde{\mathscr{Z}^{\epsilon,\beta}_{|W,\mathsf{M}}}(\mathbf{u}, 0) = \sum_{\mathbf{m}' \in \mathbf{N}^{\mathscr{A}_U} \times \mathbf{N}^{\mathsf{M}}} \tilde{\mathfrak{a}}^{\beta}_{\mathbf{m}'} \mathbf{u}^{\mathbf{m}'} = \sum_{B \subset \mathscr{A}_U} \sum_{\mathbf{m}' \in \mathbf{N}^{\mathscr{A}_U} \times \mathbf{N}^{\mathsf{M}}} \tilde{\mathfrak{a}}^{\beta, B}_{\mathbf{m}'} \mathbf{u}^{\mathbf{m}'}$$

such that in $\widehat{\mathscr{M}_{k,r}}$

$$\tilde{\mathfrak{a}}^{\beta, \varnothing}_{\mathbf{m}'} \mathbf{L}^{-\langle \rho - \epsilon, \mathbf{m}' + \mathbf{e}^{\beta} \rangle_{\mathscr{A}_U \sqcup \mathsf{M}}}$$

converges $(\rho - \epsilon)$-weight-linearly to the non-zero effective element

$$\left(\frac{[\mathrm{Pic}^0(\mathscr{C})] \mathbf{L}^{1-g}}{\mathbf{L} - 1} \right)^{\mathrm{rk}(\mathrm{Pic}(U))} \times \prod_{v \in \mathscr{C}_0} \left((1 - \mathbf{L}^{-1})^{\mathrm{rk}(\mathrm{Pic}(U))} \int_{\mathscr{L}^{\epsilon}(\mathscr{X}_{\mathscr{O}_v}, E^{\circ}_{\beta_v})} \mathbf{L}^{-\mathrm{ord}_{\mathscr{L}_\epsilon} |\omega_X|} \right)$$
$$\times \prod_{s \in S} \left((1 - \mathbf{L}^{-1})^{|\mathsf{M}_s| + \mathrm{rk}(\mathrm{Pic}(U))} \int_{\mathscr{L}(\mathscr{D}_{\mathsf{M}_s}, E^{\circ}_{\beta_s})} \mathbf{L}^{-\mathrm{ord}_{\mathscr{L}_\epsilon} |\omega_X|} \right)$$

with error term of weight bounded by

$$-\frac{1}{\langle \rho - \epsilon, \frac{1+\epsilon}{1-\epsilon} \rangle_{\mathscr{A}_U}} \langle \rho - \epsilon, \mathbf{m}' \rangle_{\mathscr{A}_U} + 2(\rho^{\beta} - \langle \rho - \epsilon, \mathbf{e}^{\beta} \rangle),$$

and for every $\varnothing \neq B \subset \mathscr{A}_U$

$$w\left(\tilde{\mathfrak{a}}^{\beta, B}_{\mathbf{m}'} \mathbf{L}^{-\langle \rho', \mathbf{m}' \rangle_{\mathscr{A}_U \sqcup \mathsf{M}}} \right) \leqslant -\frac{1}{\langle \rho - \epsilon, \frac{1+\epsilon}{1-\epsilon} \rangle_{\mathscr{A}_U}} \langle \rho - \epsilon, \mathbf{m}' \rangle_B + 2\rho^{\beta}$$

for all $\mathbf{m}' \in \mathbf{N}^{\mathscr{A}_U} \times \mathbf{N}^{\mathsf{M}}$.

Now we can adapt the case $\mathscr{U} = \mathscr{X}$ to control the contribution of $\mathscr{H}_{\mathsf{M}}(\mathbf{u},0)$. One immediately gets a decomposition

$$\mathscr{H}_{\mathsf{M}}(\mathbf{u},0) = \sum_{\substack{\varnothing\neq A\subset\mathscr{A}_U \\ b\in\{0,1\}^{\mathscr{A}_U\setminus A}\times\{1\}^A}} [\mathrm{Pic}^0(\mathscr{C})]^{|\mathscr{A}_U\setminus A|}\frac{(-\mathbf{L}^{1-g})^{|\mathscr{A}_U|-|b|}}{(1-\mathbf{L})^{|\mathscr{A}_U\setminus A|}}\mathscr{H}_{\mathsf{M}}^{A,b}(\mathbf{u},\xi)$$

where for every $A\subset\mathscr{A}_U$ non-empty and $b\in\{0,1\}^{\mathscr{A}_U\setminus A}\times\{1\}^A$

$$\mathscr{H}_{\mathsf{M}}^{A,b}(\mathbf{u},0) = \sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}_U}\times\mathbf{N}^{\mathsf{M}}} \mathbf{L}^{\langle\rho'-b,\mathbf{m}'\rangle_{\mathscr{A}_U\sqcup\mathsf{M}}}\prod_{\alpha\in A}\left([\mathrm{Sym}^{m_\alpha}\mathscr{C}]-[\mathrm{Pic}^0(\mathscr{C})]\frac{\mathbf{L}^{m_\alpha-g-1}-1}{\mathbf{L}-1}\right)\mathbf{u}^{\mathbf{m}'}.$$

Adapting the argument from the $\mathscr{U}=\mathscr{X}$ case, one deduces the following proposition.

Proposition 4.4.9 ($\mathscr{U}\subset\mathscr{X}$, $\mathscr{C}_0\subset\mathscr{C}$, $\xi=0$)**.** *For all choice of maximal faces* $\mathsf{M}\in\mathrm{Cl}_S^{\max}(X,D)$*, there is a decomposition*

$$\begin{aligned}&\mathscr{H}_{\mathsf{M}}(\mathbf{u},0)\left(\prod_{v\in\mathscr{C}_0}\mathscr{F}_v^{\beta}(\mathbf{u}_0,0)\right)\left(\prod_{s\in S}\mathscr{P}_{\mathsf{M}_s}^{\beta_s}(\mathbf{u}_0,\mathbf{u}_s)\prod_{\alpha\in\mathscr{A}_U}\frac{1-(\mathbf{L}_s^{\rho_\alpha-1}u_\alpha)^{m_\alpha}}{1-\mathbf{L}_s^{\rho_\alpha-1}u_\alpha}\right)\\ &=\sum_{\substack{\varnothing\neq A\subset\mathscr{A}_U\\ B\subset\mathscr{A}_U\setminus A}}\sum_{\mathbf{m}'\in\mathbf{N}^{\mathscr{A}_U}\times\mathbf{N}^{\mathsf{M}}}\left(\mathfrak{h}_{\mathsf{M}}^{\beta,A,B}\right)_{\mathbf{m}'}\mathbf{u}^{\mathbf{m}'}\end{aligned}$$

such that for all $\varnothing\neq A\subset\mathscr{A}_U$ *and* $B\subset\mathscr{A}_U\setminus A$

$$w\left(\left(\mathfrak{h}_{\mathsf{M}}^{\beta,A,B}\right)_{\mathbf{m}'}\mathbf{L}^{-\langle\rho',\mathbf{m}'\rangle_{\mathscr{A}_U\sqcup\mathsf{M}}}\right)\leqslant-\frac{1}{\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle_{\mathscr{A}_U}}\langle\rho-\epsilon,\mathbf{m}'\rangle_{A\sqcup B}+2(\rho^{\beta_v}+|A|(g-1)_+)$$

for all $\mathbf{m}'\in\mathbf{N}^{\mathscr{A}_U}\times\mathbf{N}^{\mathsf{M}}$.

4.4.2.4. *S-integral points and non-trivial characters.* Concerning places $s\in S$, recall that the local factor of $Z^\beta(\mathbf{t})$ is given by the formula

$$Z_s^\beta(\mathbf{t},\xi)=\sum_{A\subset\mathscr{A}}\mathbf{L}^{\rho^\beta}\int_{\Omega_s(A,\beta)}\prod_{\alpha\in A}(\mathbf{L}^{\rho_\alpha}t_\alpha)^{\mathrm{ord}_s(x_\alpha)}\mathbf{e}(\langle x,\xi\rangle)\mathrm{d}x.$$

Let $\mathrm{Cl}_s(X,D)_0$ be the subcomplex of $\mathrm{Cl}_s(X,D)$ where we only keep vertices $\alpha\in\mathscr{A}$ such that $d_\alpha(\xi)=0$. Then Proposition 5.3.1 of [CLL16] can be rewritten as follows.

Proposition 4.4.10. *Assume first that f_ξ extends to a regular map $\mathscr{X}_{F_s}\to\mathbf{P}^1_{F_s}$. Then there is a family $(P_{\mathsf{M}_s^0}^{\beta_s})_{\mathsf{M}_s^0\in\mathrm{Cl}^{\max}(X,D)_0}$ of polynomials with coefficients in $\mathscr{E}xp\mathscr{M}_k$ such that*

$$Z_s^\beta(\mathbf{t},\xi_s)=\sum_{\mathsf{M}_s^0\in\mathrm{Cl}_s^{\max}(X,D)_0}\frac{P_{\mathsf{M}_s^0}^{\beta_s}(\mathbf{t},\xi_s)}{\prod_{\alpha\in\mathscr{A}_U^0(\xi)\sqcup\mathsf{M}_s^0}(1-\mathbf{L}^{\rho_\alpha-1}t_\alpha)}.$$

If f_ξ does not extend to a regular map $\mathscr{X}_{F_s}\to\mathbf{P}^1_{F_s}$ then the argument of Chambert-Loir and Loeser in [CLL16] consists in using a resolution of indeterminacies, that is a proper birational morphism $\varpi:\mathscr{Y}_{\mathscr{O}_s}\to\mathscr{X}_{\mathscr{O}_s}$ such that ϖ^*f extends to a regular map from $\mathscr{Y}_{\mathscr{O}_s}$ to $\mathbf{P}^1_{\mathscr{O}_v}$. Moreover one can assume that the generic fibre $\mathscr{Y}_{F_s}\to\mathscr{X}_{F_s}$ is invariant under the action of G_{F_v}, and this can be done locally uniformly with respect to ξ [CLL16, Lemma 6.5.1]. Let $\mathscr{D}'_{\alpha,s}$ be the strict transform of $\mathscr{D}_{\alpha,s}$ through φ and

$(\mathscr{E}_\gamma)_{\gamma\in\Gamma}$ be the set of horizontal exceptional divisors of ϖ. There exists a family $(m_{\gamma,\alpha})$ of nonnegative integers such that

$$\pi^*(\mathscr{D}_{\alpha,s})_{F_s} = (\mathscr{D}'_{\alpha,s})_{F_s} + \sum_{\gamma\in\Gamma} m_{\gamma,\alpha}(\mathscr{E}_\gamma)_{F_s}$$

as well as positive integers ν_γ such that

$$K_{\mathscr{Y}_{F_s}/\mathscr{X}_{F_s}} = \sum_{\gamma\in\Gamma} (\nu_\gamma - 1)\,(\mathscr{E}_\gamma)_{F_s}$$

and v a bounded constructible function on $\mathscr{L}(\mathscr{Y}_{\mathscr{O}_s})$ such that

$$\operatorname{ord}_{K_{\mathscr{Y}_{\mathscr{O}_s}/\mathscr{X}_{\mathscr{O}_s}}} = \sum_{\gamma\in\Gamma}(\nu_\gamma - 1)\operatorname{ord}_{\mathscr{E}_\gamma} + v.$$

Chambert-Loir and Loeser then use the change of variable formula [**CL10**, Theorem 13.2.2] and obtain an integral over $\mathscr{L}(\mathscr{Y})$, which in our case gives

$$\begin{aligned} Z_s^\beta(\mathbf{t},\xi_s) = \sum_{A\subset\mathscr{A}} \mathbf{L}^{\rho^\beta}\times \\ \times \sum_{\substack{\mathbf{p}\in\mathbf{Z}^{\mathscr{A}}\\ \text{(finite sum)}}} \mathbf{L}^{\langle\rho',\mathbf{p}\rangle}\mathbf{t}^{\mathbf{p}} \int_{W_\mathbf{p}\cap\varpi^*\Omega_s(A,\beta)} \mathbf{L}^{\langle\rho',\mathbf{ord}_{\mathscr{D}'}(y)\rangle}\mathbf{t}^{\mathbf{ord}_{\mathscr{D}'}(y)}\times \\ \times\prod_{\gamma\in\Gamma}\left(\mathbf{L}^{\nu_\gamma-1}\mathbf{L}^{\langle\rho',\mathbf{m'}_\gamma\rangle}\mathbf{t}^{\mathbf{m'}_\gamma}\right)^{\mathbf{ord}_{\mathscr{E}_\gamma}(y)}\mathbf{L}^{v(y)}\mathbf{e}(\varpi^* f_{\xi_s}(y))\mathbf{d}y \end{aligned}$$

where $(W_\mathbf{p})$ is a finite partition of $\mathscr{L}(\mathscr{Y})$.

Then one applies Proposition 4.4.10 to $\varpi^* f_{\xi_v}$ and to the integral over $W_\mathbf{p}\cap\varpi^*\Omega_v(A,\beta)$ to get a similar result: $Z_s(\mathbf{t},\xi_s)$ is rational with denominators given by products of polynomials of the form $(1-\mathbf{L}^{\langle\rho',\mathbf{a}\rangle}\mathbf{t}^\mathbf{a})$ for some $\mathbf{a}\in\mathbf{N}^{\mathscr{A}}$. One has to justify that this procedure does not change the set $\mathscr{A}$ and the relevant faces of Clemens complex. The argument is given in the very last paragraph of [**CLL16**] and in [**CLT12**, §3.4]: the F_s-Clemens complex $\mathrm{Cl}_{F_s}(Y, Y\setminus\cup_{\alpha\in\mathscr{A}_U}(\mathscr{D}'_{\alpha,s})_{F_s})$ has vertices coming from X, corresponding to the strict transforms $\mathscr{D}'_\alpha$ for $\alpha\in\mathscr{A}_D$, and vertices corresponding to the divisors $\mathscr{E}_\gamma$ contracted by ϖ. The divisor of $\varpi^* f_\xi$ on $\mathscr{Y}_{F_s}$ takes the form

$$\Xi'_\xi - \sum_{\alpha\in\mathscr{A}} d_\alpha(\xi)(\mathscr{D}'_{\alpha,s})_{F_s} - \sum_{\gamma\in\Gamma} e_\gamma(\mathscr{E}_\gamma)_{F_s}$$

where Ξ'_ξ is the strict transform of the hyperplane $\langle\xi,x\rangle = 0$. Since Y_{F_s} is an equivariant compactification of G_{F_s}, the integers e_γ are all non-negative. Furthermore $K_{\mathscr{Y}_{F_s}/\mathscr{X}_{F_s}}$ is a linear combination of the $(\mathscr{E}_\gamma)_{F_s}$ with non-negative coefficients. Consequently, looking back on the formula giving $Z_s^\beta(\mathbf{t},\xi_s)$ above, one remarks that only the vertices coming from X_{F_s} will contribute to the poles.

Replacing $\xi = 0$ by $\xi\neq 0$, $\mathscr{A}_U$ by $\mathscr{A}_U^0(\xi)$ and $\mathrm{Cl}_s(X,D)$ by $\mathrm{Cl}_s(X,D)_0$ everywhere in the previous paragraph, including in the definitions of $\widetilde{\mathscr{Z}_\mathsf{M}^\beta}$, $\widetilde{\mathscr{G}_\mathsf{M}}$ and $\widetilde{\mathscr{H}_\mathsf{M}}$, one gets the following proposition. Since $\mathrm{Cl}_s^{\max}(X,D)_0$ is a proper subset of $\mathrm{Cl}_s^{\max}(X,D)$ for every $s\in S$ [**CLT12**, Lemma 3.5.4], this proposition implies that non-trivial characters do not contribute asymptotically, included in the case $\mathscr{U}\neq\mathscr{X}$.

Proposition 4.4.11. *Let ξ be a non-trivial character. For all*

$$\mathsf{M}^0\in\mathrm{Cl}_S^{\max}(X,D)_0\subset\mathcal{P}(\mathscr{A}_D)^S$$

and all choice of vertical components β, there exist decompositions

$$\widetilde{\mathscr{Z}^{\beta}_{\mathsf{M}^0}}(\mathbf{u},\xi) = \sum_{B\subset \mathscr{A}^0_U(\xi)} \sum_{\mathbf{m}'\in \mathbf{N}^{\mathscr{A}_U}\times \mathbf{N}^{\mathsf{M}^0}} \left(\tilde{\mathfrak{a}}^{\beta,\xi,B}_{\mathsf{M}^0}\right)_{\mathbf{m}'} \mathbf{u}^{\mathbf{m}'}$$

$$\mathscr{H}_{\mathsf{M}^0}(\mathbf{u},\xi) = \sum_{\substack{\varnothing\neq A\subset \mathscr{A}^0_U(\xi)\\ B\subset \mathscr{A}^0_U(\xi)\\ A\cap B=\varnothing}} \sum_{\mathbf{m}'\in \mathbf{N}^{\mathscr{A}_U}\times \mathbf{N}^{\mathsf{M}^0}} \left(\mathfrak{h}^{\beta,\xi,A,B}_{\mathsf{M}^0}\right)_{\mathbf{m}'} \mathbf{u}^{\mathbf{m}'}$$

such that for all $B\subset \mathscr{A}^0_U(\xi)$

$$w\left(\left(\tilde{\mathfrak{a}}^{\beta,\xi,B}_{\mathsf{M}^0}\right)_{\mathbf{m}'} \mathbf{L}^{-\langle \rho-\epsilon,\mathbf{m}'\rangle_{\mathscr{A}_U\sqcup \mathsf{M}^0}}\right) \leqslant -\delta\langle \rho-\epsilon,\mathbf{m}'\rangle_{B\sqcup(\mathscr{A}_U\setminus \mathscr{A}^0_U(\xi))} + 2\rho^{\beta}$$

and for all $\varnothing\neq A\subset \mathscr{A}^0_U(\xi)$ and $B\subset \mathscr{A}^0_U(\xi)\setminus A$

$$w\left(\left(\mathfrak{h}^{\beta,\xi,A,B}_{\mathsf{M}^0}\right)_{\mathbf{m}'} \mathbf{L}^{-\langle \rho-\epsilon,\mathbf{m}'\rangle_{\mathscr{A}_U\sqcup \mathsf{M}^0}}\right) \leqslant -\delta\langle \rho-\epsilon,\mathbf{m}'\rangle_{A\sqcup B} + 2(\rho^{\beta}+|A|(g-1)_+)$$

for all $\mathbf{m}'\in \mathbf{N}^{\mathscr{A}_U}\times \mathbf{N}^{\mathsf{M}^0}$, where $\delta = \min\limits_{\alpha\in\mathscr{A}_U} \frac{1-\epsilon_\alpha}{\rho_\alpha-\epsilon_\alpha}$.

4.4.3. Summation over all the characters and convergence. In this subsection we perform the final step of our proof: we have to permute a motivic summation with taking a limit.

4.4.3.1. *Expression of $\left[\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid \mathsf{m}_S)^{\beta}\right]$ in $\mathscr{M}_k$.* The $\mathbf{m}'$-th coefficient of the Zeta function $\mathscr{Z}^{\epsilon,\beta}_{|W}(\mathbf{u})$ is by definition the class $\left[\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid \mathsf{m}_S)^{\beta}\right]$ of the moduli space of Campana sections we are interested in. Recall that we have a decomposition

$$\mathscr{Z}^{\epsilon,\beta}_{|W}(\mathbf{u}) = \mathbf{L}^{n(1-g)}\left(\mathscr{Z}^{\epsilon,\beta}_{|W}(\mathbf{u},0) + \sum_{\xi\in V\setminus\{0\}} \mathscr{Z}^{\epsilon,\beta}_{|W}(\mathbf{u},\xi)\right).$$

where $V = L(\tilde{E})^n$ is the n-th power of a finite dimensional Riemann-Roch space. The class of $\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid \mathsf{m}_S)^{\beta}$ in $\mathscr{M}_k$ is given by

$$\left[\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid \mathsf{m}_S)^{\beta}\right] = \mathbf{L}^{(1-g)n}\sum_{\xi\in V}\sum_{\mathsf{m}\in \mathrm{S}^{\mathbf{n}^{\beta}_U}_{/k}(\mathscr{C})} \theta^{*}_{\mathbf{n}^{\beta}}\mathscr{F}(\mathbf{1}_{H^{\epsilon}(\mathsf{m}+\mathsf{m}_S,\beta)\cap W})(\xi)$$

where the two sums here are motivic sums (given by the projections to $S^{\mathbf{n}^{\beta}_U}(\mathscr{C})$ and then to k), $\tilde{E}$ is the divisor defined by (4.2.3.18) page 96 and

$$\theta_{\mathbf{n}^{\beta}} : V\times \mathrm{Sym}^{\mathbf{n}^{\beta}}_{/k}(\mathscr{C}) \longrightarrow \mathscr{A}_{\mathbf{n}^{\beta}}(\nu-s,\nu-s',N,0)$$

is the morphism introduced in Section 2.2.5. By definition,

$$\sum_{\mathsf{m}\in \mathrm{Sym}^{\mathbf{n}^{\beta}_U}_{/k}(\mathscr{C})} \theta^{*}_{\mathbf{n}^{\beta}}\mathscr{F}(\mathbf{1}_{H^{\epsilon}(\mathsf{m}+\mathsf{m}_S,\beta)\cap W})(\xi) \in \mathscr{E}xp\mathscr{M}_V$$

is the coefficient of degree $(\mathbf{n}_U,\mathsf{m}_S)$ of

$$\prod_{v\in\mathscr{C}} \mathscr{Z}^{\epsilon,\beta_v}_{|W_v}(\mathbf{u},\xi).$$

4.4.3.2. *Uniform convergence.* By Remark 4.3.6, we already know the existence of a finite constructible partition of $V \setminus \{0\}$ over which the weight-linear convergence of $\prod_{v\in\mathscr{C}_1} F_v(\mathbf{t},\xi)$ with respect to $\rho-\epsilon$ is uniform in ξ. Concerning places in S, Proposition 4.4.10 and [**CLL16**, Lemma 6.5.1] allows us to resolve indeterminacies of f_ξ uniformly on a partition (which is actually finite for the same reason). Taking a partition refining both previous partitions, we get a finite partition P of $V\setminus\{0\}$ over which $\xi\mapsto\mathscr{A}_U^0(\xi)$ and $\xi\mapsto\mathscr{A}_U^1(\xi)$ are constant, as well as the corresponding degrees for $\alpha\in\mathscr{A}_D$, which means that $(\rho-\epsilon)$-weight-linear convergence is uniform over such a partition. This partition does not depend on β or W. This provides a decomposition

$$\left[\operatorname{Hom}_{\mathscr{C}}^{\mathbf{n}}(\mathscr{C},\mathscr{X}\mid\epsilon\mid W\mid \mathsf{m}_S)^\beta\right] = \mathbf{L}^{(1-g)n}\sum_{\mathsf{m}\in\operatorname{Sym}_{/k}^{\mathbf{n}_U^\beta}(\mathscr{C})}\theta^*_{\mathbf{n}^\beta}\mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m}+\mathsf{m}_S,\beta)\cap W})(0)$$
$$+\mathbf{L}^{(1-g)n}\sum_{\substack{P\\ \text{stratum}}}\sum_{\xi\in P}\sum_{\mathsf{m}\in\operatorname{Sym}_{/k}^{\mathbf{n}_U^\beta}(\mathscr{C})}\theta^*_{\mathbf{n}^\beta}\mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m}+\mathsf{m}_S,\beta)\cap W})(\xi).$$

All the results of §4.4.2 hold uniformly on each stratum P, which means that we can pass from the decomposition of $\prod_{v\in\mathscr{C}}\mathscr{Z}^{\epsilon,\beta_v}_{|W_v}(\mathbf{u},\xi)$ given by Proposition 4.4.11 to a decomposition of

$$\sum_{\xi\in V}\prod_{v\in\mathscr{C}}\mathscr{Z}^{\epsilon,\beta_v}_{|W_v}(\mathbf{u},\xi)$$

from which one deduces a decomposition of its coefficient

$$\sum_{\xi\in P}\sum_{\mathsf{m}\in\operatorname{Sym}_{/k}^{\mathbf{n}_U^\beta}(\mathscr{C})}\theta^*_{\mathbf{n}^\beta}\mathscr{F}(\mathbf{1}_{H^\epsilon(\mathsf{m}+\mathsf{m}_S,\beta)\cap W})(\xi)$$

for every $(\mathbf{n}_U,\mathsf{m}_S)\in\mathbf{N}^{\mathscr{A}_U}\times\mathbf{N}^{\mathscr{A}_D\times S}$. Summing over the finite set of stratum P and adding the term coming from the trivial character, we get a decomposition of $\left[\operatorname{Hom}_{\mathscr{C}}^{\mathbf{n}}(\mathscr{C},\mathscr{X}\mid\mathsf{m}_S)^\beta\right]$.

4.4.4. Final result. We combine the uniform convergence argument above and the results of Propositions 4.4.8, 4.4.9 and 4.4.11, fixing a choice of vertical components $\beta\in\prod_v\mathscr{B}_v^{\mathscr{U}}$.

Note that the summation over V adds a term $\dim V = n\dim L(\tilde{E})$ to the upper bounds on weights, where the divisor $\tilde{E}$ is given by (4.2.3.18) page 96.

Proposition 4.4.12. *Let $\mathsf{M}\in\operatorname{Cl}_S^{\max}(X,D)$ be any choice of maximal faces over the places $s\in S$ and $\beta\in\prod_v\mathscr{B}_v^{\mathscr{U}}$ a choice of vertical components. When $\min(m'_{\alpha,s})$ becomes arbitrarily large, the $\mathbf{m}'$-th coefficient of*

$$\mathscr{Z}^{\epsilon,\beta}_{|W,\mathsf{M}}(\mathbf{u}) = \mathbf{L}^{(1-g)n}\sum_{\xi\in V}\mathscr{Z}^{\epsilon,\beta}_{|W,\mathsf{M}}(\mathbf{u},\xi),$$

normalised by $\mathbf{L}^{\langle\rho-\epsilon,\mathbf{m}'+\mathbf{e}^{\beta}\rangle}$*, converges in* $\widehat{\mathscr{E}xp\mathscr{M}_{k,r}}$ *to a non-zero effective element of* $\widehat{\mathscr{M}_{k,r}}$ *which can be written as the following motivic Euler product*

$$\mathbf{L}^{(1-g)n}\left(\frac{[\mathrm{Pic}^0(\mathscr{C})]\mathbf{L}^{1-g}}{\mathbf{L}-1}\right)^{\mathrm{rk}(\mathrm{Pic}(U))}$$
$$\times\prod_{v\in\mathscr{C}_0}\left(1-\mathbf{L}^{-1}\right)^{\mathrm{rk}(\mathrm{Pic}(U))}\int_{\mathscr{L}^{\epsilon}(\mathscr{X}_{\mathcal{O}_v},E_{\beta_v}|W_v)}\frac{\mathbf{L}^{-\operatorname{ord}_{\mathscr{L}_{\rho-\epsilon}}(x)}}{\mathbf{L}^{\operatorname{ord}_{\omega_X}(x)}}$$
$$\times\prod_{s\in S}(1-\mathbf{L}^{-1})^{\mathrm{rk}(\mathrm{Pic}(U))+|\mathsf{M}_s|}\int_{\mathscr{L}((\mathscr{D}_{\mathcal{O}_s})_{\mathsf{M}_s},E^{\circ}_{\beta_s})}\frac{\mathbf{L}^{-\operatorname{ord}_{\mathscr{L}_{\rho-\epsilon}}(x)}}{\mathbf{L}^{\operatorname{ord}_{\omega_X}(x)}}$$

with an error term of weight bounded by

$$-\frac{1}{\langle\rho-\epsilon,\frac{1+\epsilon}{1-\epsilon}\rangle}\min_{\varnothing\neq A\subset\mathscr{A}_U}\langle\rho-\epsilon,\mathbf{m}'\rangle_A+2(\rho^{\beta}+r(g-1)_+-\langle\rho-\epsilon,\mathbf{e}^{\beta}\rangle+n(1-g))+n\dim L(\tilde{E}).$$

Remark 4.4.13. Over places $v\in\mathscr{C}_0$, as we pointed out in Remark 4.3.3, up to the factor $(1-\mathbf{L}^{-1})^{\mathrm{rk}\,\mathrm{Pic}(U)}$, the local term can be interpreted as the motivic integral

$$\int_{H^{\epsilon}(F_v,\beta_v)\cap W_v}\mathbf{L}^{-(g,\mathscr{L}_{\rho-\epsilon})}|\omega_X|=\int_{\mathscr{L}^{\epsilon}(\mathscr{U}_v,E^{\circ}_{\beta_v}|W_v)}\mathbf{L}^{-\operatorname{ord}_{\mathscr{L}_{\rho-\epsilon}}(x)}\mathbf{L}^{-\operatorname{ord}_{\omega_X}(x)}.$$

This remark can be adapted for places $s\in S$ if one replaces $(\mathscr{U}_{\mathcal{O}_v},E^{\circ}_{\beta_v})$ by $\mathscr{D}_{\mathsf{M}_s},E^{\circ}_{\beta_v}$ and add a controlling term.

Let $\lambda=1/\tilde{m}$ with $\tilde{m}\in\mathbf{Z}_{>0}$ and set $\lambda_\alpha=\lambda$ if $\alpha\in\mathsf{M}_s$ and zero otherwise, as well as

$$\mathscr{L}_{\rho-\epsilon+\lambda}=\mathscr{L}_{\rho-\epsilon}+\sum_{\alpha\in\mathscr{A}}\lambda_\alpha\mathscr{L}_\alpha.$$

Then one considers

$$\begin{aligned}
&\int_{\mathscr{L}(\mathscr{X}_{\mathcal{O}_s},E^{\circ}_{\beta_s}\cap\mathscr{D}_{\mathsf{M}_s})}\mathbf{L}^{-\operatorname{ord}_{\mathscr{L}_{\rho-\epsilon+\lambda}}(x)}\mathbf{L}^{-\operatorname{ord}_{\omega_X}(x)}\\
&=\int_{\mathscr{L}(\mathscr{X}_{\mathcal{O}_s},E^{\circ}_{\beta_s}\cap\mathscr{D}_{\mathsf{M}_s})}\mathbf{L}^{-\sum_{\alpha\in\mathscr{A}_U\sqcup\mathsf{M}_s}(\rho_\alpha-\epsilon_\alpha+\lambda_\alpha)\left(\operatorname{ord}_{\mathscr{D}_\alpha}(x)+e_\alpha^{\beta_s}\right)}\mathbf{L}^{\rho^{\beta_s}+\sum_{\alpha\in\mathscr{A}_U\sqcup\mathsf{M}_s}\rho_\alpha\operatorname{ord}_{\mathscr{D}_\alpha}(x)}\\
&=\mathbf{L}^{\rho^{\beta_s}-\langle\rho-\epsilon-\lambda,\mathbf{e}^{\beta}\rangle}\int_{\mathscr{L}(\mathscr{X}_{\mathcal{O}_s},E^{\circ}_{\beta_s}\cap\mathscr{D}_{\mathsf{M}_s})}\mathbf{L}^{\sum_{\alpha\in\mathscr{A}_U\sqcup\mathsf{M}_s}(\epsilon_\alpha-\lambda_\alpha)\operatorname{ord}_{\mathscr{D}_\alpha}(x)}\\
&=\mathbf{L}^{\rho^{\beta_s}-\langle\rho-\epsilon-\lambda,\mathbf{e}^{\beta}\rangle}\sum_{A\subset\mathscr{A}_U}\int_{\Omega_s(\mathsf{M}_s\cup A,\beta)}\mathbf{L}^{\sum_{\alpha\in\mathscr{A}_U\sqcup\mathsf{M}_s}(\epsilon_\alpha-\lambda_\alpha)\operatorname{ord}_{\mathscr{D}_\alpha}(x)}.
\end{aligned}$$

This last family of integrals can be computing using the isomorphism

$$\Theta:\Delta_v(A,\beta)\times\mathscr{L}(\mathbf{A}^1,0)^A\times\mathscr{L}(\mathbf{A}^1,0)^{n-|A|}\longrightarrow\Omega_v(A,\beta)$$

introduced at the beginning of Section 4.1.2, together with the motivic volumes computed in Section 4.1.4.2:

$$\begin{aligned}
&\int_{\Omega_s(A\cup \mathsf{M}_s,\beta)} \mathbf{L}^{\sum_{\alpha\in\mathsf{M}_s}(\epsilon_\alpha-\lambda_\alpha)\operatorname{ord}_{\mathscr{D}_\alpha}(x)}\\
&=\sum_{\mathbf{m}'\in\mathbf{N}_{>0}^{A\cup\mathsf{M}_s}}\int_{\substack{\Delta_s(A\cup\mathsf{M}_s,\beta)\times\mathscr{L}(\mathbf{A}^1,0)^{A\cup\mathsf{M}_s}\times\mathscr{L}(\mathbf{A}^1,0)^{n-|A|-|\mathsf{M}_s|}\\ \operatorname{ord}_s(x_\alpha)=m_\alpha}}\mathbf{L}^{|(\epsilon-\lambda)\mathbf{m}'_{\mathsf{M}_s}|}\\
&=\sum_{\mathbf{m}'\in\mathbf{N}_{>0}^{A\cup\mathsf{M}_s}}[\Delta_s(A\cup\mathsf{M}_s,\beta)]\mathbf{L}^{|(\epsilon-\lambda)\mathbf{m}'_{\mathsf{M}_s}|}(1-\mathbf{L}^{-1})^{|A\cup\mathsf{M}_s|}\mathbf{L}^{-|\mathbf{m}'|}\mathbf{L}^{-n+|A|+|\mathsf{M}_s|}\\
&=[\Delta_s(A\cup\mathsf{M}_s,\beta)]\left(\frac{\mathbf{L}^{-\lambda}}{1-\mathbf{L}^{-\lambda}}\right)^{|\mathsf{M}_s|}(1-\mathbf{L}^{-1})^{|A\cup\mathsf{M}_s|}\mathbf{L}^{-n+|A|+|\mathsf{M}_s|}\prod_{\alpha\in A}\frac{\mathbf{L}^{\epsilon_\alpha-1}}{1-\mathbf{L}^{\epsilon_\alpha-1}}
\end{aligned}$$

This last equality shows that it makes sense take the limit as $\lambda\to 0$, equivalently $\tilde{m}\to\infty$, of

$$(1-\mathbf{L}^{-\lambda})^{|\mathsf{M}_s|}\int_{\mathscr{L}(\mathscr{X}_{\mathcal{O}_s},E^\circ_{\beta_s}\cap\mathscr{D}_{\mathsf{M}_s})}\mathbf{L}^{-\operatorname{ord}_{\mathscr{L}_{\rho-\epsilon+\lambda}}(x)}\mathbf{L}^{-\operatorname{ord}_{\omega_X}(x)}$$

which gives

$$\begin{aligned}
&\mathbf{L}^{\rho^{\beta_s}-\langle\rho-\epsilon,\mathbf{e}^\beta\rangle}\sum_{A\subset\mathscr{A}_U}[\Delta_s(A\cup\mathsf{M}_s,\beta)](1-\mathbf{L}^{-1})^{|A\cup\mathsf{M}_s|}\mathbf{L}^{-n+|A|+|\mathsf{M}_s|}\prod_{\alpha\in A}\frac{\mathbf{L}^{\epsilon_\alpha-1}}{1-\mathbf{L}^{\epsilon_\alpha-1}}\\
&=\mathbf{L}^{\rho^{\beta_s}-\langle\rho-\epsilon,\mathbf{e}^\beta\rangle}(1-\mathbf{L}^{-1})^{|\mathsf{M}_s|}\sum_{A\subset\mathscr{A}_U}[\Delta_s(A\cup\mathsf{M}_s,\beta)]\mathbf{L}^{-(n-|\mathsf{M}_s|)+|A|}\prod_{\alpha\in A}\frac{\mathbf{L}^{\epsilon_\alpha-1}(1-\mathbf{L}^{-1})}{1-\mathbf{L}^{\epsilon_\alpha-1}}
\end{aligned}$$

and one checks, using the previous decompositions of arcs spaces and associated isomorphisms, that this is

$$(1-\mathbf{L}^{-1})^{\operatorname{rk}(\operatorname{Pic}(U))+|\mathsf{M}_s|}\int_{\mathscr{L}((\mathscr{D}_{\mathcal{O}_s})_{\mathsf{M}_s},E^\circ_{\beta_s})}\frac{\mathbf{L}^{-\operatorname{ord}_{\mathscr{L}_{\rho-\epsilon}}(x)}}{\mathbf{L}^{\operatorname{ord}_{\omega_X}(x)}}$$

up to the factor $(1-\mathbf{L}^{-1})^{\operatorname{rk}\operatorname{Pic}(U)}$.

Remark 4.4.14. A geometric interpretation of Proposition 4.4.12 is as follows.

For any $\mathsf{M}\in\operatorname{Cl}_S^{\max}(X,D)$, let $M^{\beta,\mathsf{M}}_{\mathbf{n},\mathsf{m}_S}\subset\operatorname{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\mathsf{m}_S)^\beta$ be the constructible moduli subspace of sections $\sigma\in\operatorname{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X}\mid\mathsf{m}_S)^\beta$ such that

$$\forall s\in S\quad\forall\alpha\in\mathsf{M}_s\quad(\sigma(\eta_{\mathscr{C}}),\mathscr{D}_\alpha)_s>0.$$

Remark that by definition of the Clemens complexes, $M^{\beta,\mathsf{M}}_{\mathbf{n},\mathsf{m}_S}$ is non-empty if and only if the support of $\mathsf{m}_s=((\sigma(\eta_{\mathscr{C}}),\mathscr{D}_\alpha)_s)_{\alpha\in\mathscr{A}_D}\in\mathbf{N}^{\mathscr{A}_D}$ is *exactly* M_s for all $s\in S$. The maximal number of $\mathscr{D}_\alpha$ a section can intersect above each point of S is encoded in $\operatorname{Cl}_S^{\max}(X,D)$. This also means that when the multidegree of a section tends to infinity, there is a unique way to associate a maximal face to the section and $M^{\beta,\mathsf{M}}_{\mathbf{n},\mathsf{m}_S}$ is actually the *right* moduli space to consider. We showed that $[M^{\beta,\mathsf{M}}_{\mathbf{n},\mathsf{m}_S}]\mathbf{L}^{-\langle\mathbf{n}^\beta_U+\mathsf{m}_S,\rho'\rangle}$ converges to the constant of Proposition 4.4.12. Since $[M^{\beta,\mathsf{M}}_{\mathbf{n},\mathsf{m}_S}]$ coincides with the coefficient of sufficiently high degree of the refined height zeta function $\mathscr{Z}^\beta_{\mathsf{M}}(\mathbf{u})$, this is the following statement and main result.

THEOREM 4.4.15. *For any* $\mathsf{M}\in\operatorname{Cl}_S^{\max}(X,D)$*, when the multidegree*

$$\mathbf{n}^\beta=(\mathbf{n}^\beta_U,\mathsf{m}_S)\in\mathbf{N}^{\mathscr{A}_U}\times\mathbf{N}^{\mathsf{M}}$$

goes arbitrarily far from the boundaries of the corresponding cone, the normalised class

$$\left[\mathrm{Hom}^{\mathbf{n}}_{\mathscr{C}}(\mathscr{C},\mathscr{X} \mid \epsilon \mid W \mid \mathsf{m}_S)^{\beta}\right] \mathbf{L}^{-\langle \rho-\epsilon, \mathbf{n}^{\beta}+\mathbf{e}^{\beta}\rangle}$$

converges, in the completion of $\mathscr{M}_{k,r}$ for the weight topology, to the motivic Euler product

$$\mathbf{L}^{(1-g)n}\left(\frac{[\mathrm{Pic}^0(\mathscr{C})]\mathbf{L}^{1-g}}{\mathbf{L}-1}\right)^{\mathrm{rk}(\mathrm{Pic}(U))}$$
$$\times \prod_{v\in\mathscr{C}_0}\left(1-\mathbf{L}^{-1}\right)^{\mathrm{rk}(\mathrm{Pic}(U))}\int_{\mathscr{L}^{\epsilon}(\mathscr{X}_{\mathcal{O}_v},E_{\beta_v}|W_v)} \frac{\mathbf{L}^{-\mathrm{ord}_{\mathscr{L}_{\rho-\epsilon}}(x)}}{\mathbf{L}^{\mathrm{ord}_{\omega_X}(x)}}$$
$$\times \prod_{s\in S}(1-\mathbf{L}^{-1})^{\mathrm{rk}(\mathrm{Pic}(U))+|\mathsf{M}_s|}\int_{\mathscr{L}((\mathscr{D}_{\mathcal{O}_s})_{\mathsf{M}_s},E^{\circ}_{\beta_s})} \frac{\mathbf{L}^{-\mathrm{ord}_{\mathscr{L}_{\rho-\epsilon}}(x)}}{\mathbf{L}^{\mathrm{ord}_{\omega_X}(x)}}$$

with an error term of weight bounded by

$$-\frac{1}{\langle \rho-\epsilon, \frac{1+\epsilon}{1-\epsilon}\rangle}\min_{\varnothing\neq A\subset\mathscr{A}_U}\langle\rho-\epsilon,\mathbf{m}'\rangle_A + 2(\rho^{\beta}+r(g-1)_+ - \langle\rho-\epsilon,\mathbf{e}^{\beta}\rangle + n(1-g)) + n\dim L(\tilde{E}).$$

CHAPTER 5

Rational curves on toric twisted products

Abstract. We show that equidistribution of rational curves holds on smooth split and projective toric varieties. Then we apply it to show that the Batyrev-Manin-Peyre principle holds for certain twisted products of toric varieties, assuming that it already holds for the base. This is [**Fai23**, §5 and §6].

5.1. Rational curves on smooth split projective toric varieties

In this section, we prove equidistribution of rational curves on smooth split projective toric varieties over any base field.

As a warm-up, we start with proving that the motivic Batyrev-Manin-Peyre principle holds for rational curves on this class of varieties, in line with the works of Bourqui [**Bou03**, **Bou09**], Bilu [**Bil23**] and Bilu-Das-Howe [**BDH22**], see Theorem 5.1.5.

Then we generalize this result, by proving equidistribution of rational curves on smooth split projective toric varieties, see Theorem 5.1.7.

5.1.1. Geometric setting. General references for toric varieties are [**Oda88**, **Ful93**, **CLS11**]. Let U be a split torus of dimension n over k. Let

$$\mathcal{X}^*(U) = \mathrm{Hom}(U, \mathbf{G}_m)$$

be its group of characters and $\mathcal{X}_*(U) = \mathrm{Hom}_{\mathbf{Z}}(\mathcal{X}^*(U), \mathbf{Z})$ its dual as a $\mathbf{Z}$-module. Let Σ be a complete and regular fan of $\mathcal{X}_*(U)$, which defines a smooth projective toric variety V_Σ over k, with open orbit isomorphic to U. Let

$$r = \mathrm{rg}\left(\mathrm{Pic}\left(V_\Sigma\right)\right)$$

be its Picard number, $\Sigma(1)$ the set of rays of the fan Σ and $(D_\alpha)_{\alpha\in\Sigma(1)}$ the set of its U-invariant divisors. Each ray $\alpha \in \Sigma(1)$ admits a minimal generator $\rho_\alpha \in \mathcal{X}_*(U)$ and the map sending a character $\chi \in \mathcal{X}^*(U)$ to the divisor

$$\sum_{\alpha\in\Sigma(1)} \langle \chi, \rho_\alpha\rangle D_\alpha$$

is part of the exact sequence [**CLS11**, Theorem 4.1.3]

$$0 \to \mathcal{X}^*(U) \to \oplus_{\alpha\in\Sigma(1)} \mathbf{Z} D_\alpha \to \mathrm{Pic}\left(V_\Sigma\right) \to 0 \qquad (5.1.1.32)$$

which provides, in particular, the equality

$$|\Sigma(1)| = n + r.$$

If σ is an element of the fan Σ, let $\sigma(1) \subset \Sigma(1)$ be the subset of rays which are faces of σ.

5.1.2. Möbius functions. Let $B_\Sigma \subset \{0,1\}^{\Sigma(1)}$ be the complement of the image of

$$\begin{array}{ccc} \Sigma & \longrightarrow & \{0,1\}^{\Sigma(1)} \\ \sigma & \longmapsto & (\mathbf{1}_{\sigma(1)}(\alpha))_{\alpha\in\Sigma(1)}. \end{array}$$

In [**Bou09**, §3.5], this set B_Σ is described explicitely as

$$B_\Sigma = \{\mathbf{n} \in \{0,1\}^{\Sigma(1)} \mid \forall\, \sigma \in \Sigma,\, \exists\, \alpha \in \Sigma(1),\, \alpha \notin \sigma(1),\ \text{and}\ n_\alpha = 1\}.$$

It has a geometric interpretation in terms of the effective divisors D_α: it corresponds to the subsets $I \subset \Sigma(1)$ such that

$$\cap_{\alpha\in I} D_\alpha = \varnothing.$$

Then, the universal torsor of V_Σ admits an explicit description which goes back to Salberger [**Sal98**]:

$$\mathcal{T}_\Sigma = \mathbf{A}^{\Sigma(1)} \setminus \left(\cup_{J\in B_\Sigma} \cap_{\alpha\in J} \{x_\alpha = 0\}\right).$$

5.1.2.1. *Local Möbius function.* Bourqui inductively defines a local Möbius function

$$\mu^0_{B_\Sigma} : \{0,1\}^{\Sigma(1)} \to \mathbf{Z}$$

through the relation

$$\mathbf{1}_{\{0,1\}^{\Sigma(1)}\setminus B_\Sigma}(\mathbf{n}) = \sum_{0\leqslant \mathbf{n}'\leqslant \mathbf{n}} \mu^0_{B_\Sigma}(\mathbf{n}')$$

for every $\mathbf{n} \in \{0,1\}^{\Sigma(1)}$. It comes with a generating polynomial

$$P_{B_\Sigma}(\mathbf{t}) = \sum_{\mathbf{n}\in\{0,1\}^{\Sigma(1)}} \mu^0_{B_\Sigma}(\mathbf{n})\mathbf{t}^{\mathbf{n}}$$

and a series

$$Q_{B_\Sigma}(\mathbf{t}) = \frac{P_{B_\Sigma}(\mathbf{t})}{\prod_{\alpha\in\Sigma(1)}(1-t_\alpha)}.$$

Let

$$A\left(B_\Sigma\right) \subset \mathbf{N}^{\Sigma(1)}$$

be the set of tuples $\mathbf{n} \in \mathbf{N}^{\Sigma(1)}$ such that there is no $\mathbf{n}' \in B_\Sigma$ with $\mathbf{n} \geqslant \mathbf{n}'$. In particular, $\mathbf{0} \in A\left(B_\Sigma\right)$. It is the set of elements of $\mathbf{N}^{\Sigma(1)}$ *not lying above* B_Σ in the sense of [**BDH22**, §4.4]. Let

$$\mu_{B_\Sigma} : \mathbf{N}^{\Sigma(1)} \to \mathbf{Z}$$

be the local Möbius function defined by the relation

$$\mathbf{1}_{A(B_\Sigma)}(\mathbf{n}) = \sum_{0\leqslant \mathbf{n}'\leqslant \mathbf{n}} \mu_{B_\Sigma}(\mathbf{n}'). \tag{5.1.2.33}$$

As Bilu-Das-Howe point out in [**BDH22**, §5.2], μ_{B_Σ} coincides with $\mu^0_{B_\Sigma}$ on $\{0,1\}^{\Sigma(1)}$ and is zero outside of this set. Hence

$$Q_{B_\Sigma}(\mathbf{t}) = \sum_{\mathbf{n}\in A(B_\Sigma)} \mathbf{t}^{\mathbf{n}}.$$

For any finite sequence

$$\mathfrak{J} = (J_1 \subsetneq J_1 \subsetneq \ldots \subsetneq J_{\ell(\mathfrak{J})})$$

of elements of B_Σ, let

$$\mathcal{D}_{\mathfrak{J}} = \{\delta \in \mathbf{N}^{B_\Sigma} \mid \delta_J > 0 \iff J \in \mathfrak{J}\}$$

be the subset of tuples whose support is exactly $\mathfrak{J}$. In particular,

$$\mathcal{D}_{\varnothing} = \{\mathbf{0}\}$$

and $\ell(\varnothing) = 0$.

Lemma 5.1.1. *The map*

$$\Phi_{loc} : \mathbf{N}^{B_\Sigma} \times \mathbf{N}^{\Sigma(1)} \longrightarrow \mathbf{N}^{\Sigma(1)}$$

given by

$$\Big((\delta_J)_{J\in B_\Sigma}, (n_\alpha)_{\alpha\in\Sigma(1)}\Big) \longmapsto \Bigg(n_\alpha + \sum_{J\ni\alpha} \delta_J\Bigg)_{\alpha\in\Sigma(1)}$$

induces a surjection

$$\bigsqcup_{\mathfrak{J}} \mathcal{D}_{\mathfrak{J}} \times A(B_\Sigma) \to \mathbf{N}^{\Sigma(1)}.$$

Proof. Given $\mathbf{n} \in \mathbf{N}^{\Sigma(1)}$ let

$$\Delta_{\Sigma(1)}(\mathbf{n}) = \min\{n_\alpha \mid \alpha \in \Sigma(1)\} \geqslant 0$$

and for all $J \in B_\Sigma$

$$\Delta_J(\mathbf{n}) = \min\left\{ n_\alpha - \sum_{J'>J} \Delta_{J'}(\mathbf{n}) \mid \alpha \in J \right\} \geqslant 0.$$

Then

$$\Phi_{\mathrm{loc}}((\Delta_J(\mathbf{n})), \mathbf{n}') = \mathbf{n}$$

where $n'_\alpha = n_\alpha - \sum_{J\ni\alpha} \Delta_J(\mathbf{n})$ defines an element of $A(B_\Sigma)$. Indeed, if $\mathbf{n}' = \mathbf{n}'' + \mathbf{b}$ with $\mathbf{b} = (\mathbf{1}_J(\alpha))$ for a certain maximal element $J \in B_\Sigma$, and $\mathbf{n}'' \in \mathbf{N}^{\Sigma(1)}$, then one gets $\Delta_J(\mathbf{n}') > 0$, contradicting the fact that $\Delta_J(\mathbf{n}') = 0$ for all $J \in B_\Sigma$ by construction. This shows that Φ_{loc} is surjective. Actually,

$$A(B_\Sigma) = \{\mathbf{d} \in \mathbf{N}^{\Sigma(1)} \mid \Delta_J(\mathbf{d}) = 0 \text{ for all } J \in B_\Sigma\}$$

and the lemma is proved. □

5.1.2.2. *Global motivic Möbius function.* For any $\mathbf{e} \in \mathbf{N}^{\Sigma(1)}$, let $(\mathbf{P}^1_k)^{\mathbf{e}}_{B_\Sigma}$ be the open subset of

$$\mathrm{Sym}^{\mathbf{e}}_{/k}(\mathbf{P}^1_k)$$

parametrizing $\Sigma(1)$-tuples of effective zero-cycles of degree e_α having disjoint supports with respect to B_Σ. More precisely,

$$(\mathbf{P}^1_k)^{\mathbf{e}}_{B_\Sigma} = \left\{ (C_\alpha) \in \mathrm{Sym}^{\mathbf{e}}_{/k}(\mathbf{P}^1_k) \;\middle|\; \forall\, J \in B_\Sigma \quad \bigcap_{\alpha\in J} \mathrm{Supp}(C_\alpha) = \varnothing \right\}.$$

Up to a tuple of multiplicative factors, this set corresponds to $\Sigma(1)$-tuples of homegeneous polynomials $P(T_0, T_1)$ of degree d_α with coefficients in k such that for all $J \in B_\Sigma$ the polynomials $(P_\alpha)_{\alpha\in J}$ have no common root in any finite extension of k, see [Bou09, Lemme 5.10].

Then, applying the definition of the motivic Euler product in Bilu's sense [Bil23], we get

$$\prod_{p\in\mathbf{P}^1_k} Q_{B_\Sigma}(\mathbf{t}) = \sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}} \left[\left(\mathbf{P}^1_k\right)^{\mathbf{d}}_{B_\Sigma} \right] \mathbf{t}^{\mathbf{d}}.$$

The formalism of pattern-avoiding zero cycles allows Bilu, Das and Howe to work with Bilu's motivic Euler product and to give a positive answer to a technical question of Bourqui [**Bou09**, Question 5], which in turn provides a lift of Bourqui's main theorem [**Bou09**, Théorème 1.1] from the localised Grothendieck ring of Chow motives $\mathscr{M}_k^\chi$ to the localised Grothendieck ring of varieties $\mathscr{M}_k$, see [**BDH22**, Lemma 4.5.4 & Remark 4.5.7]. Indeed, since

$$Q_{B_\Sigma}(\mathbf{t}) = P_{B_\Sigma}(\mathbf{t}) \prod_{\alpha\in\Sigma(1)} (1-t_\alpha)^{-1}$$

one obtains, by taking motivic Euler products (in Bilu's sense)

$$\sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}} \left[\left(\mathbf{P}^1_k\right)^{\mathbf{d}}_{B_\Sigma}\right]\mathbf{t}^{\mathbf{d}} = \prod_{p\in\mathbf{P}^1_k} Q_{B_\Sigma}(\mathbf{t}) = \prod_{p\in\mathbf{P}^1_k} P_{B_\Sigma}(\mathbf{t}) \times \prod_{\alpha\in\Sigma(1)} Z_{\mathbf{P}^1_k}(t_\alpha) \tag{5.1.2.34}$$

where $Z_{\mathbf{P}^1_k}(t)$ is Kapranov's zeta function of the projective line

$$Z_{\mathbf{P}^1_k}(t) = \sum_{e\geqslant 0}\left[\mathrm{Sym}^e_{/k}(\mathbf{P}^1_k)\right]t^e.$$

Bourqui's construction [**Bou09**, Section 3.3], applied to the projective line over k and the set B_Σ, provides a motivic global Möbius function

$$\mu_\Sigma : \mathbf{N}^{\Sigma(1)} \to \mathscr{M}_k$$

given by the relation

$$\left[(\mathbf{P}^1_k)^{\mathbf{e}}_{B_\Sigma}\right] = \mu_\Sigma(\mathbf{e}')\left[\mathrm{Sym}^{\mathbf{e}-\mathbf{e}'}_{/k}(\mathbf{P}^1_k)\right]$$

for all $\mathbf{e}\in\mathbf{N}^{\Sigma(1)}$, which is nothing else than what one obtains by considering the coefficient of multidegree $\mathbf{e}$ in expression (5.1.2.34).

It follows from the definitions that the motivic global Möbius function is linked to the local one by the relation

$$\prod_{p\in\mathbf{P}^1_k}\left(\sum_{\mathbf{m}\in\mathbf{N}^{\Sigma(1)}} \mu_{B_\Sigma}(\mathbf{m})\mathbf{t}^{\mathbf{m}}\right) = \sum_{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}} \mu_\Sigma(\mathbf{e})\mathbf{t}^{\mathbf{e}}.$$

5.1.3. Motivic Tamagawa number. By the following Proposition and Remark, the constant $\tau(V_\Sigma)$ is well-defined in $\widehat{\mathscr{M}_k}^{\dim}$.

Proposition 5.1.2. *The motivic Euler product*

$$\left(\prod_{p\in\mathbf{P}^1_k} P_{B_\Sigma}(\mathbf{t})\right)\left(\mathbf{L}_k^{-1}\right) = \prod_{p\in\mathbf{P}^1_k}\left(\sum_{\mathbf{m}\in\mathbf{N}^{\Sigma(1)}} \mu_{B_\Sigma}(\mathbf{m})\mathbf{L}_p^{-|\mathbf{m}|}\right) = \sum_{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}} \mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-|\mathbf{e}|}$$

is well-defined in $\widehat{\mathscr{M}_k}^{\dim}$.

Proof. By [**Bou09**, Lemme 3.8], the valuation of $P_{B_\Sigma}(T)-1$ is at least equal to 2. Thus by [**BDH22**, Lemma 4.2.5] the formal motivic Euler product $\prod_{p\in\mathbf{P}^1_k} P_{B_\Sigma}(\mathbf{t})$ converges at $\mathbf{t}=\mathbf{L}^{-1}$ for the dimensional filtration. □

Remark 5.1.3. The local factor of the motivic Euler product above is actually

$$(P_{B_\Sigma}(\mathbf{t}))_{|t_\alpha=\mathbf{L}_p^{-1}} = \sum_{\mathbf{m}\in\mathbf{N}^{\Sigma(1)}} \mu_{B_\Sigma}(\mathbf{m})\mathbf{L}_p^{-|\mathbf{m}|} = \frac{[V_\Sigma\otimes\kappa(p)]}{\mathbf{L}_p^{\dim(V_\Sigma)}}\left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))} = \frac{[\mathcal{T}_\Sigma\otimes\kappa(p)]}{\mathbf{L}_p^{|\Sigma(1)|}}.$$

Indeed, we can interpret $[\mathcal{T}_\Sigma \otimes \kappa(p)]\mathbf{L}_p^{-|\Sigma(1)|}$ as the motivic density of $\kappa(p)$-arcs of $\mathbf{A}_k^{\Sigma(1)}$ with origin in $\mathcal{T}_\Sigma$, and $\mathbf{L}_p^{-|\mathbf{m}|}$ as the motivic density of the subspace $V_{\mathbf{m}}$ of $\kappa(p)$-arcs with $\Sigma(1)$-tuple of valuations greater than $\mathbf{m}$:

$$V_{\mathbf{m}} = \{x \in \mathbf{A}_k^{\Sigma(1)}(\kappa(p)[[t]]) \mid \forall\, \alpha \in \Sigma(1),\ x_\alpha \in t^{m_i}\kappa(p)[[t]]\}.$$

We consider as well the subspace of arcs with given valuation:

$$\begin{aligned} V_{\mathbf{m}}^\circ &= \{x \in \mathbf{A}_k^{\Sigma(1)}(\kappa(p)[[t]]) \mid \forall\, \alpha \in \Sigma(1),\ x_\alpha \in t^{m_i}\kappa(p)[[t]] \text{ and } x_\alpha \notin t^{m_i+1}\kappa(p)[[t]]\} \\ &= V_{\mathbf{m}} \setminus \bigcup_{\alpha\in\Sigma(1)} V_{\mathbf{m}+\mathbf{1}_\alpha}. \end{aligned}$$

The arc-space of $\mathcal{T}_\Sigma$ is the space of arcs whose $\Sigma(1)$-tuple of valuations does not lie above B_Σ:

$$\mathrm{Gr}_\infty(\mathcal{T}_\Sigma \otimes \kappa(p)) = \bigsqcup_{\mathbf{m}\in A(B_\Sigma)} V_{\mathbf{m}}^\circ = \bigsqcup_{\mathbf{m}\in A(B_\Sigma)} \left(V_{\mathbf{m}} \setminus \bigcup_{\alpha\in\Sigma(1)} V_{\mathbf{m}+\mathbf{1}_\alpha}\right)$$

and thus

$$\begin{aligned} [\mathcal{T}_\Sigma \otimes \kappa(p)]\,\mathbf{L}_p^{-|\Sigma(1)|} &= \sum_{\mathbf{m}\in A(B_\Sigma)} \mu_V \left(V_{\mathbf{m}} \setminus \bigcup_{\alpha\in\Sigma(1)} V_{\mathbf{m}+\mathbf{1}_\alpha}\right) \\ &= \sum_{\mathbf{m}\in A(B_\Sigma)} \sum_{J\subset\Sigma(1)} (-1)^{|J|}\mu_V\left(\bigcap_{\alpha\in J} V_{\mathbf{m}+\mathbf{1}_\alpha}\right) \\ &= \sum_{\mathbf{m}\in A(B_\Sigma)} \mathbf{L}_p^{-|\mathbf{m}|} \prod_{\alpha\in\Sigma(1)} \left(1-\mathbf{L}_p^{-1}\right) \\ &= \left((Q_{B_\Sigma}(\mathbf{t})) \prod_{\alpha\in\Sigma(1)} (1-t_\alpha)\right)_{|t_\alpha=\mathbf{L}_p^{-1}} \\ &= (P_{B_\Sigma}(\mathbf{t}))_{|t_\alpha=\mathbf{L}_p^{-1}} \end{aligned}$$

as expected.

By compatibility of formal motivic Euler products with changes of variables of the form $t'_\alpha = \mathbf{L}_k^a t_\alpha$ with a an integer [**Bil23**, §3.6.4], together with the compatibility with partial specialisation [**BH21**, Lemma 6.5.1], what we get by taking the corresponding motivic Euler product is exactly the motivic Tamagawa number of V_Σ given by Definition 3.1.2 and Notation 2.4.15:

$$\left(\prod_{p\in\mathbf{P}_k^1} P_{B_\Sigma}(\mathbf{t})\right)_{|t_\alpha=\mathbf{L}_k^{-1}} = \left(\prod_{p\in\mathbf{P}_k^1}\left(1+\left(\frac{[\mathcal{T}_\Sigma\otimes\kappa(p)]}{\mathbf{L}_p^{|\Sigma(1)|}}-1\right)t\right)\right)_{|t=1} = \tau(V_\Sigma).$$

5.1.4. Non-constrained rational curves. Let

$$\mathrm{Hom}_k^\delta(\mathbf{P}_k^1, V_\Sigma)_U$$

be the quasi-projective scheme parametrizing morphisms $\mathbf{P}_k^1 \to V_\Sigma$ of multidegree $\delta \in \mathrm{Pic}(X)^\vee$ intersecting the dense open subset $U \subset V_\Sigma$ (see Definition 2.3.5 and Lemma 2.3.6). It is empty whenever $\delta \notin \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ (see the remark after Notation 5.4 in [**Bou09**]), so we will always assume that $\delta \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ in the remaining of this section.

Through the injection $\mathrm{Pic}(V_\Sigma)^\vee \hookrightarrow \mathbf{Z}^{\Sigma(1)}$ given by the exact sequence (5.1.1.32), $\mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ can be see as the submonoid of tuples $(d_\alpha)_{\alpha\in\Sigma(1)} \in \mathbf{N}^{\Sigma(1)}$ such that

$$\sum_{\alpha\in\Sigma(1)} d_\alpha \langle \chi, \rho_\alpha\rangle = 0$$

for all $\chi \in \mathcal{X}^*(U)$. Note that this submonoid is denoted by $\mathbf{N}^{\Sigma(1)}_{(*)}$ in Bourqui's work [**Bou09**, Notation 5.3].

For every $\mathbf{d} \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$, let

$$\widetilde{(\mathbf{P}^1_k)^{\mathbf{d}}_{B_\Sigma}}$$

be the inverse image of the open subset

$$\left(\mathbf{P}^1_k\right)^{\mathbf{d}}_{B_\Sigma} \subset \mathrm{Sym}^{\mathbf{d}}_{/k}\left(\mathbf{P}^1_k\right) \simeq \prod_{\alpha\in\Sigma(1)} \mathbf{P}^{d_\alpha}_k$$

through the $\mathbf{G}_m^{\Sigma(1)}$-torsor

$$\prod_{\alpha\in\Sigma(1)} \left(\mathbf{A}^{d_\alpha+1}_k \setminus \{0\}\right) \longrightarrow \prod_{\alpha\in\Sigma(1)} \mathbf{P}^{d_\alpha}_k.$$

One of Bourqui's key results is the following proposition.

Proposition 5.1.4 ([**Bou09**, Proposition 5.14]). *For every* $\mathbf{d} \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$,

$$\widetilde{(\mathbf{P}^1_k)^{\mathbf{d}}_{B_\Sigma}}/T_{\mathrm{NS}}$$

represents the functor $\mathbf{Hom}^{\mathbf{d}}_k(\mathbf{P}^1, V_\Sigma)_U$.

THEOREM 5.1.5. *The normalised class*

$$\left[\mathrm{Hom}^\delta_k(\mathbf{P}^1_k, V_\Sigma)_U\right] \mathbf{L}_k^{-\,\delta\,\cdot\,\omega^{-1}_{V_\Sigma}}$$

tends to the non-zero effective element

$$\tau(V_\Sigma) = \frac{\mathbf{L}_k^{\dim(V_\Sigma)}}{\left(1-\mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))}} \prod_{p\in\mathbf{P}^1_k} \frac{[V_\Sigma \otimes_k \kappa(p)]}{\mathbf{L}_p^{\dim(V_\Sigma)}} \left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))} \in \widehat{\mathscr{M}_k}^{\dim}$$

when $\delta \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ *goes arbitrary far from the boundary of the dual of the effective cone of* V_Σ.

Moreover the error term

$$\tau(V_\Sigma) - \left[\mathrm{Hom}^\delta_k(\mathbf{P}^1_k, V_\Sigma)_U\right] \mathbf{L}_k^{-\,\delta\,\cdot\,\omega^{-1}_{V_\Sigma}}$$

has virtual dimension at most

$$-\frac{1}{4}\min_{\alpha\in\Sigma(1)}(\delta_\alpha) + \dim(V_\Sigma).$$

PROOF. First note that since $\sum_{\alpha\in\Sigma(1)} D_\alpha$ is an anticanonical divisor of V_Σ, the anticanonical degree of a curve of multidegree $\mathbf{d} \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ is $|\mathbf{d}| = \sum_{\alpha\in\Sigma(1)} d_\alpha$. By Proposition 5.1.4, we have the relation

$$[\mathrm{Hom}^{\mathbf{d}}_k(\mathbf{P}^1_k, V_\Sigma)_U] = (\mathbf{L}_k - 1)^{\dim(V_\Sigma)} \left[\left(\mathbf{P}^1_k\right)^{\mathbf{d}}_{B_\Sigma}\right]$$

for every $\mathbf{d} \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$. Therefore studying the asymptotic behaviour of $[\mathrm{Hom}_k^{\mathbf{d}}(\mathbf{P}^1_k, V_\Sigma)_U]\mathbf{L}_k^{-|\mathbf{d}|}$ when $\min_\alpha(d_\alpha) \to \infty$ goes back to studying the one of $\left[(\mathbf{P}^1_k)^{\mathbf{d}}_{B_\Sigma}\right]\mathbf{L}^{-|\mathbf{d}|}$. Moreover it is convenient to drop the assumption $\mathbf{d} \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$.

Then (5.1.2.34) gives

$$\sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}} \left[\left(\mathbf{P}^1_k\right)^{\mathbf{d}}_{B_\Sigma}\right]\mathbf{t}^{\mathbf{d}} = \prod_{p\in\mathbf{P}^1_k} P_{B_\Sigma}(\mathbf{t}) \times \prod_{\alpha\in\Sigma(1)} Z^{\mathrm{Kapr}}_{\mathbf{P}^1_k}(t_\alpha)$$

and we proceed as in [**Fai22**, §4.1]. First, recall that

$$Z^{\mathrm{Kapr}}_{\mathbf{P}^1_k}(t) = (1-t)^{-1}(1-\mathbf{L}_k t)^{-1}.$$

For any $A \subset \Sigma(1)$, set

$$Z_A(\mathbf{t}) = \prod_{\alpha\in A}(1-t_\alpha)^{-1}\prod_{\alpha\notin A}(1-\mathbf{L}_k t_\alpha)^{-1}$$

so that

$$\prod_{\alpha\in\Sigma(1)} Z^{\mathrm{Kapr}}_{\mathbf{P}^1_k}(t_\alpha) = \prod_{\alpha\in\Sigma(1)} \frac{1}{1-\mathbf{L}_k}\left(\frac{1}{1-t_\alpha} - \frac{\mathbf{L}_k}{1-\mathbf{L}_k t_\alpha}\right) = \sum_{A\subset\Sigma(1)} \frac{(-\mathbf{L}_k)^{|\Sigma(1)|-|A|}}{(1-\mathbf{L}_k)^{|\Sigma(1)|}} Z_A(\mathbf{t}). \tag{5.1.4.35}$$

The coefficient of order $\mathbf{d} \in \mathbf{N}^{\Sigma(1)}$ of

$$Z_A(\mathbf{t}) \times \prod_{p\in\mathbf{P}^1_k} P_{B_\Sigma}(\mathbf{t})$$

for a given $A \subset \Sigma(1)$ is the finite sum

$$\sum_{\substack{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}\\ \mathbf{e}\leqslant\mathbf{d}}} \mu_\Sigma(\mathbf{e})\mathbf{L}_k^{\sum_{\alpha\in\Sigma(1)\setminus A} d_\alpha - e_\alpha} = \mathbf{L}_k^{|\mathbf{d}_{\Sigma(1)\setminus A}|} \sum_{\substack{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}\\ \mathbf{e}\leqslant\delta}} \mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-\left|\mathbf{e}_{\Sigma(1)\setminus A}\right|}.$$

Normalizing by $\mathbf{L}_k^{|\mathbf{d}|}$ and writing $\mathbf{e}_{\Sigma(1)\setminus A} = \mathbf{e} - \mathbf{e}_A$ one gets the sum

$$\sum_{\substack{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}\\ \mathbf{e}\leqslant\mathbf{d}}} \mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-|\mathbf{e}|}\mathbf{L}_k^{|\mathbf{e}_A|-|\mathbf{d}_A|}. \tag{5.1.4.36}$$

If $A = \varnothing$, this is exactly the $\mathbf{d}$-th partial sum

$$\sum_{\substack{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}\\ \mathbf{e}\leqslant\mathbf{d}}} \mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-|\mathbf{e}|}$$

and when $\min_{\alpha\in\Sigma(1)} d_\alpha$ goes to infinity, we know by Proposition 5.1.2 that this sum converges to

$$\sum_{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}} \mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-|\mathbf{e}|} = \prod_{p\in\mathbf{P}^1_k} \frac{[V_\Sigma\otimes\kappa(p)]}{\mathbf{L}_p^n}\left(1-\mathbf{L}_p^{-1}\right)^r \qquad \text{(by Remark 5.1.3)}$$

in $\widehat{\mathscr{M}_k}^{\dim}$. By the proof of [**BDH22**, Lemma 4.2.5] we have $\dim(\mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-|\mathbf{e}|}) \leqslant -\frac{1}{2}|\mathbf{e}|$ for any $\mathbf{e} \in \mathbf{N}^{\Sigma(1)}$. Thus the error term

$$\sum_{\substack{\mathbf{e}\in\mathbf{N}^{\Sigma(1)}\\ \mathbf{e}\nleqslant\mathbf{d}}} \mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-|\mathbf{e}|}$$

has virtual dimension at most $-\frac{1}{2}\min_{\alpha\in\Sigma(1)}(d_\alpha+1)$.
If $A\neq\varnothing$, then by Lemma 2.4.21 (taking $\mathbf{m}=\mathbf{e}$, $\mathfrak{c}_{\mathbf{m}}=\mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-\mathbf{e}}$, $a=1/2$ and $b=0$), the sum (5.1.4.36) has virtual dimension at most

$$-\frac{1}{4}\min_{\alpha\in A}(d_\alpha)$$

hence it becomes negligible in comparison with the term given by $A=\varnothing$ as $\min(d_\alpha)\to\infty$.

Finally, putting all together, one concludes that the normalised class

$$\left[\mathrm{Hom}_k^\delta(\mathbf{P}^1_k,V_\Sigma)_U\right]\mathbf{L}_k^{-\delta\cdot\omega_{V_\Sigma}^{-1}}$$

tends to

$$\frac{\mathbf{L}_k^{\dim(V_\Sigma)}}{\left(1-\mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))}}\prod_{p\in\mathbf{P}^1_k}\frac{[V_\Sigma\otimes\kappa(p)]}{\mathbf{L}_p^{\dim(V_\Sigma)}}\left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))}$$

in $\widehat{\mathscr{M}_k}^{\dim}$ when $\delta\in\mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ goes arbitrary far from the boundary $\partial\,\mathrm{Eff}(V_\Sigma)^\vee$, with error term of virtual dimension bounded by $-\frac{1}{4}\min_{\alpha\in\Sigma(1)}(\delta_\alpha)+\dim(V_\Sigma)$. □

Corollary 5.1.6. *Let*

$$\mathscr{F}(\mathbf{t})=\sum_{\delta\in\mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}}\left[\mathrm{Hom}_k^\delta(\mathbf{P}^1_k,V_\Sigma)_U\right]\mathbf{t}^\delta\prod_{\alpha\in\Sigma(1)}(1-\mathbf{L}_kt_\alpha).$$

Then $\mathscr{F}(\mathbf{t})$ *converges at* $t_\alpha=\mathbf{L}_k^{-1}$ *to* $\tau_{\mathbf{P}^1_k}(V_\Sigma)$.

Proof. Let $\mathfrak{b}_{\mathbf{d}}$ be the coefficient of multidegree $\mathbf{d}$ of $\mathscr{F}(\mathbf{t})$. Since

$$\mathscr{F}(\mathbf{t})\prod_{\alpha\in\Sigma(1)}(1-\mathbf{L}_kt_\alpha)^{-1}=\mathscr{F}(\mathbf{t})\sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}}\mathbf{L}_k^{|\mathbf{d}|}\mathbf{t}^{\mathbf{d}}$$

we have the relation

$$\left[\mathrm{Hom}_k^\delta(\mathbf{P}^1_k,V_\Sigma)_U\right]=\sum_{\mathbf{d}+\mathbf{d}'=\delta}\mathfrak{b}_{\mathbf{d}}\mathbf{L}_k^{|\mathbf{d}'|}$$

for every $\delta\in\mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$, which becomes after normalisation

$$\left[\mathrm{Hom}_k^\delta(\mathbf{P}^1_k,V_\Sigma)_U\right]\mathbf{L}_k^{-\delta\cdot\omega_{V_\Sigma}^{-1}}=\sum_{\mathbf{e}\leqslant\delta}\mathfrak{b}_{\mathbf{e}}\mathbf{L}_k^{-|\mathbf{e}|}.$$

This is exactly the δ-th partial sum of the series $\mathscr{F}(\mathbf{L}_k^{-1})$. Since $\left[\mathrm{Hom}_k^\delta(\mathbf{P}^1_k,V_\Sigma)_U\right]\mathbf{L}_k^{-\delta\cdot\omega_{V_\Sigma}^{-1}}$ converges to $\tau_{\mathbf{P}^1_k}(V_\Sigma)$ when $\min(\delta_\alpha)$ tends to infinity, the claim follows. □

5.1.5. Equidistribution. In the remainder of this section, we prove equidistribution of rational curves on smooth split projective toric varieties.

Theorem 5.1.7. *Let*

$$\mathscr{S}=\coprod_{p\in|\mathscr{S}|}\mathscr{S}_p$$

be a zero-dimensional subscheme of $\mathbf{P}^1_k$, $(m_p)_{p\in|\mathscr{S}|}$ *non-negative integers such that*

$$\ell(\mathscr{S}_p)=(m_p+1)[\kappa(p):k]$$

for all $p\in|\mathscr{S}|$, *and* W *a constructible subset of* $\mathrm{Hom}_k(\mathscr{S},V_\Sigma)\simeq\prod_{p\in|\mathscr{S}|}\mathrm{Gr}_{m_p}(V_\Sigma\otimes\kappa(p))$.

Then, when $\delta \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ goes arbitrary far from the boundary of the dual of the effective cone, the normalised class

$$\left[\mathrm{Hom}^\delta_k(\mathbf{P}^1_k, V_\Sigma \mid W)_U\right] \mathbf{L}_k^{-\delta \cdot \omega^{-1}_{V_\Sigma}} \in \mathscr{M}_k$$

tends to the non-zero effective element

$$\tau(V_\Sigma \mid W) = \frac{\mathbf{L}_k^{\dim(V_\Sigma)}}{\left(1-\mathbf{L}_k^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))}} \prod_{p \notin |\mathscr{S}|} \left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))} \frac{[V_\Sigma \otimes \kappa(p)]}{\mathbf{L}_p^{\dim(V_\Sigma)}}$$
$$\times [W] \prod_{p \in |\mathscr{S}|} \left(1-\mathbf{L}_p^{-1}\right)^{\mathrm{rg}(\mathrm{Pic}(V_\Sigma))} \mathbf{L}_p^{-(m_p+1)\dim(V_\Sigma)}$$

in the completion $\widehat{\mathscr{M}_k}^{\dim}$.

Moreover, the error term

$$\tau(V_\Sigma \mid W) - \left[\mathrm{Hom}^\delta_k(\mathbf{P}^1_k, V_\Sigma \mid W)_U\right] \mathbf{L}_k^{-\delta \cdot \omega^{-1}_{V_\Sigma}} \in \widehat{\mathscr{M}_k}^{\dim}$$

has virtual dimension smaller than

$$-\frac{1}{4} \min_{\alpha \in \Sigma(1)} (\delta_\alpha) + \ell(\mathscr{S}) + (1-\ell(\mathscr{S}))(\dim(V_\Sigma)-1) + \dim(W)$$

for all $\delta \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$.

Corollary 5.1.8. *For any non-zero multidegree $\delta \in \mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$ such that*

$$\min_{\alpha \in \Sigma(1)} (\delta_\alpha) \geqslant 8\ell(\mathscr{S}) - 4,$$

the moduli space $\mathrm{Hom}^\delta_k(\mathbf{P}^1_k, V_\Sigma \mid W)_U$ has dimension

$$\delta \cdot \omega^{-1}_{V_\Sigma} + \dim(V_\Sigma)(1-\ell(\mathscr{S})) + \dim(W)$$

as expected.

Remark 5.1.9. The upper bound on the dimension of the error term we give can be made uniform in the set W of conditions, since the dimension of W is bounded by $\ell(\mathscr{S})\dim(V_\Sigma)$.

The remainder of this section is devoted to the proof of Theorem 5.1.7. We see $\mathbf{A}^2_k \setminus \{0\}$ as the universal $\mathbf{G}_m$-torsor

$$\mathbf{A}^2_k \setminus \{0\} \to \mathbf{P}^1_k.$$

Given a cocharacter $\chi : \mathbf{G}_m \to \mathbf{G}_m^{\Sigma(1)}$, or equivalently a tuple $\mathbf{d} = (d_\alpha) \in \mathbf{Z}^{\Sigma(1)}$ (we will switch freely between both notations), we consider the functor from k-schemes to sets

$$\mathbf{Hom}^\chi\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) : S \rightsquigarrow \mathrm{Hom}^\chi_S\left(\mathbf{A}^2_S \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_S\right),$$

of χ-equivariant morphisms, as well as its restriction

$$\mathbf{Hom}^\chi\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right)^* : S \rightsquigarrow \mathrm{Hom}^\chi_S\left(\mathbf{A}^2_S \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_S\right),$$

to χ-equivariant morphisms *with no trivial coordinate* and its second restriction

$$\mathbf{Hom}^\chi\left(\mathbf{A}^2_k \setminus \{0\}, \mathscr{T}_\Sigma\right)^* : S \rightsquigarrow \mathrm{Hom}^\chi_S\left(\mathbf{A}^2_S \setminus \{0\}, \mathscr{T}_\Sigma \times_k S\right)^*$$

to *non-degenerated* χ-equivariant morphisms with no trivial coordinate. These functors are represented respectively by the product

$$\prod_{\alpha\in\Sigma(1)} \mathbf{A}_k^{d_\alpha+1},$$

its restriction

$$\prod_{\alpha\in\Sigma(1)} \mathbf{A}_k^{d_\alpha+1} \setminus \{0\},$$

and the open subset

$$\widetilde{(\mathbf{P}_k^1)}^{\mathbf{d}}_{B_\Sigma} \subset \prod_{\alpha\in\Sigma(1)} \mathbf{A}_k^{d_\alpha+1} \setminus \{0\}$$

(defined page 134 just before Proposition 5.1.4).

If χ lies in $\mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$, composition by $\pi : \mathcal{T}_\Sigma \to V_\Sigma$ provides a map of functors

$$\pi_* : \mathbf{Hom}^\chi\left(\mathbf{A}_k^2 \setminus \{0\}, \mathcal{T}_\Sigma\right)^* \longrightarrow \mathbf{Hom}_k^{\mathbf{d}}\left(\mathbf{P}_k^1, V_\Sigma\right)$$

and we recover the T_{NS}-torsor

$$\widetilde{(\mathbf{P}_k^1)}^{\mathbf{d}}_{B_\Sigma} \longrightarrow \mathrm{Hom}_k^{\mathbf{d}}\left(\mathbf{P}_k^1, V_\Sigma\right)_U$$

of Proposition 5.1.4.

5.1.5.1. *Restricting to $\mathscr{S}$.* Let $\iota : \mathscr{S} \hookrightarrow \mathbf{P}_k^1$ be a zero-dimensional subscheme of $\mathbf{P}_k^1$. In what follows, we will work with the restriction to $\mathscr{S}$ of the universal torsor $\mathbf{A}^2\setminus\{0\} \to \mathbf{P}^1$, defined over any base-scheme S by the following Cartesian square.

$$\begin{tikzcd}
(\mathbf{A}_S^2 \setminus \{0\})_{|\mathscr{S}} \arrow[r] \arrow[d, "\mathrm{pr}_{\mathscr{S}}"] \arrow[dr, phantom, "\ulcorner"] & \mathbf{A}_S^2 \setminus \{0\} \arrow[d] \\
\mathscr{S} \times_k S \arrow[r, hook, "\iota_S"] & \mathbf{P}_S^1
\end{tikzcd}$$

For any cocharacter $\chi \in \mathcal{X}_*(U)$, we consider the set of χ-equivariant S-morphisms

$$\mathrm{Hom}_S^\chi\left((\mathbf{A}_S^2 \setminus \{0\})_{|\mathscr{S}}, \mathbf{A}_S^{\Sigma(1)}\right)$$

as well as its subset of morphisms landing in the universal torsor $\mathcal{T}_\Sigma \subset \mathbf{A}_k^{\Sigma(1)}$

$$\mathrm{Hom}_S^\chi\left((\mathbf{A}_S^2 \setminus \{0\})_{|\mathscr{S}}, \mathcal{T}_{\Sigma,S}\right).$$

The groups $\mathbf{G}_m^{\Sigma(1)}(S)$ and $T_{\mathrm{NS}}(S)$ act on these sets *via* their action on the target.

Lemma 5.1.10. *For all $\chi \in \mathcal{X}_*(\mathbf{G}_m^{\Sigma(1)})$, the functors*

$$S \rightsquigarrow \mathrm{Hom}_S^\chi\left((\mathbf{A}_S^2 \setminus \{0\})_{|\mathscr{S}}, \mathbf{A}_S^{\Sigma(1)}\right)$$

and

$$S \rightsquigarrow \mathrm{Hom}_S^\chi((\mathbf{A}_S^2 \setminus \{0\})_{|\mathscr{S}}, \mathcal{T}_{\Sigma,S})$$

are respectively represented by the finite products

$$\prod_{p\in|\mathscr{S}|} \mathrm{Gr}_{m_p}\left(\mathbf{A}_{\kappa(p)}^{\Sigma(1)}\right)$$

and

$$\prod_{p\in|\mathscr{S}|} \mathrm{Gr}_{m_p}(\mathcal{T}_\Sigma \times_k \kappa(p))$$

of jet schemes.

Proof. Since the restriction of $\mathbf{A}^2_k \setminus \{0\} \to \mathbf{P}^1_k$ to $\mathscr{S}$ is a trivial bundle, we may fix a section $s : \mathscr{S} \to (\mathbf{A}^2_k \setminus \{0\})_{|\mathscr{S}}$. Then composition by s induces a map

$$\begin{array}{rcl} \mathrm{Hom}^\chi_S\left((\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}}, \mathbf{A}^{\Sigma(1)}_S\right) & \longrightarrow & \mathbf{A}^{\Sigma(1)}_k(\mathscr{S} \times_k S) \\ f & \longmapsto & f \circ (s, \mathrm{id}_S) \end{array}$$

which is functorial in S.

Now remark that an element of $\mathrm{Hom}^\chi_S\left((\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}}, \mathbf{A}^{\Sigma(1)}_S\right)$ is entirely determined by its restriction to the image of s: indeed, for all $y \in (\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}}(S)$, one has the relation

$$f(y) = \chi\left(y \cdot (s \circ \mathrm{pr}_{\mathscr{S}}(y))^{-1}\right) f(s \circ \mathrm{pr}_{\mathscr{S}}(y))$$

where $y \cdot (s \circ \mathrm{pr}_{\mathscr{S}}(y))^{-1} \in \mathbf{G}_{m,\mathscr{S}}(S)$ is given by the $\mathbf{G}_{m,\mathscr{S}}$-torsor structure. An $\mathscr{S} \times_k S$-point of $\mathbf{A}^{\Sigma(1)}_S$ is the datum of such a restriction, hence it provides a unique χ-equivariant morphism. By definition of Greenberg schemes, the conclusion follows. $\square$

Composition by s and $\pi : \mathcal{T}_\Sigma \to V_\Sigma$ provides a map (functorial in S)

$$\begin{array}{rcl} \mathrm{Hom}^\chi_S\left((\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}}, \mathcal{T}_{\Sigma,S}\right) & \longrightarrow & V_\Sigma(\mathscr{S} \times_k S) \\ f & \longmapsto & (\pi, \mathrm{id}_S) \circ f \circ (s, \mathrm{id}_S). \end{array}$$

Two χ-equivariant morphisms $\tilde{\varphi}, \tilde{\varphi}' : (\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}} \to \mathbf{A}^{\Sigma(1)}_S$ induce the same $(\mathscr{S} \times_k S)$-point of V_Σ if and only if there is an element $a \in T_{\mathrm{NS}}(\mathscr{S} \times_k S)$ such that $\tilde{\varphi} = a \cdot \tilde{\varphi}'$. This map defines a $T_{\mathrm{NS},\mathscr{S}}$-torsor over $\mathrm{Hom}(\mathscr{S}, V_\Sigma)$.

Definition 5.1.11. For any constructible subset $\widetilde{W} \subset \mathrm{Hom}^\chi_S\left((\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}}, \mathbf{A}^{\Sigma(1)}_S\right)$, we denote by

$$\mathrm{Hom}^\chi_S\left(\mathbf{A}^2_S \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_S \mid \widetilde{W}\right)^{(*)}$$

the subsets of $\mathrm{Hom}^\chi_S\left(\mathbf{A}^2_S \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_S\right)^{(*)}$ of morphisms whose restriction to $(\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}}$ belongs to $\widetilde{W}$ (in the sequel the exponent $^{(*)}$ will be a convenient notation to say that one can restrict to morphisms having no trivial coordinate).

We will say that a χ-equivariant S-morphism *is non-degenerate* above $\mathscr{S}$ if its pull-back to $(\mathbf{A}^2_S \setminus \{0\})_{|\mathscr{S}}$ has image in $\mathcal{T}_{\Sigma,S} \subset \mathbf{A}^{\Sigma(1)}_S$.

This defines subfunctors of $\mathbf{Hom}^\chi\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right)$.

5.1.5.2. *Euclidian division.* Fixing coordinates on $\mathbf{A}^2_k$, we can see χ-equivariant morphisms $\mathbf{A}^2_k \setminus \{0\} \to \mathbf{A}^{\Sigma(1)}_k$ as $\Sigma(1)$-tuples of homogenous polynomials in two indeterminates t_0 and t_1. Let us temporally choose a generator ϖ of the ideal defining $\mathscr{S}$ in $\mathbf{P}^1_k$, that is to say a non-trivial homogeneous polynomial of degree $\ell(\mathscr{S})$ in two indeterminates. Up to changing coordinates on $\mathbf{P}^1_k$ we may assume furthermore that $[0:1]$ does not belong to $\mathscr{S}$ (that is to say, t_0 does not divide ϖ).

Then, the Euclidian division of a polynomial $P(t)$ of degree at most d by $\varpi(t) = \varpi(1, t)$, is the unique decomposition of the form

$$P(t) = Q(t)\varpi(t) + R(t)$$

where $Q(t)$ is of degree at most $\max(0, d-\ell(\mathscr{S}))$ and R of degree stricly smaller than $\ell(\mathscr{S})$. This provides a Euclidian division of $P(t_0,t_1)=t_0^d P(t_1/t_0)$ of the form

$$\begin{aligned} P(t_0,t_1) &= Q(t_0,t_1)\varpi(t_0,t_1)+R(t_0,t_1) \\ &= t_0^d Q(t_1/t_0)\varpi(t_1/t_0)+t_0^d R(t_1/t_0). \end{aligned}$$

Note that the products $t_0^d Q(t_1/t_0)\varpi(t_1/t_0)$ and $t_0^d R(t_1/t_0)$ are homogeneous polynomials of degree d and that they do not depend on the choice of ϖ. The first one uniquely defines an element of

$$\mathrm{Hom}^d\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}_k \mid \mathrm{res}_{\mathscr{S}}=0\right)$$

while the second one uniquely defines an element of

$$\mathrm{Hom}^d\left(\left(\mathbf{A}^2_k\setminus\{0\}\right)_{|\mathscr{S}},\mathbf{A}_k\right),$$

since in that case the k-vector space

$$k[t]/(\varpi(1,t))\simeq\prod_{p\in|\mathscr{S}|}\mathbf{A}^1_{\kappa(p)}(\kappa(p)[t]/(t^{m_p+1}))\simeq\prod_{p\in|\mathscr{S}|}\mathrm{Gr}_{m_p}\left(\mathbf{A}^{\Sigma(1)}_{\kappa(p)}\right)(\kappa(p))$$

provides a concrete incarnation of such a space of morphisms. Remember as well that taking the restriction to $\mathscr{S}$ is a linear operation.

One can perform this Euclidian division simultaneously for all the coordinates of an element of $\mathrm{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right)$. Recall that we fixed a section $s:\mathscr{S}\to(\mathbf{A}^2\setminus\{0\})_{|\mathscr{S}}$ in Lemma 5.1.10 so that

$$\prod_{p\in|\mathscr{S}|}\mathrm{Gr}_{m_p}\left(\mathbf{A}^{\Sigma(1)}_{\kappa(p)}\right)$$

represents the functor $S\rightsquigarrow\mathrm{Hom}^\chi_S\left((\mathbf{A}^2_S\setminus\{0\})_{|\mathscr{S}},\mathbf{A}^{\Sigma(1)}_S\right)$. We can see elements of this product of arcs spaces as tuples $(r_\alpha(t))_{\alpha\in\Sigma(1)}$ of polynomials of degree at most $\ell(\mathscr{S})-1$. From these remarks, we deduce the following lemma.

Lemma 5.1.12. *For every $\chi\in\mathbf{N}^{\Sigma(1)}\subset\mathcal{X}_*(\mathbf{G}_m^{\Sigma(1)})\simeq\mathbf{Z}^{\Sigma(1)}$, Euclidian decomposition corresponds to the exact sequence of vector spaces over k*

$$0\to\ker(\mathrm{res}_{\mathscr{S}})\to\mathrm{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right)\xrightarrow{\mathrm{res}_{\mathscr{S}}}\mathrm{Hom}^\chi\left((\mathbf{A}^2_k\setminus\{0\})_{|\mathscr{S}},\mathbf{A}^{\Sigma(1)}_k\right)\to\mathrm{coker}(\mathrm{res}_{\mathscr{S}})\to 0$$

and $\mathrm{res}_{\mathscr{S}}$ is a piecewise trivial fibration. Moreover if $\chi_\alpha\geqslant\ell(\mathscr{S})-1$ then the α-th coordinate of $\mathrm{res}_{\mathscr{S}}$ is surjective.

Hence for all constructible subset $\widetilde{W}\subset\prod_{p\in|\mathscr{S}|}\mathrm{Gr}_{m_p}\left(\mathbf{A}^{\Sigma(1)}_{\kappa(p)}\right)$ we have the dimensional upper bound

$$\dim_k\left(\mathrm{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\widetilde{W}\right)\right)\leqslant\dim\left(\widetilde{W}\right)+\sum_{\alpha\in\Sigma(1)}\min(0,\chi_\alpha-\ell(\mathscr{S})+1). \tag{5.1.5.37}$$

Assume that $\chi\geqslant\ell(\mathscr{S})-1$. If we fix once and for all a section

$$\mathrm{Im}(\mathrm{res}_{\mathscr{S}})\hookrightarrow\mathrm{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right),$$

then for all $g \in \mathbf{G}_m^{\Sigma(1)}(\mathscr{S})$ *and constructible subset* $\widetilde{W} \subset \prod_{p\in|\mathscr{S}|} \mathrm{Gr}_{m_p}\left(\mathbf{A}^{\Sigma(1)}_{\kappa(p)}\right)$ *there is a commutative diagram*

$$\begin{array}{ccc} \mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)} \mid \widetilde{W}\right)^{(*)} & \xrightarrow{\mathrm{res}_{\mathscr{S}}} & \mathrm{Hom}^{\chi}\left((\mathbf{A}^2_k \setminus \{0\})_{|\mathscr{S}}, \mathbf{A}_k^{\Sigma(1)}\right) \\ {\scriptstyle \tau_g}\big\downarrow & & {\scriptstyle g\cdot}\big\downarrow \\ \mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)} \mid g\cdot\widetilde{W}\right)^{(*)} & \xrightarrow{\mathrm{res}_{\mathscr{S}}} & \mathrm{Hom}^{\chi}\left((\mathbf{A}^2_k \setminus \{0\})_{|\mathscr{S}}, \mathbf{A}_k^{\Sigma(1)}\right) \end{array}$$

in which the vertical arrows are isomorphisms. The first one, τ_g*, sends a morphism of Euclidian decomposition*

$$(q_\alpha, r_\alpha)_{\alpha\in\Sigma(1)}$$

to

$$(q_\alpha, g_\alpha \cdot r_\alpha)_{\alpha\in\Sigma(1)}.$$

5.1.5.3. *Degeneracies of equivariant morphisms.* Up to a multiplicative constant, rational curves of degree $d \in \mathbf{N}$ in $\mathbf{P}^n_k$ are given by $(n+1)$-tuples of degree d homogeneous polynomials with no common factor. Conversely, the parameter space of $(n+1)$-tuples of degree d homogeneous polynomials admits a decomposition into disjoint subspaces corresponding to the degree of the common factor.

In this paragraph we extend this decomposition to our setting.

Definition 5.1.13. Given a B_Σ-tuple of non-negative integers $\underline{\delta} \in \mathbf{N}^{B_\Sigma}$, B_Σ-tuples of zero-cycles on $\mathbf{P}^1_k$ of degree $\underline{\delta}$ will be called *coarse degeneracy tuples of degree* $\underline{\delta}$. They are parametrised by

$$\mathrm{Sym}^{\underline{\delta}}_{/k}(\mathbf{P}^1_k).$$

Remark 5.1.14. For any partition $\mu = (m_i)_{i\in\mathbf{N}^*}$ of $\underline{\delta} \in \mathbf{N}^{B_\Sigma}$, the restricted symmetric product

$$\mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k)_*$$

defined page 64 is isomorphic to the locally closed subset of $\mathrm{Sym}^{\underline{\delta}}_{/k}(\mathbf{P}^1_k) \simeq \prod_{J\in B_\Sigma} \mathbf{P}_k^{\delta_J}$ parametrising B_Σ-tuples of zero-cycles on $\mathbf{P}^1_k$ with multiplicities given by μ, that is to say: n_1 (geometric) points appearing once, n_2 points appearing twice, and so on. This can be easily rephrased in terms of roots of homogeneous polynomials in an algebraic closure of k. In any case, we get a decomposition of

$$\mathrm{Sym}^{\underline{\delta}}_{/k}(\mathbf{P}^1_k)$$

into locally closed subsets of the form

$$\bigsqcup_{\mu \text{ partition of } \underline{\delta}} \mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k)_*.$$

Moreover, this decomposition is compatible with the restriction to the complement of $|\mathscr{S}|$, meaning that

$$\mathrm{Sym}^{\underline{\delta}}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)$$

admits a decomposition into the disjoint union

$$\bigsqcup_{\mu \text{ partition of } \underline{\delta}} \mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*.$$

Definition 5.1.15. A *representative of a coarse degeneracy tuple* $(\Delta_J)_{J\in B_\Sigma}$ *of degree* $\underline{\delta}$ is an element of

$$\mathrm{Hom}^{\underline{\delta}}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{B_\Sigma}_k\right)^*$$

whose image by the $\mathbf{G}_m^{B_\Sigma}$ torsor

$$\mathrm{Hom}^{\underline{\delta}}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{B_\Sigma}_k\right)^* \longrightarrow \mathrm{Sym}^{\underline{\delta}}_{/k}(\mathbf{P}^1_k)$$

is $(\Delta_J)_{J\in B_\Sigma}$.

Definition 5.1.16. A *degeneracy tuple of degree* $\underline{\delta}$ is a coarse degeneracy tuple $(\Delta_J)_{J\in B_\Sigma}$ of degree $\underline{\delta}$ *such that* Δ_J *and* $\Delta_{J'}$ *are coprime as soon as* $J \not\subseteq J'$ *and* $J' \not\subseteq J$.

Lemma 5.1.17. *Let* $(\mathscr{P}^1_k)_{B_\Sigma}$ *be the family indexed by* $\mathbf{N}^{B_\Sigma} \setminus \{\mathbf{0}\}$ *whose* $\underline{\delta}$*th-term is*
- $\mathbf{P}^1_k \xrightarrow{\mathrm{id}} \mathbf{P}^1_k$ *whenever the support of* $\underline{\delta}$ *is given by a finite chain* $\mathfrak{J} = J_1 \subsetneq \ldots \subsetneq J_{\ell(\mathfrak{J})}$ *of elements of* B_Σ *– that is to say,* $\underline{\delta} \in D_{\mathfrak{J}}$*, where* $D_{\mathfrak{J}}$ *was defined just before* *Lemma 5.1.1 page 131,*
- *and empty otherwise.*

Then the disjoint union

$$\mathrm{Sym}^{\underline{\delta}}_{\mathbf{P}^1_k}\left((\mathscr{P}^1_k)_{B_\Sigma}\right) = \bigsqcup_{\substack{\mu \\ \text{partition of } \underline{\delta}}} \mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}\left((\mathscr{P}^1_k)_{B_\Sigma}\right)_*$$

parametrises degeneracy tuples of degree $\underline{\delta}$ *and the maps*

$$\mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}\left((\mathscr{P}^1_k)_{B_\Sigma}\right)_* \longrightarrow \mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k\right)_*.$$

are open embeddings.

Proof. The elements of $\mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}((\mathscr{P}^1_k)_{B_\Sigma})_*$ are degeneracy tuples for any partition μ. We check that the converse is true. Let Δ be a degeneracy tuple of degree $\underline{\delta}$ and p a closed point of $\mathbf{P}^1_k$ appearing in the decomposition of one of the Δ_J. For all $J \in B_\Sigma$, let $m_J(p)$ be the multiplicity of Δ_J at p. Let J_1 be minimal such that $m_{J_1}(p) > 0$. Then $m_J(p) > 0$ implies that J contains J_1, since Δ_J and Δ_{J_1} are coprime as soon as $J \not\subseteq J_1$ and $J_1 \not\subseteq J$. The chain $\mathfrak{J}_p$ for the point p is given by

$$\mathfrak{J}_p = \{J \in B_\Sigma \mid m_J(p) > 0\}.$$

Let $\mu_J = (m^J_i)_{i\in\mathbf{N}^*}$ be the partition of δ_J given by

$$m^J_i = |\{p \in \mathbf{P}^1_k \mid m_J(p) = i\}|$$

and $\mu = (\mu_J)_{J\in B_\Sigma}$. By definition of symmetric products (see Section 2.4.2.1), Δ lies in $\mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}((\mathscr{P}^1_k)_{B_\Sigma})_*$. □

Definition 5.1.18. We will say that a coarse degeneracy tuple *avoids* $\mathscr{S}$ if it has support outside $|\mathscr{S}|$. Degeneracy tuples of degree $\underline{\delta}$ avoiding $\mathscr{S}$ are parametrised by the restriction

$$\begin{aligned}&\bigsqcup_{\substack{\mu\\ \text{partition of } \underline{\delta}}} \left(\mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}\left((\mathscr{P}^1_k)_{B_\Sigma}\right)_* \cap \mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)\\ \simeq &\bigsqcup_{\substack{\mu\\ \text{partition of } \underline{\delta}}} \mathrm{Sym}^{\mu}_{\mathbf{P}^1_k\setminus|\mathscr{S}|}\left((\mathscr{P}^1_k)^{\widehat{|\mathscr{S}|}}_{B_\Sigma}\right)_*\end{aligned}$$

where $(\mathscr{P}^1_k)^{\widehat{|\mathscr{S}|}}_{B_\Sigma}$ is the restriction of the family $(\mathscr{P}^1_k)_{B_\Sigma}$ to the complement of $|\mathscr{S}|$.

A morphism in $\mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}\right)^*$ defines a $\Sigma(1)$-tuple

$$(C_\alpha)_{\alpha\in\Sigma(1)} = \left(\sum_{p\in|\mathbf{P}^1_k|} n_{\alpha,p}[p]\right)_{\alpha\in\Sigma(1)}$$

of zero-cycles on $\mathbf{P}^1_k$, given by the $\mathbf{G}_m^{\Sigma(1)}$-torsor

$$\mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}\right)^* \to \prod_{\alpha\in\Sigma(1)} \mathbf{P}^{\chi_\alpha}_k \simeq \mathrm{Sym}^{\chi}_{/k}(\mathbf{P}^1_k).$$

Moreover, given any non-empty subset $J \subset \Sigma(1)$, the greatest common divisor

$$\gcd(C_\alpha)_{\alpha\in J} = \sum_{p\in|\mathbf{P}^1_k|} \min_{\alpha\in J}(n_{\alpha,p})[p]$$

of a J-tuple of zero-cycles of degrees χ_α, $\alpha \in J$, defines an algebraic map

$$\prod_{\alpha\in J} (\mathbf{P}^1_k)^{\langle\chi_\alpha\rangle} \to \bigsqcup_{0\leqslant d\leqslant \min(\chi_\alpha)} \mathrm{Sym}^d_{/k}(\mathbf{P}^1_k) \subset \mathrm{Div}(\mathbf{P}^1_k).$$

Definition 5.1.19. Let $(C_\alpha)_{\alpha\in\Sigma(1)} \in (\mathrm{Div}(\mathbf{P}^1_k))^{\Sigma(1)}$ be a $\Sigma(1)$-tuple of effective 0-cycles on $\mathbf{P}^1_k$. We inductively define the 0-cycle $(\Delta_J((C_\alpha)))_{J\in B_\Sigma}$ by setting

$$\Delta_{\Sigma(1)}((C_\alpha)) = \gcd((C_\alpha)_{\alpha\in\Sigma(1)})$$

and for any $J \in B_\Sigma$

$$\Delta_J((C_\alpha)) = \gcd((C_\alpha)_{\alpha\in J}) - \sum_{\substack{J'\in B_\Sigma\\ J'\supsetneq J}} \Delta_{J'}((C_\alpha)).$$

Lemma 5.1.20. *For every $(C_\alpha)_{\alpha\in\Sigma(1)} \in (\mathrm{Div}(\mathbf{P}^1_k))^{\Sigma(1)}$, the B_Σ-tuple $(\Delta_J((C_\alpha)))_{J\in B_\Sigma}$ is a degeneracy tuple.*

Proof. It is sufficient to check the claim when the support of the $(C_\alpha)_{\alpha\in\Sigma(1)}$ is reduced to a point. Then $(\Delta_J)_{J\in B_\Sigma}$ is the tuple of non-negative integers described in the proof of Lemma 5.1.1, and there is a $\mathfrak{J} = (J_1 \subsetneq \ldots \subsetneq J_{\ell(\mathfrak{J})})$ such that $\Delta_J \neq 0$ if and only if $J \in \mathfrak{J}$. This shows that $\Delta_J \wedge \Delta_{J'} \neq 1$ implies $J \subset J'$ or $J' \subset J$. □

Applying this construction to the $\Sigma(1)$-tuple of a χ-equivariant morphism, we get a map

$$\mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right)^* \xrightarrow{\Delta} \left(\mathrm{Div}(\mathbf{P}^1_k)\right)^{B_\Sigma}$$

which specialises to a morphism

$$\mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k\setminus\{0\}\xrightarrow{f}\mathbf{A}^{\Sigma(1)}_k\;\middle|\;\mathbf{deg}(\Delta(f))=\delta\right)^*\xrightarrow{\Delta}\mathrm{Sym}^{\delta}_{\mathbf{P}^1_k}\left((\mathscr{P}^1_k)_{B_\Sigma}\right)$$

for every $\underline{\delta}\in\mathbf{N}^{B_\Sigma}$.

5.1.5.4. *Degenerations versus reduction modulo $\mathscr{S}$.* Equivariant morphisms with image contained in $\mathcal{T}_\Sigma$ are the morphisms $\mathbf{A}^2_k\setminus\{0\}\to\mathbf{A}^{\Sigma(1)}_k$ with no degeneracies. Our goal now is to adapt the Möbius inversion technique to the context of constrained curves. More precisely, in this paragraph we provide a relation between the classes of (possibly degenerate) equivariant morphisms and the ones obtained by adding coarse degeneracies to non-degenerate morphisms. Then we specialise this relation to morphisms with contraints. Finally, we approximate and inverse it.

Let W be a constructible subset of $\mathrm{Hom}(\mathscr{S},V_\Sigma)$. Its preimage through the map

$$\mathrm{Hom}^{\chi}\left((\mathbf{A}^2_S\setminus\{0\})_{|\mathscr{S}},\mathcal{T}_{\Sigma,S}\right)\longrightarrow\mathrm{Hom}(\mathscr{S},V_\Sigma)$$

will be written $\widetilde{W}$. It is a Zariski-locally trivial $T_{\mathrm{NS},\mathscr{S}}$-torsor above W.

Remark 5.1.21. Let $f\in\mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\widetilde{W}\right)^*$. The degeneracy tuple of f,

$$(\Delta_J(f))_{J\in B_\Sigma},$$

avoids $\mathscr{S}$.

Up to replacing $\mathscr{S}$ by $\mathscr{S}\cup\{[1:0]\}$ and W by $W\times V_\Sigma$, we can always assume that $[1:0]=\infty$ is a closed point of $\mathscr{S}$. Then $(\Delta_J(f))_{J\in B_\Sigma}$ can be identified with the unique B_Σ-tuple of δ_J-equivariant morphisms $\mathbf{A}^2_k\setminus\{0\}\to\mathbf{A}^1_k$ sending $(1,0)$ to 1. This is a convention we adopt from now on: one can think of $D_J(f)$ as a unitary polynomial in one variable, of degree δ_J. We also adopt this convention for coarse degeneracy tuples and zero-cycles of $\mathbf{P}^1_k$ avoiding $|\mathscr{S}|$ in general.

The morphism obtained by multiplication

$$(\Delta_\alpha(f))_{\alpha\in\Sigma(1)}:\left\{\begin{array}{ccc}\mathbf{A}^2_k\setminus\{0\}&\to&\mathbf{A}^{\Sigma(1)}_k\\(x,y)&\longmapsto&\left(\prod_{J\ni\alpha}\Delta_J(f)(x,y)\right)_{\alpha\in\Sigma(1)}\end{array}\right.$$

is $(\sum_{J\ni\alpha}\delta_J)_{\alpha\in\Sigma(1)}$-invariant and its reduction modulo $\mathscr{S}$ is an element of $\mathbf{G}_m^{\Sigma(1)}(\mathscr{S})$.

Moreover we fix for every $d\in\mathbf{N}$ a section of

$$\mathrm{Hom}^d\left(\mathbf{A}^2_k\setminus 0,\mathbf{A}^1_k\right)\xrightarrow{\mathrm{res}_{\mathscr{S}}}\mathrm{Im}(\mathrm{res}_{\mathscr{S}})\subset\mathrm{Hom}^d\left(\left(\mathbf{A}^2_k\setminus 0\right)_{|\mathscr{S}},\mathbf{A}^1_k\right)$$

so that we are able to see reduction modulo $\mathscr{S}$ as morphisms in $\mathrm{Hom}^d(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^1_k)$, as we did at the beginning of Section 5.1.5.2.

In the following we will freely use the convenient notation

$$D_\alpha=\prod_{J\ni\alpha}D_J$$

for the α-coordinate of the image of D by the corresponding morphism

$$\left(\prod_{J\ni\alpha}\cdot\right)_{\alpha\in\Sigma(1)}:\mathrm{Sym}^{\delta}_{/k}(\mathbf{P}^1_k\setminus|\mathscr{S}|)\longrightarrow\mathrm{Sym}^{\left(\sum_{J\ni\alpha}\delta_\alpha\right)_{\alpha\in\Sigma(1)}}_{/k}(\mathbf{P}^1_k\setminus|\mathscr{S}|),$$

as well as

$$\overline{D}^{-1}=\left(\overline{D_\alpha}^{-1}\right)_{\alpha\in\Sigma(1)}$$

for the inverse of the restriction to $\mathscr{S}$ of any point $D \in \mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}\left((\mathscr{P}^1_k)^{\widehat{|\mathscr{S}|}}_{B_\Sigma}\right)_*$.

Consider the morphism

$$\begin{array}{rcl} \Phi^{\chi}_{\underline{\delta}} : \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) \times \mathrm{Sym}^{\underline{\delta}}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|) & \longrightarrow & \mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) \\ \left((f_\alpha)_{\alpha\in\Sigma(1)}, (D_J)_{J\in B_\Sigma}\right) & \longmapsto & (f_\alpha D_\alpha)_{\alpha\in\Sigma(1)}. \end{array}$$

In other words,

$$\Phi^{\chi}_{\underline{\delta}} = \Phi^{\chi}_{\chi-\chi'} \circ \left(\mathrm{pr}_1, \left(\prod_{J\ni\alpha} \cdot\right)_{\alpha\in\Sigma(1)}\right)$$

where $\Phi^{\chi}_{\chi-\chi'}$ is the morphism of coordinatewise multiplication of a χ'-equivariant morphism with a $(\chi-\chi')$-equivariant one, hence $\Phi^{\chi}_{\underline{\delta}}$ is constant on each fibre of $(\prod_{J\ni\alpha}\cdot)_{\alpha\in\Sigma(1)}$. For any partition μ of $\underline{\delta}$, let

$$\Phi^{\chi}_{\mu} : \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) \times \mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \longrightarrow \mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right)$$

be the restriction of $\Phi^{\chi}_{\underline{\delta}}$ to coarse degeneracy tuples whose multiplicities are given by μ. Its image is a constructible subset. For any coarse degeneracy tuple D avoiding $\mathscr{S}$ (or any $(\chi-\chi')$-equivariant morphism), consider the induced map

$$\begin{array}{rcl} \Phi^{\chi}_{D} : \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) & \longrightarrow & \mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) \\ (f_\alpha)_{\alpha\in\Sigma(1)} & \longmapsto & (f_\alpha D_\alpha)_{\alpha\in\Sigma(1)}. \end{array}$$

Note that Φ^{χ}_{μ} is a global version of the map Φ_{loc} given in the proof of Lemma 5.1.1 page 131, and that Φ^{χ}_{D} is an injective linear map between k-vector spaces of finite dimension; in particular, it induces an isomorphism

$$\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right)^{(*)} \simeq \mathrm{Im}(\Phi_D)^{(*)}.$$

We will sometimes use the concise notations

$$S^{\mu}_{\widehat{|\mathscr{S}|}*} \qquad \text{and} \qquad H^{\chi}$$

respectively for $\mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$ and Hom^{χ}, when necessary.

Let

$$\begin{aligned} &\left(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^{(*)} \times \mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)_{\widetilde{W}} \\ &= \left(\mathrm{H}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^{(*)} \times S^{\mu}_{\widehat{|\mathscr{S}|}*}\right)_{\widetilde{W}} \end{aligned}$$

be the preimage of $\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \widetilde{W}\right)^{(*)}$ by Φ^{χ}_{μ}. The projection onto the second factors endows it with the structure of a $\mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$-variety. We are going to compare its class with the one of

$$\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma \,\middle|\, \widetilde{W}\right)^{*} \times \mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*.$$

The previous development about degeneracy tuples shows in particular that

$$\mathrm{H}^{\chi}(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \mathcal{T}_{\Sigma,\mathscr{S}})^{(*)}$$

and

$$\mathrm{Hom}^{\chi}(\mathbf{A}^2 \setminus \{0\}, \mathbf{A}^{\Sigma(1)} \mid \widetilde{W})^{(*)}$$

are covered by the images through the multiplication morphisms $\Phi^{\chi}_{\underline{\delta}}$ of respectively all the products

$$\mathrm{H}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^{(*)} \times S^{\mu}_{\widehat{|\mathscr{S}|}\,*}$$

and

$$\left(\mathrm{H}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^{(*)} \times S^{\mu}_{\widehat{|\mathscr{S}|}\,*}\right)_{\widetilde{W}}$$

for χ' and $\underline{\delta}$ such that $\chi' + (\sum_{J\ni\alpha}\delta_J) = \chi$. The relation between all these spaces is summarised by the commutative diagram given page 147.

Remark 5.1.22. The relation we are looking for is encoded by the inverse of the motivic Euler product

$$\prod_{p\notin|\mathscr{S}|} P_{B_\Sigma(\mathbf{t})}.$$

Indeed, by inverting the relation (5.1.2.33) page 130 defining μ_{B_Σ}, one gets

$$\begin{aligned}
\sum_{\mathbf{n}\in\mathbf{N}^{\Sigma(1)}} \mathbf{t}^{\mathbf{n}} &= \frac{Q_{B_\Sigma}(\mathbf{t})}{1-(1-P_{B_\Sigma}(\mathbf{t}))} \\
&= Q_{B_\Sigma}(\mathbf{t}) \sum_{m\in\mathbf{N}} \left(\sum_{\mathbf{n}\in B_\Sigma} -\mu_{B_\Sigma}(\mathbf{n})\mathbf{t}^{\mathbf{n}} \right)^m \\
&= \left(\sum_{\mathbf{n}\in A(B_\Sigma)} \mathbf{t}^{\mathbf{n}} \right) \left(1 + \sum_{m\in\mathbf{N}^*} \sum_{\mathbf{n}_1,\dots,\mathbf{n}_m\in B_\Sigma} (-1)^m \mu_{B_\Sigma}(\mathbf{n}_1)\cdots\mu_{B_\Sigma}(\mathbf{n}_m)\mathbf{t}^{\mathbf{n}_1+\dots+\mathbf{n}_m} \right)
\end{aligned} \tag{5.1.5.38}$$

(it is useful to remark that $\mu_{B_\Sigma}(\mathbf{0}) = 1$ and $\mu_{B_\Sigma}(\mathbf{n}) = 0$ whenever $\mathbf{n}\notin B_\Sigma\cup\{\mathbf{0}\}$). Let us quickly interpret this relation in terms of (symmetric products of) families above $\mathbf{P}^1_k$.

Let

$$(\mathbf{P}^1_k)_{A(B_\Sigma)}$$

be the family of $\mathbf{P}^1_k$-varieties indexed by $\mathbf{N}^{\Sigma(1)}\setminus\{\mathbf{0}\}$ and given by $\mathbf{P}^1_k \xrightarrow{\mathrm{id}} \mathbf{P}^1_k$ if $\mathbf{n}\in A(B_\Sigma)$ and $\varnothing$ otherwise. Similarly, let

$$(\mathbf{P}^1_k)_{\sqcup_{m\in\mathbf{N}^*} B^m_\Sigma}$$

be the trivial family of $\mathbf{P}^1_k$-varieties $(\mathbf{P}^1_k \xrightarrow{\mathrm{id}} \mathbf{P}^1_k)$ indexed by $\sqcup_{m\in\mathbf{N}^*} B^m_\Sigma$.

For any $\mathbf{n}\in\mathbf{N}^{\Sigma(1)}$, $\mathbf{n}'_0\in A(B_\Sigma)$, $m\in\mathbf{N}$ and $\mathbf{n}'_1,\dots,\mathbf{n}'_m\in B_\Sigma$ such that

$$\mathbf{n} = \mathbf{n}'_0 + \mathbf{n}'_1 + \dots + \mathbf{n}'_m,$$

and any partitions $\kappa_1 = (k_{\mathbf{n}})$ and $\kappa_2 = (k_{\mathfrak{J}})$ respectively of $\mathbf{n}'_0$ and $(m,\mathbf{n}'_1,\dots,\mathbf{n}'_m)$, there is a canonical morphism

$$\mathrm{Sym}^{\kappa_1}_{\mathbf{P}^1_k/k}((\mathbf{P}^1_k)_{A(B_\Sigma)}) \times \mathrm{Sym}^{\kappa_2}_{\mathbf{P}^1_k/k}((\mathbf{P}^1_k)_{\sqcup_{m\in\mathbf{N}^*} B^m_\Sigma}) \longrightarrow \mathrm{Sym}^{\varpi}_{\mathbf{P}^1_k/k}(\mathbf{P}^1_k)$$

sending tuples of zero-cycles of the form

$$\sum_{\mathbf{n}\in A(B_\Sigma)} (n_\alpha(x^1_\alpha + \dots + x^{k_{\mathbf{n}}}_\alpha))_{\alpha\in\Sigma(1)} \in \left(\prod_{\mathbf{n}\in A(B_\Sigma)} \mathrm{Sym}^{k_{\mathbf{n}}}_{\mathbf{P}^1_k/k}(\mathbf{P}^1_k) \right)_*$$

$$\begin{tikzcd}
 & \left(\mathrm{H}^{\chi'}(\mathbf{A}_k^2 \setminus \{0\}, \mathcal{T}_\Sigma)^* \times S^{\mu}_{\widehat{|\mathscr{S}|}*}\right)_{\widetilde{W}} \arrow[rr, "\Phi_\mu^\chi"] \arrow[dl, hook] \arrow[dd, "(\mathrm{res}_{\mathscr{S}}, \mathrm{pr}_2)"] & & \mathrm{H}^{\chi}(\mathbf{A}_k^2 \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)} \mid \widetilde{W})^* \arrow[dl, hook] \arrow[dd, "\mathrm{res}_{\mathscr{S}}"] \\
\mathrm{H}^{\chi'}(\mathbf{A}_k^2 \setminus \{0\}, \mathcal{T}_\Sigma)^* \times S^{\mu}_{\widehat{|\mathscr{S}|}*} \arrow[rr, "\Phi_\mu^\chi"] \arrow[dd, "(\mathrm{res}_{\mathscr{S}}, \mathrm{pr}_2)"'] & & \mathrm{H}^{\chi}(\mathbf{A}_k^2 \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)} \mid \mathcal{T}_{\Sigma,\mathscr{S}})^* \arrow[dd] & \\
 & \left(\mathrm{Hom}(\mathscr{S}, \mathcal{T}_\Sigma) \times S^{\mu}_{\widehat{|\mathscr{S}|}*}\right)_{\widetilde{W}} \arrow[rr, "(\varphi, D) \mapsto \varphi\overline{D}"] \arrow[dl] & & \widetilde{W} \arrow[dl, hook] \\
\mathrm{Hom}(\mathscr{S}, \mathcal{T}_\Sigma) \times S^{\mu}_{\widehat{|\mathscr{S}|}*} \arrow[rr, "(\varphi, D) \mapsto \varphi\overline{D}"] & & \mathrm{Hom}(\mathscr{S}, \mathcal{T}_\Sigma) &
\end{tikzcd}$$

and

$$\sum_{m\in\mathbf{N}^*}\sum_{\mathfrak{J}\in B_\Sigma^m} x_{\mathfrak{J}}^1+...+x_{\mathfrak{J}}^{k_{\mathfrak{J}}} \in \left(\prod_{\substack{m\in\mathbf{N}^*\\ \mathfrak{J}\in B_\Sigma^m}} \mathrm{Sym}^{k_{\mathfrak{J}}}_{\mathbf{P}^1_k/k}(\mathbf{P}^1_k)\right)_*$$

to their sum whose α-coordinate is

$$\begin{aligned}&\sum_{\mathbf{n}\in A(B_\Sigma)} n_\alpha(x_{1,\alpha}+...+x_{k_{\mathbf{n}},\alpha})\\ &+\sum_{m\in\mathbf{N}^*}\sum_{\mathfrak{J}\in B_\Sigma^m} |\{i\in\{1,...,m\}\mid \alpha\in\mathfrak{J}_i\}|(x_{\mathfrak{J}}^1+...+x_{\mathfrak{J}}^{k_{\mathfrak{J}}}).\end{aligned}$$

The partition $\varpi=(m_{\mathbf{n}})$ is given explicitely by

$$m_{\mathbf{n}} = \sum_{\substack{\mathbf{n}'_0\in A(B_\Sigma)\\ \mathbf{n}'_1,...,\mathbf{n}'_m\in\sqcup_{m\in\mathbf{N}^*}B_\Sigma^m\\ \mathbf{n}=\mathbf{n}'_0+\sum_m \mathbf{n}'_1+...+\mathbf{n}'_m}} k_{\mathbf{n}'_0}+k_{(\mathbf{n}'_1,...\mathbf{n}'_m)}.$$

This addition is nothing else than the zero-cycles version of the multiplication of polynomials, after a change of indeterminate, given by the Φ^χ_μ's. It factors through Φ^χ_μ, for a certain partition μ of a certain B_Σ-tuple $\underline{\delta}\in\mathbf{N}^{B_\Sigma}$ given by

$$\mu(J)_n = \sum_{\substack{\mathfrak{J}\in\sqcup_{m\in\mathbf{N}^*}B_\Sigma^m\\ |\{i|\mathfrak{J}_i=J\}|=n}} k_{\mathfrak{J}}.$$

Indeed, there is a morphism

$$\left(\prod_{\mathfrak{J}\in\sqcup_{m\in\mathbf{N}^*}B_\Sigma^m} \mathrm{Sym}^{k_{\mathfrak{J}}}_{\mathbf{P}^1_k/k}(\mathbf{P}^1_k)\right)_* \to \mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1)_*$$

sending a zero-cycle

$$\sum_{\mathfrak{J}\in\sqcup_{m\in\mathbf{N}^*}B_\Sigma^m} x_{\mathfrak{J}}^1+...+x_{\mathfrak{J}}^{k_{\mathfrak{J}}}$$

to the coarse degeneracy tuple

$$\left(\sum_{n\in\mathbf{N}^*} n\left(\sum_{\substack{\mathfrak{J}\in\sqcup_{m\in\mathbf{N}^*}B_\Sigma^m\\ |\{i|\mathfrak{J}_i=J\}|=n}} x_{\mathfrak{J}}^1+...+x_{\mathfrak{J}}^{k_{\mathfrak{J}}}\right)\right)_{J\in B_\Sigma}.$$

As before, one can restrict to zero-cycles avoiding $|\mathscr{S}|$.

For every $m\in\mathbf{N}^*$ and $\mathfrak{J}=(J_1,...,J_m)\in B_\Sigma^m$ let us set

$$\lambda_{B_\Sigma}(\mathfrak{J}) = (-1)^m\mu_{B_\Sigma}(J_1)\cdots\mu_{B_\Sigma}(J_m)$$

and $\mathbf{t}_{\mathfrak{J}} = \mathbf{t}_{J_1} \cdots \mathbf{t}_{J_m}$. Then by the very definition of the motivic Euler product Notation 2.4.9 page 66,

$$\begin{aligned}
&\prod_{p \notin |\mathscr{S}|} \left(\sum_{m \in \mathbf{N}} \left(\sum_{J \in B_\Sigma} -\mu_{B_\Sigma}(J) \mathbf{t}_J \right)^m \right) \\
&= \prod_{p \notin |\mathscr{S}|} \left(1 + \sum_{m \in \mathbf{N}^*} \sum_{\mathfrak{J} \in B_\Sigma^m} \lambda_{B_\Sigma}(\mathfrak{J}) \mathbf{t}_{\mathfrak{J}} \right) \\
&= \sum_{\kappa \in \mathbf{N}^{(\sqcup_{m \in \mathbf{N}^*} B_\Sigma^m)}} \mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k/k} \left(\left(\lambda_{B_\Sigma}(\mathfrak{J}) \left[\mathbf{P}^1_k \setminus |\mathscr{S}| \right] \right)_{\mathfrak{J} \in \sqcup_{m \in \mathbf{N}^*} B_\Sigma^m} \right)_* \mathbf{t}^{\kappa}_{(\mathfrak{J})}.
\end{aligned}$$

Remark 5.1.23. Let $\mathfrak{a}$ be a class in the Grothendieck ring of varieties over a certain S-variety X. Then for any $k \in \mathbf{N}^*$,

$$\begin{aligned}
\mathrm{Sym}^k_{X/S}(2\mathfrak{a})_* &= \sum_{k_1+k_2=k} \mathrm{Sym}^{k_1,k_2}(\mathfrak{a}, \mathfrak{a})_* \\
&= (k+1)\, \mathrm{Sym}^n(\mathfrak{a})_*
\end{aligned}$$

and more generally for any $k, \ell \in \mathbf{N}^*$,

$$\begin{aligned}
\mathrm{Sym}^k_{X/S}(\ell\mathfrak{a})_* &= \sum_{k_1+\ldots+k_\ell=k} \mathrm{Sym}^{k_1,\ldots,k_\ell}_{X/S}(\mathfrak{a}, \ldots, \mathfrak{a})_* \\
&= \binom{k+\ell-1}{\ell-1} \mathrm{Sym}^k_{X/S}(\mathfrak{a})_*.
\end{aligned}$$

Furthemore, if $\mathscr{Q} \subset \mathbf{N}^{(\mathbf{N}^*)}$ is the set of partitions of integers (in the usual sense) without holes, then for any $k \in \mathbf{N}^*$

$$\mathrm{Sym}^k_{X/S}(-\mathfrak{a})_* = \sum_{\substack{\kappa=(k_i)_{i\in\mathbf{N}} \in \mathscr{Q} \\ \sum_i k_i = k}} (-1)^{|\{i \in \mathbf{N} | k_i > 0\}|} \mathrm{Sym}^{\kappa}_{X/S}(\mathfrak{a})_*$$

above $\mathrm{Sym}^k_{/S}(X)*$ [**BH21**, Example 6.1.4].

Using these relations, we see that for all $\kappa = (k_{(J_1,\ldots,J_m)}) \in \mathbf{N}^{(\sqcup_{m\in\mathbf{N}^*} B_\Sigma^m)}$ the class

$$\mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k/k} \left(\left(\lambda_{B_\Sigma}(J_1, \ldots, J_m) \left[\mathbf{P}^1_k \setminus |\mathscr{S}| \right] \right)_{(J_1,\ldots,J_m) \in \sqcup_{m\in\mathbf{N}^*} B_\Sigma^m} \right)_*$$

is a linear combination with integer coefficients of symmetric products of the form

$$\left(\boxtimes_{\substack{(J_1,\ldots,J_m) \\ \in \sqcup_{m\in\mathbf{N}^*} B_\Sigma^m \\ \lambda_{B_\Sigma}(J_1,\ldots,J_m) \\ \neq 0}} \binom{k_{(J_1,\ldots,J_m)} + |\lambda_{B_\Sigma}(J_1, \ldots, J_m)| - 1}{|\lambda_{B_\Sigma}(J_1, \ldots, J_m)| - 1} \mathrm{Sym}^{k_{(J_1,\ldots,J_m)}}_{\mathbf{P}^1_k/k} (\mathrm{sign}(\lambda_{B_\Sigma}(J_1, \ldots, J_m))[\mathbf{P}^1 \setminus |\mathscr{S}|]) \right)_*$$

above $\mathrm{Sym}^{\kappa}_{/k}(\mathbf{P}^1)_*$. If $\lambda_{B_\Sigma}(J_1, \ldots, J_m)$ is negative, we replace

$$\mathrm{Sym}^{k_{(J_1,\ldots,J_m)}}_{\mathbf{P}^1_k/k} (\mathrm{sign}(\lambda_{B_\Sigma}(J_1, \ldots, J_m))[\mathbf{P}^1 \setminus |\mathscr{S}|])$$

by the sum

$$\sum_{\substack{\kappa'=(k'_i)_{i\in\mathbf{N}} \in \mathscr{Q} \\ \sum_i k'_i = k_{(J_1,\ldots,J_m)}}} (-1)^{|\{i \in \mathbf{N} | k'_i > 0\}|} [\mathrm{Sym}^{\kappa'}_{/k}(\mathbf{P}^1 \setminus |\mathscr{S}|)]$$

in the expression above. In the end, the elementary pieces of this linear combination are restricted symmetric products

$$\mathrm{Sym}^{\ell_1,\dots,\ell_p}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$$

for a certain tuple of integers $(\ell_1,\dots,\ell_p)$ with certain length p, which can be viewed as constructible subsets of $\mathrm{Sym}^{\kappa}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$.

Remark 5.1.24. For all χ, there is a piecewise $\mathbf{G}_m^{\Sigma(1)}$-equivariant isomorphism between $\mathrm{Hom}^{\chi}(\mathbf{A}^2 \setminus \{0\}, \mathbf{A}^{\Sigma(1)})$ and $\mathbf{G}_m^{\Sigma(1)} \times \mathrm{Sym}^{\chi}_{/k}(\mathbf{P}^1_k)$ sending $\mathrm{Hom}^{\chi}(\mathbf{A}^2 \setminus \{0\}, \mathcal{T}_\Sigma)$ to $\mathbf{G}_m^{\Sigma(1)} \times (\mathbf{P}^1_k)^{\chi}_{B_\Sigma}$. Furthermore this piecewise isomorphism commutes with the addition of cycles and multiplication of the corresponding polynomials. Alternatively, one can argue by local triviality with respect to the Zariski topology.

Therefore, instead of working with families defined over $\mathbf{P}^1_k$, we can work with the same families defined over $\mathbf{P}^1_k \times \mathbf{G}_m^{\Sigma(1)}$ relatively to $\mathbf{G}_m^{\Sigma(1)}$. It means that one can replace $\mathrm{Sym}_{\mathbf{P}^1_k/k}$ by $\mathrm{Sym}_{\mathbf{P}^1_{\mathbf{G}_m^{\Sigma(1)}}/\mathbf{G}_m^{\Sigma(1)}}$ everywhere in the previous identities. Applying the motivic Euler product notation to the local relation (5.1.5.38) given page 146, this modification provides the relation

$$\begin{aligned}&\left[\mathrm{Hom}^{\chi}(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)})\right]\\ &= \sum_{\chi',\kappa} \left[\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma) \boxtimes_{\mathrm{Sym}^{\chi}(\mathbf{P}^1_k)\times \mathbf{G}_m} \mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k/k}\left(\left(\lambda_{B_\Sigma}(\mathfrak{J})\left[\mathbf{P}^1_k \setminus |\mathscr{S}|\right]\right)_{\mathfrak{J}\in \sqcup_{m\in\mathbf{N}^*} B_\Sigma^m}\right)_*\right]\end{aligned}$$

in $K_0\mathbf{Var}_{\mathrm{Hom}^{\chi}(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}_k^{\Sigma(1)})} \simeq K_0\mathbf{Var}_{\mathrm{Sym}^{\chi}(\mathbf{P}^1_k)\times\mathbf{G}_m}$. Since we are working piecewisely above $\mathrm{Hom}^{\chi}(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)})$, one can restrict to morphisms with prescribed restriction to $\mathscr{S}$. By the previous remarks, it is straightforward to extend our subscript notation

$$(\cdots)_{\widetilde{W}}$$

for the preimages of $\mathrm{Hom}^{\chi}(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)} \mid \widetilde{W})$ by the Φ^{χ}_{μ}'s. This proves the following proposition.

Proposition 5.1.25. *For all $\chi \in \mathbf{N}^{\Sigma(1)}$,*

$$\begin{aligned}&\left[\mathrm{Hom}^{\chi}(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}_k^{\Sigma(1)} \mid \widetilde{W})\right]\\ &= \sum_{\chi',\kappa}\left[\left(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma) \boxtimes \mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k/k}\left(\left(\lambda_{B_\Sigma}(\mathfrak{J})\left[\mathbf{P}^1_k \setminus |\mathscr{S}|\right]\right)_{\mathfrak{J}\in \sqcup_{m\in\mathbf{N}^*} B_\Sigma^m}\right)_*\right)_{\widetilde{W}}\right]\end{aligned}$$

where the sum is over all χ', κ summing to χ in the sense of Remark 5.1.22.

Our goal now is to suitably approximate the class

$$\left[\left(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma) \boxtimes \mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k/k}\left(\left(\lambda_{B_\Sigma}(\mathfrak{J})\left[\mathbf{P}^1_k \setminus |\mathscr{S}|\right]\right)_{\mathfrak{J}\in \sqcup_{m\in\mathbf{N}^*} B_\Sigma^m}\right)_*\right)_{\widetilde{W}}\right]$$

by the detwisted class

$$\left[\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma \mid \widetilde{W})\right]\left[\mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k/k}\left(\left(\lambda_{B_\Sigma}(\mathfrak{J})\left[\mathbf{P}^1_k \setminus |\mathscr{S}|\right]\right)_{\mathfrak{J}\in \sqcup_{m\in\mathbf{N}^*} B_\Sigma^m}\right)_*\right]$$

above $\mathrm{Sym}^{\kappa}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$ or $\mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$.

Remark 5.1.26. Let $\mu_1, \mu_2 \leqslant \mu$ be two partitions such that $\mu_1 + \mu_2 = \mu$. If

$$D_1 \in \mathrm{Sym}^{\mu_1}_{/k}\left(\mathbf{P}^1_k\right)_* \qquad \text{and} \qquad D_2 \in \mathrm{Sym}^{\mu_2}_{/k}\left(\mathbf{P}^1_k\right)_*$$

are two coarse degeneracy tuples, we have factorisations

$$\Phi^{\chi}_{D_1 D_2} = \Phi^{\chi}_{D_1} \circ \Phi^{\chi_1}_{D_2} = \Phi^{\chi}_{D_2} \circ \Phi^{\chi_2}_{D_1}$$

(where the definitions of χ_1 and χ_2 are clear from the context) from which we get

$$\mathrm{Im}\left(\Phi^{\chi}_{D_1+D_2}\right) \subset \mathrm{Im}\left(\Phi^{\chi}_{D_1}\right) \cap \mathrm{Im}\left(\Phi^{\chi}_{D_2}\right).$$

The other inclusion holds if D_1 and D_2 are coprime, that is to say if (D_1, D_2) belongs to the mixed restricted symmetric product

$$\mathrm{Sym}^{\mu_1,\mu_2}_{/k}\left(\mathbf{P}^1_k\right)_* = \mathrm{Sym}^{\mu_1,\mu_2}_{/k}\left(\mathbf{P}^1_k, \mathbf{P}^1_k\right)_*$$

as defined in [**Bil23**, §3.3.2]. It is important to stress that even if D_1 and D_2 are both degeneracy tuples, their sum $D_1 + D_2$ has no reason to be a degeneracy tuple (Definition 5.1.18), it is only a coarse one (Definition 5.1.13). For the purposes of notation we will adopt the following definition.

Definition 5.1.27. Keeping the previous notations, let D_1 and D_2 be (representatives of) two coarse generacy tuples.

Then $\mathrm{lcm}(D_1, D_2)$ is the tuples of zero-cycles (respectively the equivariant morphism) given by

$$\mathrm{lcm}(D_1, D_2) = \mathrm{lcm}\left(\prod_{J \ni \alpha} D_{1,J}, \prod_{J \ni \alpha} D_{2,J}\right)_{\alpha \in \Sigma(1)}.$$

Going back to our remark, in general,

$$\mathrm{Im}(\Phi^{\chi}_{D_1}) \cap \mathrm{Im}(\Phi^{\chi}_{D_2}) = \mathrm{Im}\left(\Phi^{\chi}_{\mathrm{lcm}(D_1,D_2)}\right). \tag{5.1.5.39}$$

Now, note that $\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma\right)^*$ is precisely the complement in $\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right)^*$ of $\bigcup_{\mu' \neq \mathbf{0}} \mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)$, while the fibre of

$$\left(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma) \times \mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)_{\widetilde{W}}$$

above $D \in \mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*$ is

$$\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \overline{D}^{-1}\widetilde{W}\right)^* \setminus \bigcup_{\mu' \neq \mathbf{0}} \mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)$$

(up to a base change to $\kappa(D)$). Indeed, the preimage of $\mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \widetilde{W}\right)^*$ by Φ_D is the subspace

$$\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \overline{D}^{-1}\widetilde{W}\right)^*$$

of χ'-equivariant morphisms whose reduction modulo $\mathscr{S}$ lies in $\overline{D}^{-1}\widetilde{W}$. Moreover, when $\chi' \geqslant \ell(\mathscr{S})$, this preimage is isomorphic to $\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \widetilde{W}\right)^*$, the isomorphism being explicitly given in Lemma 5.1.12: it sends a element of Euclidian decomposition modulo ϖ

$$\left(q_\alpha, \overline{D_\alpha}^{-1} w_\alpha\right)_{\alpha \in \Sigma(1)}$$

with $(w_\alpha) \in \widetilde{W}$, to

$$(q_\alpha, w_\alpha)_{\alpha\in\Sigma(1)},$$

but *there is no reason this isomorphism behaves well with respect to degeneracies.*

We are going to compare the classes of

$$\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \overline{D}^{-1}\widetilde{W}\right)^{(*)} \cap \mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)$$

and

$$\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \widetilde{W}\right)^{(*)} \cap \mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)$$

for every partition μ'. Note again that these sets are empty for partitions μ' of sufficiently large tuples.

Remark 5.1.28. Looking at Euclidian decompositions when $\chi'' \geqslant \ell(\mathscr{S})$, we remark that the compositions

$$\begin{aligned}\mathrm{Hom}^{\chi''}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) \ni (q_\alpha, \overline{r_\alpha}) &\xmapsto{\Phi^{\chi'}_{D'}} \left(q_\alpha D'_\alpha + u'_\alpha, \overline{r_\alpha D'_\alpha}\right) \\ &\underset{1:1}{\xmapsto{\text{Lemma 5.1.12}}} \left(q_\alpha D'_\alpha + u'_\alpha, \overline{r_\alpha D_\alpha D'_\alpha}\right)\end{aligned}$$

where u'_α is the quotient of $\overline{r_\alpha}D'_\alpha$ by ϖ, and

$$(q_\alpha, \overline{r_\alpha}) \underset{1:1}{\xmapsto{\text{Lemma 5.1.12}}} \left(q_\alpha, \overline{r_\alpha D_\alpha}\right) \xmapsto{\Phi^{\chi'}_{D'}} \left(q_\alpha D'_\alpha + u''_\alpha, \overline{r_\alpha D_\alpha D'_\alpha}\right)$$

where u''_α is the quotient of $\overline{r_\alpha D_\alpha}D'_\alpha$ by ϖ, differ the one from the other by $u'_\alpha - u''_\alpha$.

This remark provides the following lemma.

Lemma 5.1.29. *For any $\chi'' \in \mathcal{X}_*(\mathbf{G}^{\Sigma(1)}_m) \simeq \mathbf{Z}^{\Sigma(1)}$ such that $\chi'' \geqslant \ell(\mathscr{S})$ and any coarse degeneracy tuple D' avoiding $\mathscr{S}$, or more generally any $(\chi' - \chi'')$-equivariant morphism avoiding $\mathscr{S}$, there exists a commutative diagram*

$$\begin{array}{ccc}\mathrm{Hom}^{\chi''}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) & \xrightarrow{\Phi^{\chi'}_{D'}} & \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) \\ {\scriptstyle\tau_{\overline{D}}}\big\downarrow & & \big\downarrow{\scriptstyle\Psi^{D}_{D'}} \\ \mathrm{Hom}^{\chi''}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) & \xrightarrow{\Phi^{\chi'}_{D'}} & \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right)\end{array}$$

in which the vertical arrows are isomorphisms and induce

$$\begin{array}{ccc}\mathrm{Hom}^{\chi''}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \overline{DD'}^{-1}\widetilde{W}\right)^{(*)} & \xrightarrow{\Phi^{\chi'}_{D'}} & \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \overline{D}^{-1}\widetilde{W}\right)^{(*)} \\ {\scriptstyle\tau_{\overline{D}}}\big\downarrow & & \big\downarrow{\scriptstyle\Psi^{D}_{D'}} \\ \mathrm{Hom}^{\chi''}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \overline{D'}^{-1}\widetilde{W}\right)^{(*)} & \xrightarrow{\Phi^{\chi'}_{D'}} & \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \widetilde{W}\right)^{(*)}.\end{array}$$

Proof. Using the previous remark, the first vertical arrow is the isomorphism induced by the linear transformation τ_g of Lemma 5.1.12 for $g = \overline{D} \in \mathbf{G}^{\Sigma(1)}_m(\mathscr{S})$

$$\begin{array}{rcl}\tau_{\overline{D}} : \mathrm{Hom}^{\chi''}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) & \longrightarrow & \mathrm{Hom}^{\chi''}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right) \\ (q_\alpha, \overline{r_\alpha}) & \longmapsto & \left(q_\alpha, \overline{r_\alpha D_\alpha}\right)\end{array}$$

while the seconds arrow is induced by the linear transformation

$$\begin{array}{rcl}\Psi^D_{D'}:\operatorname{Hom}^{\chi'}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right) & \longrightarrow & \operatorname{Hom}^{\chi'}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right)\\ (q_\alpha,\overline{r_\alpha}) & \longmapsto & \left(q_\alpha-u'_\alpha+u''_\alpha,\overline{r_\alpha D_\alpha}\right)\end{array}$$

where u'_α is the quotient of

$$\left(\overline{r_\alpha}\cdot\overline{D'_\alpha}^{-1}\right)\cdot D'_\alpha$$

by ϖ while u''_α is the one of

$$\left(\overline{r_\alpha}\cdot\overline{D_\alpha}\cdot\overline{D'_\alpha}^{-1}\right)\cdot D'_\alpha.$$

Since both D and D' avoid $\mathscr{S}$, $\overline{r'_\alpha D_\alpha}=0$ if and only if $\overline{r'_\alpha}=0$ if and only if $\overline{r'_\alpha}\cdot\overline{D'_\alpha}^{-1}=0$. Assume that the image of $(q'_\alpha,\overline{r'_\alpha})$ is zero. In that case, $\overline{r'_\alpha}=0$ and both u_α and u'_α are zero by the previous equivalence, and so is q'_α. This shows that $\Phi^D_{D'}$ is a linear isomorphism.

By construction, as an isomorphism of schemes, $\Phi^D_{D'}$ sends $\operatorname{Hom}^{\chi'}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\overline{D}^{-1}\widetilde{W}\right)^*$ to $\operatorname{Hom}^{\chi'}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\widetilde{W}\right)^*$ while leaving the constructible subset $\operatorname{Im}(\Phi^{\chi'}_{D'})$ stable, by Remark 5.1.28. □

Now we make D' vary. From now on, we use the compact notations

$$\begin{aligned}H^\chi&=\operatorname{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right)\\ H^\chi_{|\widetilde{W}}&=\operatorname{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\widetilde{W}\right).\end{aligned}$$

The morphism

$$\left(\mathrm{pr}_1,\Phi^{\chi'}_{\mu'}\right):\operatorname{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\times\mathrm{H}^{\chi''}\longrightarrow\operatorname{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\times\mathrm{H}^{\chi'}$$

will be denoted $\widetilde{\Phi^{\chi'}_{\mu'}}$. It is convenient to assume μ' to be either a partition of a certain tuple $\underline{\delta'}$ of degrees of degeneracy tuple, or a partition of $\chi'-\chi''$. In doing so, our argument will be compatible with taking the lowest common multiple of a finite set of degeneracy tuples, in the sense of Remark 5.1.26. Indeed, the "detwisting" morphisms will only depend on the α-coordinates, that is to say the images of degeneracy tuples by $(\prod_{J\ni\alpha}\cdot)_{\alpha\in\Sigma(1)}$.

Let

$$\left(\operatorname{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\times\operatorname{Hom}^{\chi''}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right)\right)_{\overline{D}^{-1}\widetilde{W}}$$

be the schematic fibre of $\mathrm{res}_{\mathscr{S}}\circ\Phi^{\chi'}_{\mu'}$ above $\overline{D}^{-1}\widetilde{W}$. Its points are given by tuples of the form

$$\left(D',\left(q_\alpha,w_\alpha\overline{D_\alpha D'_\alpha}^{-1}\right)_{\alpha\in\Sigma(1)}\right)$$

with $w\in\widetilde{W}$. Such tuples are sent by the isomorphism $\widetilde{\tau_D}:(D',f)\mapsto(D',\tau_{\overline{D}}(f))$ to

$$\left(D',\tau_{\overline{D}}\left(\left(q_\alpha,w_\alpha\overline{D_\alpha D'_\alpha}^{-1}\right)_{\alpha\in\Sigma(1)}\right)\right)=\left(D',\left(q_\alpha,w_\alpha\overline{D'_\alpha}^{-1}\right)_{\alpha\in\Sigma(1)}\right)$$

lying by definition in the schematic fibre of $\mathrm{res}_{\mathscr{S}}\circ\Phi^{\chi'}_{\mu'}$ above $\widetilde{W}$

$$\left(\operatorname{Sym}^{\mu'}_{\mathbf{P}^1_k}\left((\mathbf{P}^1_k)_{B_\Sigma}\right)_*\times\operatorname{Hom}^{\chi''}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\right)^*\right)_{\widetilde{W}}.$$

Let $\Psi^D_{\mu'}$ be the isomorphism sending

$$\left(D',\left(q_\alpha,w_\alpha\overline{D_\alpha}^{-1}\right)_{\alpha\in\Sigma(1)}\right)\in\operatorname{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\times\operatorname{Hom}^{\chi'}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\overline{D}^{-1}\widetilde{W}\right)^*$$

to

$$\left(D', \Psi^D_{D'}\left(\left(q_\alpha, w_\alpha \overline{D'_\alpha}^{-1}\right)_{\alpha\in\Sigma(1)}\right)\right) \in \mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \widetilde{W}\right)^*$$

where

$$\Psi^D_{D'}\left(q_\alpha, w_\alpha \overline{D'_\alpha}^{-1}\right) = (q_\alpha - u_\alpha + u'_\alpha, w_\alpha)$$

as in Lemma 5.1.29, while leaving $\mathrm{Im}\left(\widetilde{\Phi^{\chi'}_{\mu'}}\right)$ stable.

We get the following lemma.

Lemma 5.1.30. *For any $\chi'' \in \mathcal{X}_*(\mathbf{G}_m^{\Sigma(1)}) \simeq \mathbf{Z}^{\Sigma(1)}$ such that $\chi'' \geqslant \ell(\mathscr{S})$, the diagram*

$$\begin{array}{ccc}
\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi''} & \xrightarrow{\widetilde{\Phi^{\chi'}_{\mu'}}} & \mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi'} \\
\widetilde{\tau_D}\big\downarrow & & \big\downarrow \Psi^D_{\mu'} \\
\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi''} & \xrightarrow{\widetilde{\Phi^{\chi'}_{\mu'}}} & \mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi'}
\end{array}$$

above $\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$, in which vertical arrows are isomorphisms leaving $\mathrm{Im}\left(\widetilde{\Phi^{\chi'}_{\mu'}}\right)$ stable, is commutative and induces

$$\begin{array}{ccc}
\left(\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi''}\right)_{\overline{D}^{-1}\widetilde{W}} & \xrightarrow{\widetilde{\Phi^{\chi'}_{\mu'}}} & \mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi'}_{\overline{D}^{-1}\widetilde{W}} \\
\widetilde{\tau_D}\big\downarrow & & \big\downarrow \Psi^D_{\mu'} \\
\left(\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi''}\right)_{|\widetilde{W}} & \xrightarrow{\widetilde{\Phi^{\chi'}_{\mu'}}} & \mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi'}_{\widetilde{W}}.
\end{array} \tag{5.1.5.40}$$

5.1.5.5. *A motivic inclusion-exclusion formula.* Let us think of the image of

$$\widetilde{\Phi^{\chi'}_{\mu'}} : \left\{ \begin{array}{rcl} \mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi''} & \longrightarrow & \mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_* \times \mathrm{H}^{\chi'} \\ (D', f) & \longmapsto & \left(D', \Phi^{\chi'}_{D'}(f)\right) \end{array}\right.$$

as

$$\coprod_{D' \in \mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*} \mathrm{Im}(\Phi_{D'})$$

while the image of $\Phi^{\chi'}_{\mu'}$ is basically

$$\bigcup_{D' \in \mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*} \mathrm{Im}(\Phi_{D'}) \subset \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k\right).$$

Loosely speaking, the difference between these two sets is given by the intersections

$$\mathrm{Im}\left(\Phi^{\chi'}_{D'_1}\right) \cap \mathrm{Im}\left(\Phi^{\chi'}_{D'_2}\right)$$

for D_1 and D_2 distincts in $\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$. By Remark 5.1.26 we know that

$$\mathrm{Im}\left(\Phi^{\chi'}_{D'_1}\right) \cap \mathrm{Im}\left(\Phi^{\chi'}_{D'_2}\right) = \mathrm{Im}\left(\Phi^{\chi'}_{\mathrm{lcm}(D'_1, D'_2)}\right).$$

Moreover, pairs of distinct coarse degeneracy tuples in $\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$ are parametrised by the restricted symmetric product

$$\mathrm{Sym}^2\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)_*$$

which admits a decomposition into a finite disjoint union of locally closed subsets

$$\bigsqcup_{\lambda}\mathrm{Sym}^2\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)^{\lambda}_*$$

with respect to the value of the partition λ, given by

$$\{D_1, D_2\} \in \mathrm{Sym}^2\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)^{\lambda}_* \iff \mathrm{lcm}(D_1, D_2) \in \mathrm{Sym}^{\lambda}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*.$$

In general one will have to keep track of all labels in order to apply the motivic inclusion-exclusion [**BH21**, Theorem 7.2.1]. For any integer $i \in \mathbf{N}^*$ we will denote by

$$\mathrm{Sym}^i\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)_*$$

the union of the

$$\mathrm{Sym}^{\varpi}\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)_*$$

on partitions $\varpi = (m_j)_{j\in\mathbf{N}^*}$ such that $\|\varpi\| = |\{j \mid m_j > 0\} = i$ and ϖ has no hole, that is to say $m_j = 0$ whenever $j > \|\varpi\|$.

In order to avoid any confusion, we recall that the subscript $*$ means "pairwise distinct".

Definition 5.1.31. We define

$$\mathrm{Sym}^i\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)^{\lambda}_*$$

for any $i \in \mathbf{N}^*$ and partition λ in the same fashion:

$$\{D_1, \dots, D_i\} \in \mathrm{Sym}^i\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)^{\lambda}_* \iff \mathrm{lcm}(D_1, \dots, D_i) \in \mathrm{Sym}^{\lambda}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*.$$

Note that fixing $i \in \mathbf{N}^*$ and μ' there is only a finite number of partitions λ for which this space is non-empty.

For any $\chi' \in \mathbf{N}^{\Sigma(1)}$ and partition λ of a certain $\underline{\delta}^{\mathrm{lcm}} \in \mathbf{N}^{B_\Sigma}$, let

$$\chi^{\mathrm{lcm}} = \chi' - \sum_{J\in B_\Sigma} \delta^{\mathrm{lcm}}_J \mathbf{1}_J$$

and for any $i \in \mathbf{N}^*$ let

$$\widetilde{\mathcal{D}_{\Phi^{\chi'}_{\mu'^i}}} = \bigsqcup_{\substack{\lambda \\ \chi^{\mathrm{lcm}}\in\mathbf{N}^{\Sigma(1)}}} \mathrm{Sym}^i\left(\mathrm{Sym}^{\mu'}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)^{\lambda}_* \times H^{\chi^{\mathrm{lcm}}}$$

be the domain of definition of the morphisms

$$\widetilde{\Phi^{\chi'}_{\mu^i}} : \left\{ \begin{array}{rcl} \widetilde{\mathcal{D}_{\Phi^{\chi'}_{\mu'^i}}} & \longrightarrow & \mathrm{Sym}^i\left(\mathrm{Sym}^{\mu'}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*\right)_* \times \mathrm{H}^{\chi'} \\ (\{D_1, \dots, D_i\}, f) & \longmapsto & \left(\{D_1, \dots, D_i\}, \Phi^{\chi}_{\mathrm{lcm}(D_1,\dots D_i)}(f)\right) \end{array} \right.$$

and

$$\Phi^{\chi'}_{\mu^i} = \mathrm{pr}_2 \circ \widetilde{\Phi^{\chi'}_{\mu^i}}.$$

Taking the union of the images of the $\Phi^{\chi'}_{\mathrm{lcm}\left(D'_1,D'_2\right)}$'s gives

$$\bigcup_{\substack{\{D'_1,D'_2\}\\ \text{distinct}}} \mathrm{Im}\left(\Phi^{\chi'}_{\mathrm{lcm}\left(D'_1,D'_2\right)}\right) = \mathrm{Im}\left(\Phi^{\chi'}_{\mu'^2}\right)$$

while the image of $\widetilde{\Phi^{\chi'}_{\mu'^2}}$ is an incarnation of the disjoint union

$$\coprod_{\substack{\{D'_1,D'_2\}\\ \text{distinct}}} \mathrm{Im}\left(\Phi^{\chi'}_{\mathrm{lcm}(D'_1,D'_2)}\right).$$

One can proceed similarly for the images respectively of $\Phi^{\chi'}_{\mu'^i}$ and $\widetilde{\Phi^{\chi'}_{\mu'^i}}$ for $i \geqslant 3$, everything being empty as soon as $i > \chi'_\alpha$ for some α. In the end, we will be able to give a meaning to an inclusion-exclusion formula of the form

$$\begin{aligned}\bigcup_{D'\in \mathrm{S}^{\mu'}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*} \mathrm{Im}(\Phi_{D'}) = &\coprod_{D'\in \mathrm{S}^{\mu'}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*} \mathrm{Im}(\Phi_{D'}) - \coprod_{\substack{\{D'_1,D'_2\}\\ \text{distinct}}} \mathrm{Im}\left(\Phi^{\chi'}_{\mathrm{lcm}(D'_1,D'_2)}\right)\\ &+ \coprod_{\substack{\{D'_1,D'_2,D'_3\}\\ \text{pairwise distinct}}} \mathrm{Im}\left(\Phi^{\chi'}_{\mathrm{lcm}(D'_1,D'_2,D'_3)}\right) - \ldots\\ &\ldots + (-1)^i \coprod_{\substack{\{D'_1,\ldots,D'_i\}\\ \text{pairwise distinct}}} \mathrm{Im}\left(\Phi^{\chi'}_{\mathrm{lcm}(D'_1,\ldots,D'_i)}\right).\end{aligned}$$

Definition 5.1.32. For any $i \in \mathbf{N}^*$, any $\chi' \in \mathbf{N}^{\Sigma(1)}$ and any partitions μ, μ', let $\widehat{\Phi^{\chi'}_{\mu,\mu'^i}}$ and $\Phi^{\chi'}_{\mu,\mu'^i}$ be the morphisms given by

$$\widehat{\Phi^{\chi'}_{\mu,\mu'^i}} : \begin{cases} \mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_* \times \widetilde{\mathcal{D}_{\Phi^{\chi'}_{\mu'^i}}} & \longrightarrow \mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_* \times \mathrm{S}^i\left(\mathrm{S}^{\mu'}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\right)_* \times H^{\chi'}\\ (D,\{D'_1,\ldots D'_i\},f) & \longmapsto \left(D,\{D'_1,\ldots D'_i\},\Phi^{\chi'}_{\mu'^i}(\{D'_1,\ldots D'_i\},f)\right).\end{cases}$$

and

$$\Phi^{\chi'}_{\mu,\mu'^i} : \begin{cases} \mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_* \times \widetilde{\mathcal{D}_{\Phi^{\chi'}_{\mu'^i}}} & \longrightarrow \mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_* \times H^{\chi'}\\ (D,\{D'_1,\ldots D'_i\},f) & \longmapsto \left(D,\Phi^{\chi'}_{\mu'^i}(\{D'_1,\ldots D'_i\},f)\right).\end{cases}$$

Proposition 5.1.33 (Motivic inclusion-exclusion)**.** *In $K_0\mathbf{Var}_{\mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*}$ we have for any $\chi' \in \mathbf{N}^{\Sigma(1)}$*

$$\left[\mathrm{Im}\left(\Phi^{\chi'}_{\mu,\mu'}\right)^{(*)}\right] = \sum_{i\in\mathbf{N}^*} (-1)^{i-1}\left[\mathrm{Im}\left(\widehat{\Phi^{\chi'}_{\mu,\mu'^i}}\right)^{(*)}\right].$$

Proof. By [**BH21**, Lemma 2.5.5], the map

$$K_0\mathbf{Var}_{\mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*} \to \prod_{D\in \mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*} K_0\mathbf{Var}_{\kappa(D)}$$

is injective, so it is enough to show that

$$\left[\mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)^{(*)}\right] = \sum_{i\in\mathbf{N}^*} (-1)^{i-1}\left[\mathrm{Im}\left(\widetilde{\Phi^{\chi'}_{\mu'^i}}\right)^{(*)}\right].$$

This is [**BH21**, Theorem 7.2.1] applied to $\mathrm{pr}_2 : \mathrm{Im}\left(\widetilde{\Phi^{\chi'}_{\mu'}}\right)^{(*)} \to \mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)^{(*)}$. $\square$

5.1.5.6. *Twisted inclusion-exclusion.* We recall the concise notation

$$S^{\mu}_{\widehat{|\mathscr{S}|}\,*}$$

for $\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*$.

For any $i \in \mathbf{N}^*$, $\chi', \chi'' \in \mathcal{X}_*(\mathbf{G}_m^{\Sigma(1)})$ such that $\chi' \geqslant \chi'' \geqslant \ell(\mathscr{S})$, and partitions μ, μ' together with a partition λ of a certain $\underline{\delta}^{\mathrm{lcm}} \in \mathbf{N}^{B_\Sigma}$ such that $\chi' - \chi'' = \sum_{J \in B_\Sigma} \delta^{\mathrm{lcm}}_J \mathbf{1}_J$, let $\widetilde{\tau}^i$ and $\Psi^{\mu}_{\mu'^i}$ be the isomorphisms above $\mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}\left((\mathbf{P}^1_k)^{\widehat{|\mathscr{S}|}}_{B_\Sigma}\right)_*$ given by

$$\widetilde{\tau}^i : \left\{ \begin{array}{rcl} S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi''} & \longrightarrow & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi''} \\ (D, \{D'_1, \dots D'_i\}, f) & \longmapsto & (D, \{D'_1, \dots D'_i\}, \tau_{\overline{D}}(f)) \end{array} \right.$$

and

$$\Psi^{\mu}_{\mu'^i} : \left\{ \begin{array}{rcl} S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi'} & \longrightarrow & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi'} \\ (D, \{D'_1, \dots D'_i\}, f) & \longmapsto & \left(D, \{D'_1, \dots D'_i\}, \Psi^{D}_{\mathrm{lcm}(D'_1, \dots, D'_i)}(f)\right). \end{array} \right.$$

Once these isomorphisms are given, the following lemma is a straightforward extension of Lemma 5.1.30.

Lemma 5.1.34. *For all positive integer i and $\chi'' \in \mathcal{X}_*(\mathbf{G}_m^{\Sigma(1)}) \simeq \mathbf{Z}^{\Sigma(1)}$ such that $\chi'' \geqslant \ell(\mathscr{S})$, up to a convenient permutation of factors, there is a commutative diagram of morphisms above $\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*$*

$$\begin{tikzcd} S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi''} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'^i}}"] \arrow[d, "\widetilde{\tau}^i"] & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \arrow[d, "\Psi^{\mu}_{\mu'^i}"] \\ S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi''} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'^i}}"] & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \end{tikzcd}$$

in which the vertical arrows are isomorphisms, inducing vertical isomorphisms

$$\begin{tikzcd} \left(S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi''}\right)_{|\widetilde{W}} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'^i}}"] \arrow[d, "\widetilde{\tau}^i"] & \left(S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \arrow[d, "\Psi^{\mu}_{\mu'^i}"] \\ S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \left(\mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \times H^{\chi''}\right)_{|\widetilde{W}} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'^i}}"] & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'}_{|\widetilde{W}} \times \mathrm{Sym}^i\left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*}\right)^{\lambda}_* \end{tikzcd} \tag{5.1.5.41}$$

For $i=1$, these two diagrams are

$$\begin{tikzcd} S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi''} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'}}"] \arrow[d, "\widetilde{\tau}"] & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'} \arrow[d, "\Psi^{\mu}_{\mu'}"] \\ S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi''} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'}}"] & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'} \end{tikzcd}$$

and

$$\begin{tikzcd} \left(S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi''}\right)_{|\widetilde{W}} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'}}"] \arrow[d, "\widetilde{\tau}"] & \left(S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'}\right)_{\widetilde{W}} \times S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \arrow[d, "\Psi^{\mu}_{\mu'}"] \\ S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times \left(S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi''}\right)_{|\widetilde{W}} \arrow[r, "\widehat{\Phi^{\chi'}_{\mu,\mu'}}"] & S^{\mu}_{\widehat{|\mathscr{S}|}\,*} \times S^{\mu'}_{\widehat{|\mathscr{S}|}\,*} \times \mathrm{H}^{\chi'}_{\widetilde{W}} \end{tikzcd}$$

For all non-zero partitions $\mu'_1, ..., \mu'_j$ and all non-negative integer i, let $\Phi^{\chi'}_{\mu,(\mu'_1,...,\mu'_j)^i}$ and $\widehat{\Phi^{\chi'}_{\mu,(\mu'_1,...,\mu'_j)^i}}$ be the compositions of respectively $\sqcup_{\mu'}\Phi^{\chi'}_{\mu,\,\mu'^i}$ and $\sqcup_{\mu'}\widehat{\Phi^{\chi'}_{\mu,\,\mu'^i}}$ together with the lcm maps

$$\mathrm{Sym}^{\mu_1}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times ... \times \mathrm{Sym}^{\mu_j}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \longrightarrow \sqcup_{\mu'}\mathrm{Sym}^{\mu'}_{\mathbf{P}^1_k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*$$

from Remark 5.1.26 and Definition 5.1.27 (here the μ''s are partitions of a certain cocharacter given by the tuple of degrees of the lcm).

Proposition 5.1.35. *In $K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*}$ we have*

$$\begin{aligned}
&\left[\mathrm{Im}\left(\Phi^{\chi'}_{\mu,(\mu'_1,...,\mu'_j)}\right)^{(*)} \cap \left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}}\right] \\
&= \left[\mathrm{pr}_{12}\left(\mathrm{Im}\left(\widehat{\Phi^{\chi'}_{\mu,(\mu'_1,...,\mu'_j)}}\right)^{(*)} \cap \left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}} \times \prod_{\ell=1}^{j}\mathrm{Sym}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)\right] \\
&= \sum_{i\in\mathbf{N}^*}(-1)^{i-1}\left[\mathrm{Im}\left(\widehat{\Phi^{\chi'}_{\mu,(\mu'_1,...,\mu'_j)^i}}\right)^{(*)} \cap \left(\left(\mathrm{S}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}} \times \mathrm{S}^{i}\left(\prod_{\ell=1}^{j}\mathrm{S}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)_*\right)\right]
\end{aligned}$$

Proof. The proof is a variant of the one of Proposition 5.1.33, obtained by pull-back to $\left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}} \times \prod_{\ell=1}^{j}\mathrm{Sym}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*$. ☐

Proposition 5.1.36. *We have*

$$\begin{aligned}
&\left[\mathrm{Hom}^{\chi}\left(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \,\middle|\, \widetilde{W}\right)^*\right] \\
&= \sum_{\chi',\,\kappa}\left[\mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma \,\middle|\, \widetilde{W}\right)^*\right] \times \left[\mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k}\left((\lambda(\mathfrak{J})\left[\mathbf{P}^1_k \setminus |\mathscr{S}|\right])_{\mathfrak{J}\in\sqcup_{m\in\mathbf{N}^*}B^m_\Sigma}\right)_*\right] + E^{\chi}
\end{aligned}$$

in $K_0\mathbf{Var}_k$, with the error term E^{χ} of bounded dimension

$$\dim_k(E^{\chi}) \leqslant -\min_{\alpha\in\Sigma(1)}(\chi_\alpha) + |\chi| + (1-\Sigma(1))\,(\ell(\mathscr{S})-1) + \dim\left(\widetilde{W}\right).$$

Proof. By Proposition 5.1.25 we already know that above $\mathrm{Hom}^{\chi}(\mathbf{A}^2 \setminus \{0\}, \mathbf{A}^{\Sigma(1)})$,

$$\begin{aligned}&\left[\mathrm{Hom}^{\chi}(\mathbf{A}^2_k \setminus \{0\}, \mathbf{A}^{\Sigma(1)}_k \mid \widetilde{W})^*\right]\\ &= \sum_{\chi',\kappa} \left[\left(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^* \boxtimes \mathrm{Sym}^{\kappa}_{\mathbf{P}^1_k/k}\left(\left(\lambda_{B_\Sigma}(\mathfrak{J})\left[\mathbf{P}^1_k \setminus |\mathscr{S}|\right]\right)_{\mathfrak{J}\in\sqcup_{m\in\mathbf{N}^*}B^m_\Sigma}\right)_*\right)_{\widetilde{W}}\right]\end{aligned}$$

for all $\chi \in \mathbf{N}^{\Sigma(1)}$. Using the decomposition made in Remark 5.1.23, we see that it is sufficient to approximate the class of

$$\left(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^* \times \mathrm{Sym}^{\kappa}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)_{\widetilde{W}}. \tag{5.1.5.42}$$

Since we are going to perform a motivic sum over points of $\mathrm{Hom}^{\chi}(\mathbf{A}^2 \setminus \{0\}, \mathbf{A}^{\Sigma(1)})^*$, we can view (5.1.5.42) as a variety above $\mathrm{Sym}^{\kappa}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$ and then take the motivic sum. Again by [**BH21**, Lemma 2.5.5] it is enough to argue fibre by fibre like in the proof of Proposition 5.1.33. Take a $D \in \mathrm{Sym}^{\kappa}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$, viewed in $\mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k \setminus |\mathscr{S}|)_*$ for a certain $\mu \in \mathbf{N}^{(\mathbf{N}^{B_\Sigma})}$ so that it makes sense to consider the morphism Φ^{χ}_D defined page 145. The class of $(\Phi^{\chi}_D(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^*))_{\widetilde{W}}$ decomposes as

$$\begin{aligned}&\left[\left(\Phi^{\chi}_D(\mathrm{Hom}^{\chi'}(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma)^*)\right)_{\widetilde{W}}\right]\\ &= \left[\Phi^{\chi}_D\left((\mathrm{H}^{\chi'}_{D^{-1}\widetilde{W}})^* \setminus \bigcup_{\mu'\neq 0} \mathrm{H}^{\chi'}_{D^{-1}\widetilde{W}} \cap \mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)\right)\right]\\ &= \left[(\mathrm{H}^{\chi'}_{D^{-1}\widetilde{W}})^* \setminus \bigcup_{\mu'\neq 0}\left(\mathrm{H}^{\chi'}_{D^{-1}\widetilde{W}} \cap \mathrm{Im}\left(\Phi^{\chi'}_{\mu'}\right)\right)\right] \qquad \text{(by injectivity of the linear map } \Phi^{\chi}_D)\\ &= \sum_{\substack{j\in\mathbf{N}\\ \text{(finite sum)}}} (-1)^j \sum_{\substack{\mu'_1,\dots,\mu'_j\neq\mathbf{0}\\ \text{pairwise distinct}}} \left[(\mathrm{H}^{\chi'}_{D^{-1}\widetilde{W}})^* \cap \mathrm{Im}\left(\Phi^{\chi'}_{\mu'_1}\right) \cap \dots \cap \mathrm{Im}\left(\Phi^{\chi'}_{\mu'_j}\right)\right]\\ &= \sum_{\substack{j\in\mathbf{N}\\ \text{(finite sum)}}} (-1)^j \sum_{\substack{\mu'_1,\dots,\mu'_j\neq\mathbf{0}\\ \text{pairwise distinct}}} \left[(\mathrm{H}^{\chi'}_{D^{-1}\widetilde{W}})^* \cap \mathrm{Im}\left(\Phi^{\chi'}_{(\mu'_1,\dots,\mu'_j)}\right)\right]. \qquad \text{(by Remark 5.1.26)}\end{aligned}$$

This shows that

$$\begin{aligned}&\left[\left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{Hom}^{\chi'}(\mathbf{A}^2 \setminus \{0\}, \mathcal{T}_\Sigma)^*\right)_{|\widetilde{W}}\right]\\ &= \sum_{\substack{j\in\mathbf{N}\\ \text{(finite sum)}}} (-1)^j \sum_{\substack{\mu'_1,\dots,\mu'_j\neq\mathbf{0}\\ \text{pairwise distinct}}} \left[\mathrm{Im}\left(\Phi^{\chi'}_{\mu,(\mu'_1,\dots,\mu'_j)}\right)^* \cap \left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'^*}\right)_{|\widetilde{W}}\right]\end{aligned}$$

in $K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{/k}(\mathbf{P}^1_k\setminus|\mathscr{S}|)_*}$.

We fix a $j \in \mathbf{N}$ together with partitions $\mu'_1, \dots, \mu'_j$ (in case $j \neq 0$) appearing in this finite sum. As a consequence of Lemma 5.1.34, for all $i \in \mathbf{N}^*$, $\chi', \chi'' \in \mathcal{X}_*(\mathbf{G}^{\Sigma(1)}_m)$ such that $\chi' \geqslant \chi'' \geqslant \ell(\mathscr{S})$, and partition λ of a certain $\underline{\delta}^{\mathrm{lcm}} \in \mathbf{N}^{B_\Sigma}$ such that $\chi' - \chi'' =$

$\sum_{J\in B_\Sigma}\delta_J^{\mathrm{lcm}}\mathbf{1}_J$,

$$\begin{aligned}&\left[\mathrm{Im}\left(\widehat{\Phi^{\chi'}_{\mu,\,(\mu'_1,\dots,\mu'_j)^i}}\right)^{(*)}\cap\left(\left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}}\times \mathrm{S}^i\left(\prod_{\ell=1}^{j}\mathrm{S}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\right)^{\lambda}_*\right)\right]\\&=\left[\mathrm{Im}\left(\widehat{\Phi^{\chi'}_{\mu,\,(\mu'_1,\dots,\mu'_j)^i}}\right)^{(*)}\cap\left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\times \mathrm{H}^{\chi'}_{|\widetilde{W}}\times \mathrm{S}^i\left(\prod_{\ell=1}^{j}\mathrm{S}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\right)^{\lambda}_*\right)\right]\end{aligned}\tag{5.1.5.43}$$

and this second class is nothing else but

$$\left[\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\times\left(\mathrm{Im}\left(\widetilde{\Phi^{\chi'}_{(\mu'_1,\dots,\mu'_j)^i}}\right)^{(*)}\cap\left(\mathrm{H}^{\chi'}_{|\widetilde{W}}\times\mathrm{Sym}^i\left(\prod_{\ell=1}^{j}\mathrm{Sym}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*\right)^{\lambda}_*\right)\right)\right].$$

We can apply this only when the condition on χ'' is fulfilled. In general, we set

$$E^{\chi',\lambda}_{\mu,\mu'_1,\dots,\mu'_j}(i)\in K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}\left(\mathbf{P}^1_k\setminus|\mathscr{S}|\right)_*}$$

to be the difference between the two terms of (5.1.5.43). By (5.1.5.37) of Lemma 5.1.12, the relative dimension of $E^{\chi',\lambda}_{\mu,\mu'_1,\dots,\mu'_j}(i)$ is bounded by

$$\dim\left(\widetilde{W}\right)+\sum_{\alpha\in\Sigma(1)}\max(0,\chi''_\alpha-\ell(\mathscr{S})+1)$$

and we know that there is at least one term of this sum which is equal to zero (since by assumption $\chi''\not\geqslant\ell(\mathscr{S})$). Hence it is bounded by

$$-\min_{\alpha\in\Sigma(1)}(\chi'_\alpha)+\ell(\mathscr{S})-1+|\chi'|+\Sigma(1)\left(1-\ell(\mathscr{S})\right)+\dim\left(\widetilde{W}\right).\tag{5.1.5.44}$$

By Proposition 5.1.35,

$$\begin{aligned}
&\left[\mathrm{Im}\left(\Phi^{\chi'}_{\mu,(\mu'_1,\dots,\mu'_j)}\right)^* \cap \left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}}\right] \\
&= \sum_{\substack{i\in\mathbf{N}^*\\ \lambda \\ \text{(finite sum)}}} (-1)^{i-1}\left[\mathrm{Im}\left(\Phi^{\chi'}_{\widehat{\mu,(\mu'_1,\dots,\mu'_j)^i_\lambda}}\right)^*\right. \\
&\qquad \left.\cap\left(\left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}\right)_{|\widetilde{W}} \times \mathrm{Sym}^i\left(\prod_{\ell=1}^{j}\mathrm{Sym}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)^{\lambda}_*\right)\right] \\
&= \sum_{\substack{i\in\mathbf{N}^*\\ \lambda \\ \text{(finite sum)}}} (-1)^{i-1}\left(\left[\mathrm{Im}\left(\Phi^{\chi'}_{\widehat{\mu,\,(\mu'_1,\dots,\mu'_j)^i_\lambda}}\right)^*\right.\right. \\
&\qquad \left.\cap\left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}_{|\widetilde{W}} \times \mathrm{Sym}^i\left(\prod_{\ell=1}^{j}\mathrm{Sym}^{\mu'_\ell}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*\right)^{\lambda}_*\right)\right] \\
&\quad \left. + E^{\chi',\lambda}_{\mu,\mu'_1,\dots,\mu'_j}(i)\right) \\
&= \left[\mathrm{Im}\left(\Phi^{\chi'}_{\mu,(\mu'_1,\dots,\mu'_j)}\right)^* \cap \left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}_{|\widetilde{W}}\right)\right] + \sum_{\substack{i\in\mathbf{N}^*\\ \lambda \\ \text{(finite sum)}}} (-1)^{i-1} E^{\chi',\lambda}_{\mu,\mu'_1,\dots,\mu'_j}(i)
\end{aligned}$$

Concerning the first term, we can go back the other way by taking again fibrewise the sum over j and $\mu'_1, \dots, \mu'_j$ to get

$$\begin{aligned}
&\sum_{\substack{j\in\mathbf{N}\\ \text{(finite sum)}}} (-1)^j \sum_{\substack{\mu'_1,\dots,\mu'_j\neq\mathbf{0}\\ \text{pairwise distinct}}} \left[\mathrm{Im}\left(\Phi^{\chi'}_{\mu,(\mu'_1,\dots,\mu'_j)}\right)^* \cap \left(\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{H}^{\chi'}_{|\widetilde{W}}\right)\right] \\
&= \left[\mathrm{Sym}^{\mu}_{/k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_* \times \mathrm{Hom}^{\chi'}\left(\mathbf{A}^2_k \setminus \{0\}, \mathcal{T}_\Sigma \,\middle|\, \widetilde{W}\right)^*\right]
\end{aligned}$$

in $K_0\mathbf{Var}_{\mathrm{Sym}^{\mu}_{\mathbf{P}^1_k}\left(\mathbf{P}^1_k \setminus |\mathscr{S}|\right)_*}$. Concerning the error term, it is given by the finite sum

$$E^{\chi'}_{\mu} = \sum_{\substack{j\in\mathbf{N}\\ \text{(finite sum)}}} (-1)^j \sum_{\substack{\mu'_1,\dots,\mu'_j\neq\mathbf{0}\\ \text{pairwise distinct}}} \sum_{\substack{i\in\mathbf{N}^*\\ \lambda \\ \chi'-\|\lambda\|\geqslant\mathbf{0} \\ \chi'-\|\lambda\|\ngeqslant\ell(\mathscr{S})}} (-1)^{i-1} E^{\chi',\lambda}_{\mu,(\mu'_1,\dots,\mu'_j)}(i).$$

Let $\varphi(\underline{\delta}) \in \mathbf{N}^{\Sigma(1)} \subset \mathcal{X}_*(\mathbf{G}_m^{\Sigma(1)})$ be the cocharacter whose α-coordinate is

$$\varphi(\underline{\delta})_\alpha = \sum_{J\ni\alpha} \delta_J.$$

Then, by (5.1.5.44) $E_{\mu}^{\chi'}$ has dimension over k bounded by

$$\begin{aligned}
&-\min_{\alpha\in\Sigma(1)}(\chi'_\alpha)+\ell(\mathscr{S})-1+|\chi'|+\Sigma(1)(1-\ell(\mathscr{S}))+\dim\left(\widetilde{W}\right)+|\underline{\delta}|\\
=&-\min_{\alpha\in\Sigma(1)}(\chi_\alpha-\varphi(\underline{\delta})_\alpha)\\
&+|\chi|-|\varphi(\underline{\delta})|+|\underline{\delta}|\\
&+(\Sigma(1)-1)(1-\ell(\mathscr{S}))+\dim\left(\widetilde{W}\right)\\
\leqslant&-\min_{\alpha\in\Sigma(1)}(\chi_\alpha)+|\chi|+(\Sigma(1)-1)(1-\ell(\mathscr{S}))+\dim\left(\widetilde{W}\right)
\end{aligned}$$

where the first equality is given by

$$\chi=\chi'+\varphi(\underline{\delta})$$

and the last inequality comes from the expression

$$|\varphi(\underline{\delta})|=\sum_{J\in B_\Sigma}|J|\delta_J$$

together with the fact that $|J|\geqslant 2$ for all $J\in B_\Sigma$, hence

$$-\min_{\alpha\in\Sigma(1)}(\chi_\alpha-\varphi(\underline{\delta})_\alpha)-|\varphi(\underline{\delta})|\leqslant-\min_{\alpha\in\Sigma(1)}(\chi_\alpha)+\max_{\alpha\in\Sigma(1)}(\varphi(\underline{\delta})_\alpha)-|\varphi(\underline{\delta})|\leqslant-\min_{\alpha\in\Sigma(1)}(\chi_\alpha)$$

and the proposition is finally proved. □

Final computation. Starting from Proposition 5.1.36 together with Lemma 5.1.1 and thanks to the multiplicative property of motivic Euler products given by Proposition 2.4.12, we are able to invert the relation

$$\begin{aligned}
&\sum_{\chi\in\mathbf{N}^{\Sigma(1)}}\left[\operatorname{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}\mid\widetilde{W}\right)^*\right]\mathbf{t}^\chi\\
&=\left(\underbrace{\prod_{p\notin|\mathscr{S}|}\left(\sum_{m\in\mathbf{N}}\left(\sum_{\mathbf{n}\in B_\Sigma}-\mu_{B_\Sigma}(\mathbf{n})\mathbf{t}^{\mathbf{n}}\right)^m\right)}_{P_{B_\Sigma}(\mathbf{t})^{-1}}\right)\left(\sum_{\chi'\in\operatorname{Eff}(V)^\vee_{\mathbf{Z}}}\left[\operatorname{Hom}^\chi\left(\mathbf{A}^2_k\setminus\{0\},\mathcal{T}_\Sigma\mid\widetilde{W}\right)^*\right]\mathbf{t}^\chi\right)\\
&\quad+\underbrace{\sum_{\chi\in\mathbf{N}^{\Sigma(1)}}E^\chi\mathbf{t}^\chi}_{E_W(\mathbf{t})}
\end{aligned}$$

where for all $\chi\in\mathbf{N}^{\Sigma(1)}$

$$E^\chi$$

has virtual dimension

$$\dim_k(E^\chi)\leqslant-\min_{\alpha\in\Sigma(1)}(\chi_\alpha)+|\chi|+(1-\Sigma(1))\left(\ell(\mathscr{S})-1\right)+\dim\left(\widetilde{W}\right),\qquad(5.1.5.45)$$

to obtain

$$\begin{aligned}&\sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}}\left[\mathrm{Hom}^{\mathbf{d}}\left(\mathbf{A}^2_k\setminus\{0\},\mathcal{T}_\Sigma\mid\widetilde{W}\right)^*\right]\mathbf{t}^{\mathbf{d}}\\&=\left(\prod_{p\notin\mathscr{S}}P_{B_\Sigma}(\mathbf{t})\right)\times\left(\sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}}\left[\mathrm{Hom}^{\mathbf{d}}\left(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\widetilde{W}\right)^*\right]\mathbf{t}^{\mathbf{d}}\right)\\&\quad-\left(\prod_{p\notin\mathscr{S}}P_{B_\Sigma}(\mathbf{t})\right)\times E_W(\mathbf{t})\end{aligned}$$

in $K_0\mathbf{Var}_k[[\mathbf{t}]]$. We define $\mu^{|\mathscr{S}|}_\Sigma(\mathbf{e})$, $\mathbf{e}\in\mathbf{N}^{\Sigma(1)}$, to be the coefficients of the motivic Euler product

$$\prod_{p\notin|\mathscr{S}|}P_{B_\Sigma}(\mathbf{t}).$$

By definition of $\widetilde{W}$, together with Proposition 5.1.4 and the equivalent description of the functor $S\rightsquigarrow\mathrm{Hom}^\chi_S(\mathbf{A}^2_S\setminus\{0\},\mathcal{T}_{\Sigma,S})^*$ we gave, we have the relation

$$(\mathbf{L}_k-1)^r\left[\mathrm{Hom}^{\mathbf{d}}_k(\mathbf{P}^1_k,V_\Sigma\mid W)_U\right]=\left[\mathrm{Hom}^{\mathbf{d}}_k(\mathbf{A}^2\setminus\{0\},\mathcal{T}_\Sigma\mid\widetilde{W})^*\right]$$

as soon as $\mathbf{d}\in\mathrm{Eff}(V)^\vee_{\mathbf{Z}}$, and by Lemma 5.1.12

$$\left[\mathrm{Hom}^{\mathbf{d}}_k(\mathbf{A}^2\setminus\{0\},\mathbf{A}^{\Sigma(1)}\mid\widetilde{W})^*\right]=\left[\widetilde{W}\right](\mathbf{L}-1)^{|\Sigma(1)|}\prod_{\alpha\in\Sigma(1)}\left[\mathbf{P}_k^{d_\alpha-\ell(\mathscr{S})}\right]$$

whenever $\mathbf{d}\geqslant\ell(\mathscr{S})$. Thus we decompose the following series into two parts:

$$\sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}}\left[\mathrm{Hom}^{\mathbf{d}}(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\widetilde{W}\right]\mathbf{t}^{\mathbf{d}}=\left[\widetilde{W}\right](\mathbf{L}-1)^{|\Sigma(1)|}\prod_{\alpha\in\Sigma(1)}t_\alpha^{\ell(\mathscr{S})}Z^{\mathrm{Kapr}}_{\mathbf{P}^1_k}(t_\alpha)+H_W(\mathbf{t})$$

where

$$H_W(\mathbf{t})=\sum_{\substack{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}\\\mathbf{d}\ngeqslant\ell(\mathscr{S})}}\left[\mathrm{Hom}^{\mathbf{d}}(\mathbf{A}^2_k\setminus\{0\},\mathbf{A}^{\Sigma(1)}_k\mid\widetilde{W})^*\right]\mathbf{t}^{\mathbf{d}}.$$

Then, we use again the decomposition (5.1.4.35) given page 135

$$\prod_{\alpha\in\Sigma(1)}Z^{\mathrm{Kapr}}_{\mathbf{P}^1_k}(t_\alpha)=\sum_{A\subset\Sigma(1)}\frac{(-\mathbf{L})^{|\Sigma(1)|-|A|}}{(1-\mathbf{L})^{|\Sigma(1)|}}Z_A(\mathbf{t})$$

of this product of Kapranov zeta functions, where for any $A\subset\Sigma(1)$

$$Z_A(\mathbf{t})=\prod_{\alpha\in A}(1-t_\alpha)^{-1}\prod_{\alpha\notin A}(1-\mathbf{L}t_\alpha)^{-1}.$$

By identification, the coefficient of order $\mathbf{d}$ of

$$\mathbf{t}^{\ell(\mathscr{S})}Z_A(\mathbf{t})\times\prod_{p\notin|\mathscr{S}|}P_{B_\Sigma}(\mathbf{t})$$

is the sum

$$\mathfrak{s}^A_{\mathbf{d}}=\sum_{\mathbf{e}\leqslant\mathbf{d}}\mu^{|\mathscr{S}|}_\Sigma(\mathbf{e})\mathbf{L}^{\sum_{\alpha\notin A}d_\alpha-\ell(\mathscr{S})-e_\alpha}$$

whenever $\mathbf{d}\geqslant\ell(\mathscr{S})$, and zero otherwise.
If $A=\varnothing$, then after dividing by $\mathbf{L}^{-|\mathbf{d}|}$ it becomes

$$\mathfrak{s}^A_{\mathbf{d}}\mathbf{L}^{-|\mathbf{d}|}=\mathbf{L}^{-|\Sigma(1)|\ell(\mathscr{S})}\sum_{\mathbf{e}\leqslant\mathbf{d}}\mu^{|\mathscr{S}|}_\Sigma(\mathbf{e})\mathbf{L}^{-|\mathbf{e}|}$$

which is, up to the factor $\mathbf{L}^{-|\Sigma(1)|\ell(\mathscr{S})}$, the $\mathbf{d}$-th partial sum of $\prod_{p\notin|\mathscr{S}|} P_{B_\Sigma}(\mathbf{L}^{-1})$. The corresponding error term

$$\mathbf{L}^{-|\Sigma(1)|\ell(\mathscr{S})}\sum_{\mathbf{e}\nleqslant\mathbf{d}}\mu_\Sigma^{|\mathscr{S}|}(\mathbf{e})\mathbf{L}^{-|\mathbf{e}|}$$

has virtual dimension at most

$$-|\Sigma(1)|\ell(\mathscr{S})-\frac{1}{2}\min_{\alpha\in\Sigma(1)}(d_\alpha+1).$$

If $A\neq\varnothing$, then one gets instead

$$\mathbf{L}^{-(|\Sigma(1)|-|A|)\ell(\mathscr{S})}\sum_{\mathbf{e}\leqslant\mathbf{d}}\mu_\Sigma^{|\mathscr{S}|}(\mathbf{e})\mathbf{L}^{-|\mathbf{e}|}\mathbf{L}^{|\mathbf{d}_A-\mathbf{e}_A|}.$$

In that case, recalling that $\dim(\mu_\Sigma(\mathbf{e})\mathbf{L}_k^{-|\mathbf{e}|})\leqslant-\frac{1}{2}|\mathbf{e}|$ for all $\mathbf{e}\in\mathbf{N}^{\Sigma(1)}$, Lemma 2.4.21 page 69 gives

$$\dim(\sum_{\mathbf{e}\leqslant\mathbf{d}}\mu_\Sigma^{|\mathscr{S}|}(\mathbf{e})\mathbf{L}^{-|\mathbf{e}|}\mathbf{L}^{|\mathbf{d}_A-\mathbf{e}_A|})\leqslant-\frac{1}{4}\min_{\alpha in A}(d_\alpha).$$

Now we consider the terms coming from H_W and E_W. The coefficient of order $\mathbf{d}$ of

$$H_W(\mathbf{t})\times\prod_{p\notin|\mathscr{S}|}P_{B_\Sigma}(\mathbf{t})$$

is

$$\mathfrak{h}_{\mathbf{d}}=\sum_{\substack{\mathbf{e}\leqslant\mathbf{d}\\ \mathbf{d}\ngeqslant\ell(\mathscr{S})+\mathbf{e}}}\mu_\Sigma^{|\mathscr{S}|}(\mathbf{e})\left[\mathrm{Hom}^{\mathbf{d}-\mathbf{e}}(\mathbf{A}_k^2\setminus\{0\},\mathbf{A}_k^{\Sigma(1)}\mid\widetilde{W})^*\right].$$

Dividing by $\mathbf{L}^{|\mathbf{d}|}$, we get

$$\mathfrak{h}_{\mathbf{d}}\mathbf{L}^{-|\mathbf{d}|}=\sum_{\substack{\mathbf{e}\leqslant\mathbf{d}\\ \mathbf{d}\ngeqslant\ell(\mathscr{S})+\mathbf{e}}}\mu_\Sigma^{|\mathscr{S}|}(\mathbf{e})\mathbf{L}^{-|\mathbf{e}|}\left[\mathrm{Hom}^{\mathbf{d}-\mathbf{e}}(\mathbf{A}_k^2\setminus\{0\},\mathbf{A}_k^{\Sigma(1)}\mid\widetilde{W})^*\right]\mathbf{L}^{-|\mathbf{d}-\mathbf{e}|}.$$

As for the coefficients of $E_W(\mathbf{t})$, see (5.1.5.45), we have the dimensional upper bound coming from (5.1.5.37) of Lemma 5.1.12

$$\begin{aligned}
&\dim\left(\mathrm{Hom}^{\mathbf{d}'}(\mathbf{A}_k^2\setminus\{0\},\mathbf{A}_k^{\Sigma(1)}\mid\widetilde{W})^*\right)\\
&\leqslant\dim(\widetilde{W})+\sum_{\alpha\in\Sigma(1)}\min(0,d'_\alpha-\ell(\mathscr{S})+1)\\
&\leqslant\dim(\widetilde{W})+|\mathbf{d}'|+|\Sigma(1)|(1-\ell(\mathscr{S}))-\min_{\alpha,d'_\alpha\leqslant|\mathscr{S}|}(d'_\alpha)+\ell(\mathscr{S})-1
\end{aligned}\tag{5.1.5.46}$$

for all $\mathbf{d}'$. Thus the dimension of $\left[\mathrm{Hom}^{\mathbf{d}-\mathbf{e}}(\mathbf{A}_k^2\setminus\{0\},\mathbf{A}_k^{\Sigma(1)}\mid\widetilde{W})\right]\mathbf{L}^{-|\mathbf{d}-\mathbf{e}|}$ in the expression of $\mathfrak{h}_{\mathbf{d}}$ above is bounded. We can be more precise and argue as we did in the proof of Lemma 2.4.21. If $2\mathbf{e}\leqslant\mathbf{d}$ then $2(\mathbf{d}-\mathbf{e})\geqslant\mathbf{d}$ and

$$\dim\left(\mu_\Sigma^{|\mathscr{S}|}(\mathbf{e})\mathbf{L}^{-|\mathbf{e}|}\left[\mathrm{Hom}^{\mathbf{d}-\mathbf{e}}(\mathbf{A}_k^2\setminus\{0\},\mathbf{A}_k^{\Sigma(1)}\mid\widetilde{W})^*\right]\mathbf{L}^{-|\mathbf{d}-\mathbf{e}|}\right)$$

is at most

$$-\frac{1}{2}\min_{\alpha\in\Sigma(1)}(d_\alpha)+\dim(\widetilde{W})+(1-\ell(\mathscr{S}))(|\Sigma(1)|-1)$$

while if $2\mathbf{e}\nleqslant\mathbf{d}$ we use the coarse upper bound deduced from (5.1.5.46)

$$\dim\left(\left[\mathrm{Hom}^{\mathbf{d}-\mathbf{e}}(\mathbf{A}_k^2\setminus\{0\},\mathbf{A}_k^{\Sigma(1)}\mid\widetilde{W})^*\right]\mathbf{L}^{-|\mathbf{d}-\mathbf{e}|}\right)\leqslant+\dim(\widetilde{W})+(1-\ell(\mathscr{S}))(|\Sigma(1)|-1)$$

together with

$$\dim(\mu_{\Sigma}^{|\mathscr{S}|}(\mathbf{e})\mathbf{L}^{-|\mathbf{e}|}) \leqslant -\frac{1}{2}|\mathbf{e}| < -\frac{1}{4}\min_{\alpha\in\Sigma(1)}(d_\alpha).$$

Therefore, for any $\mathbf{d}$ we have

$$\dim(\mathfrak{h}_{\mathbf{d}}\mathbf{L}^{-|\mathbf{d}|}) \leqslant -\frac{1}{4}\min_{\alpha\in\Sigma(1)}(d_\alpha) + \dim(\widetilde{W}) + (1-\ell(\mathscr{S}))(|\Sigma(1)|-1)$$

and using the exact same argument we get the same bound for the error term $\mathfrak{e}_{\mathbf{d}}\mathbf{L}^{-|\mathbf{d}|}$coming from $E_W(\mathbf{t})$, where

$$\sum_{\mathbf{d}\in\mathbf{N}^{\Sigma(1)}} \mathfrak{e}_{\mathbf{d}}\mathbf{t}^{\mathbf{d}} = E_W(\mathbf{t})\prod_{p\notin|\mathscr{S}|} P_{B_\Sigma}(\mathbf{t}).$$

Now we rewrite the motivic density of $\widetilde{W}$ as follows:

$$\begin{aligned}\left[\widetilde{W}\right]\mathbf{L}^{-|\Sigma(1)|\ell(\mathscr{S})} &= [W]\mathbf{L}^{-\ell(\mathscr{S})\dim(V_\Sigma)} \times [T_{\mathrm{NS},\mathscr{S}}]\mathbf{L}^{-r\ell(\mathscr{S})}\\ &= \frac{[W]}{\mathbf{L}^{\ell(\mathscr{S})\dim(V_\Sigma)}}\prod_{p\in|\mathscr{S}|}\left(1-\mathbf{L}_p^{-1}\right)^r\end{aligned}$$

Putting all together, we get

$$\begin{aligned}&\left[\mathrm{Hom}_k^{\mathbf{d}}(\mathbf{P}^1_k, V_\Sigma \mid W)_U\right]\mathbf{L}^{-|\mathbf{d}|}\\ &= (\mathbf{L}-1)^{-r}\left(\mathbf{L}^{|\Sigma(1)|}[\widetilde{W}]\sum_{A\subset\Sigma(1)}(-\mathbf{L})^{-|A|}\mathfrak{s}^A_{\mathbf{d}}\mathbf{L}^{-|\mathbf{d}|} + \mathfrak{h}_d\mathbf{L}^{-|\mathbf{d}|} + \mathfrak{e}_{\mathbf{d}}\mathbf{L}^{-|\mathbf{d}|}\right)\\ &= \frac{\mathbf{L}^{\dim(V_\Sigma)}}{(1-\mathbf{L}^{-1})^r}[W]\mathbf{L}^{-\ell(\mathscr{S})\dim(V_\Sigma)}\prod_{p\in|\mathscr{S}|}(1-\mathbf{L}_p^{-1})^r\prod_{p\notin|\mathscr{S}|}P_{B_\Sigma}(\mathbf{L}^{-1})\\ &+ \text{ an error term of dimension at most:}\\ &\qquad -\frac{1}{4}\min_{\alpha\in\Sigma(1)}(d_\alpha) + (1-\ell(\mathscr{S}))(\dim(V_\Sigma)-1) + \dim(W)\end{aligned}$$

for all $\mathbf{d}\in\mathrm{Eff}(V_\Sigma)^\vee_{\mathbf{Z}}$. This concludes the proof of Theorem 5.1.7. □

5.2. Twisted products of toric varieties

The goal of this section is to apply the notion of equidistribution of (rational) curves to the case of a certain kind of twisted products. It provides a going-up theorem answering the following question in a particular setting: given a fibration, if a Batyrev-Manin-Peyre principle holds for the base and for the fibres, does it hold for the entire fibration ?

First, we recall the construction and geometric properties of such a twisted product, as it is done in [**CLT01b**]. Then we study the moduli space of rational curves and apply the change of model Theorem 3.2.6 to this context.

5.2.1. Generalities on twisted products. In this section we adapt the framework of [**CLT01b**] and [**ST99**] to the study of rational curves. Concerning torsors, we will refer to [**CTS87**].

In this paragraph S is a scheme, G is a linear flat group scheme over S, with connected fibres, and $g : \mathscr{B} \to S$ a flat scheme over S. Recall that a G-torsor over $\mathscr{B}$ is a scheme

$\mathscr{T} \to \mathscr{B}$ over $\mathscr{B}$ which is faithfully flat and locally of finite presentation, endowed with a G-action $\tau : G \times_S \mathscr{T} \to \mathscr{T}$ over $\mathscr{B}$ such that the induced morphism

$$(\tau, \mathrm{pr}_2) : G \times_S \mathscr{T} \to \mathscr{T} \times_{\mathscr{B}} \mathscr{T}$$

is an isomorphism. Moreover, in this article we will only consider torsors which are locally trivial for the Zariski topology.

5.2.1.1. *Twisted products and twisted invertible sheaves.* Following [**CLT01b**, §2], let $f : X \to S$ be a flat (quasi-compact and quasi-separated) S-scheme endowed with an action of G/S. Let $\mathscr{T} \to \mathscr{B}$ be a G-torsor locally trivial for the Zariski topology. We construct a fibration $\pi : \mathscr{X} = \mathscr{T} \times^G X \to \mathscr{B}$ locally isomorphic to X over $\mathscr{B}$ in the following manner. Let $(U_i)_{i\in I}$ be a Zariski-covering of $\mathscr{B}$ together with trivialisation $\phi_i : G \times_S U_i \to \mathscr{T}_{U_i}$. For all $i, j \in I$ there exists a unique section g_{ij} of G over $U_i \cap U_j$ such that $\phi_i = g_{ij}\phi_j$ on $U_i \cap U_j$. This data provides a cocycle whose class in $H^1(\mathscr{B}_{\mathrm{Zar}}, G)$ represents the isomorphism class of $\mathscr{T}$ as a G-torsor. Then we set $\mathscr{X}_i = X \times_S U_i$. The action of G/S on X/S induces actions of g_{ij} over $X \times (U_i \cap U_j)$ and isomorphisms $\varphi_{ij} : \mathscr{X}_{j|U_i\cap U_j} \simeq \mathscr{X}_{i|U_i\cap U_j}$. Gluing *via* the φ_{ij}'s defines $\pi : \mathscr{X} \to \mathscr{B}$. Up to a unique isomorphism, this construction does not depend on the choice of the open sets (U_i).

There exists a functor ϑ from the category of G-linearised quasi-coherent sheaves over X to the category of quasi-coherent sheaves over $\mathscr{X}$ [**CLT01b**, Construction 2.1.7] which is compatible with the standard operations for sheaves (direct sum, tensor product, localisation). It sends a G-linearised quasi-coherent sheave over X to its twisted version over $\mathscr{X}$, the gluing isomorphisms being given by the $\varphi_{i,j}$'s. This functor induces a map on the isomorphism classes. In particular, such a map sends $\Omega^1_{X/S}$ to $\Omega^1_{\mathscr{X}/\mathscr{B}}$ [**CLT01b**, Proposition 2.1.8].

When $X = S$ then $\mathscr{X} = \mathscr{B}$ and this functor ϑ is written $\eta_{\mathscr{T}}$. It induces a group morphism $\mathcal{X}^*(G) \to \mathrm{Pic}(\mathscr{B})$, also written $\eta_{\mathscr{T}}$, sending χ to the (isomorphism class of the) line bundle on $\mathscr{B}$ obtained *via* the gluing morphisms

$$(u, t) \in (U_i \cap U_j) \times_S \mathbf{A}^1_S \mapsto (u, \chi(g_{ij})t) \in (U_j \cap U_i) \times_S \mathbf{A}^1_S$$

where $(g_{ij}) \in H^1(\mathscr{B}_{\mathrm{Zar}}, G)$ is the cocycle defined above.

We define $\mathrm{Pic}^G(X)$ to be the group of isomorphism classes of G-linearised invertible sheaves on X. If X/S is smooth, the canonical sheaf over X/S is endowed with a canonical G-linearisation and

$$\omega_{\mathscr{X}/S} \simeq \vartheta(\omega_{X/S}) \otimes \pi^*\omega_{\mathscr{B}/S}$$

by [**CLT01b**, Proposition 2.1.8]. The forgetful functor ϖ induces a forgetful morphism

$$\varpi : \mathrm{Pic}^G(X) \to \mathrm{Pic}(X).$$

Let

$$\iota : \mathcal{X}^*(G) \to \mathrm{Pic}^G(X)$$

be the group morphism sending χ to the (isomorphism class of the) trivial bundle

$$X \times_S \mathbf{A}^1_S$$

together with the action of G given by

$$g \cdot (x, t) = (g \cdot x, \chi(g)t)$$

for all $(x, t) \in X \times_S \times \mathbf{A}^1_S$ and $g \in G$.

Putting ι, $\eta_{\mathscr{T}}$, ϑ and π^* together, we get morphisms

$$X^*(G) \xrightarrow{(\iota,-\eta_{\mathscr{T}})} \operatorname{Pic}^G(X) \oplus \operatorname{Pic}(\mathscr{B})$$

and

$$\operatorname{Pic}^G(X) \oplus \operatorname{Pic}(\mathscr{B}) \xrightarrow{\vartheta\otimes\pi^*} \operatorname{Pic}(\mathscr{X}).$$

For all character χ, there is a canonical isomorphism of invertible sheaves on $\mathscr{X}$

$$\vartheta(\iota(\chi)) \simeq \pi^*\eta_{\mathscr{T}}(\chi)$$

by [**CLT01b**, Proposition 2.1.11]. If X and $\mathscr{B}$ are smooth over S, then the relative canonical bundle $\omega_{X/S}$ of X/S admits a G-linearisation and there is an isomorphism

$$\omega_{\mathscr{X}/S} \simeq \vartheta(\omega_{X/S}) \otimes \pi^*(\omega_{\mathscr{B}/S}),$$

by [**CLT01b**, Proposition 2.1.8].

5.2.1.2. *Twisted products over a field: an exact sequence.* Assume that S is the spectrum of a field k, that $\mathscr{B}$ is a smooth proper and geometrically integral variety over k and G is a multiplicative group. Then, there is an exact sequence [**CTS87**, (2.0.2) & Theorem 1.5.1]

$$0 \longrightarrow H^1(k,G) \longrightarrow H^1(\mathscr{B},G) \longrightarrow \operatorname{Hom}(\mathcal{X}^*(G),\operatorname{Pic}(\overline{\mathscr{B}})) \longrightarrow H^2(k,G) \longrightarrow H^2(\mathscr{B},G).$$

Assume moreover that $\mathscr{B}$ admits an open subset U such that $\operatorname{Pic}(\overline{U}) = 0$. Then by [**CTS87**, Remark 2.2.7 & Proposition 2.2.8], in the previous exact sequence $H^2(k,G) \to H^2(\mathscr{B},G)$ is injective and the resulting short exact sequence

$$0 \longrightarrow H^1(k,G) \longrightarrow H^1(\mathscr{B},G) \longrightarrow \operatorname{Hom}(\mathcal{X}^*(G),\operatorname{Pic}(\overline{\mathscr{B}})) \longrightarrow 0 \tag{5.2.1.47}$$

splits. It is the case if the base $\mathscr{B}$ has a k-rational point, the splitting being given by the evaluation map $H^1(\mathscr{B},G) \to H^1(k,G)$ at this point.

5.2.1.3. *H90 multiplicative groups.* Still, we take S to be the spectrum of a field k and G is a linear connected group over k.

Definition 5.2.1. We will say that G is an H90 multiplicative algebraic group if
— $H^1(k,G)$ is trivial;
— G is multiplicative and solvable over k.
If G acts on a projective k-variety V, we will always assume that every line bundle on V admits a G-linearisation.

Example 5.2.2. If k has cohomological dimension at most 1 and G is a linear connected group which is solvable over k, then by [**Ser94**, Théorème 1'] the first cohomology group $H^1(k,G)$ is trivial.

Example 5.2.3. If V is a split smooth toric variety and G is its torus, then G is an H90 multiplicative algebraic group, by Hilbert 90.

5.2.2. Twisted models of X over $\mathbf{P}^1_k$. From now on we assume that X is Fano-like, that G is H90 multiplicative and acts on X, and that $\mathscr{B}(k)$ is Zariski-dense in $\mathscr{B}$. Moreover, we assume that the Picard groups of $\mathscr{B}$ and X coincide respectively with their geometric Picard group: $\operatorname{Pic}(\overline{\mathscr{B}}) \simeq \operatorname{Pic}(\mathscr{B})$ and $\operatorname{Pic}(\overline{X}) \simeq \operatorname{Pic}(X)$.

Then the sequence

$$0 \longrightarrow \mathcal{X}^*(G) \xrightarrow{(\iota,-\eta_{\mathscr{T}})} \operatorname{Pic}^G(X) \oplus \operatorname{Pic}(\mathscr{B}) \xrightarrow{\vartheta\otimes\pi^*} \operatorname{Pic}(\mathscr{X}) \longrightarrow 0 \tag{5.2.2.48}$$

is exact by [**CLT01b**, Théorème 2.2.4]. As a corollary, we get exact sequences

$$0 \longrightarrow \mathcal{X}^*(G) \xrightarrow{\iota} \mathrm{Pic}^G(X) \xrightarrow{\varpi} \mathrm{Pic}(X) \longrightarrow 0$$

(by taking $\mathscr{B} = \mathrm{Spec}(k)$ in (5.2.2.48)) and

$$0 \longrightarrow \mathrm{Pic}(\mathscr{B}) \xrightarrow{\pi^*} \mathrm{Pic}(\mathscr{X}) \xrightarrow{\widetilde{\varpi}} \mathrm{Pic}(X) \longrightarrow 0. \tag{5.2.2.49}$$

The map $\widetilde{\varpi} : \mathrm{Pic}(\mathscr{X}) \to \mathrm{Pic}(X)$ above is the map sending the class of a line bundle of the form

$$\vartheta(L) \otimes \pi^*(\mathscr{L}),$$

with $\mathscr{L}$ is a line bundle on $\mathscr{B}$ and L a G-linearised line bundle on X, to the one of $\varpi(L)$.

5.2.2.1. *Pulling-back.* Let $f : \mathbf{P}^1_k \to \mathscr{B}$ be a rational curve on $\mathscr{B}$. It induces a morphism $\mathbf{deg}\, f : \mathrm{Pic}(\mathscr{B}) \to \mathrm{Pic}(\mathbf{P}^1_k) \simeq \mathbf{Z}$. Let $\alpha \in \mathrm{Hom}(\mathcal{X}^*(G), \mathrm{Pic}(\mathscr{B}))$ be the type of the G-torsor $\mathscr{T} \to \mathscr{B}$ (see [**CTS87**, §2] for the definition of type). The pulling-back operation

$$\begin{tikzcd} \mathscr{T}_f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{T} \arrow[d] \\ \mathbf{P}^1_k \arrow[r, "f"] & \mathscr{B} \end{tikzcd}$$

induces a G-torsor $\mathscr{T}_f$ whose type is given by $(\mathbf{deg}\, f) \circ \alpha \in \mathcal{X}_*(G)$, together with functors on quasi-coherent sheaves $\vartheta_f = f^*_{\mathscr{X}} \circ \vartheta$ and $\eta_{\mathscr{T}_f} = f^* \circ \eta_{\mathscr{T}}$.

In our situation, type and class coincide by (5.2.1.47), so that $\eta_{\mathscr{T}}$ and α can be identified. Then, remark that the pull-back

$$\begin{tikzcd} \mathscr{X}_f \arrow[r, "f_{\mathscr{X}}"] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X} \arrow[d] \\ \mathbf{P}^1_k \arrow[r, "f"] & \mathscr{B}. \end{tikzcd}$$

only depends on the multidegree of f since it is exactly the twisted product obtained by starting from the G-torsor $\mathscr{T}_f \to \mathbf{P}^1_k$ of class $f^* \circ \eta_{\mathscr{T}} = \mathbf{deg}(f) \circ \alpha$.

In order to compare degrees of line bundles on models of X above $\mathbf{P}^1_k$ coming from different f of same multidegree $\delta_{\mathscr{B}}$, we need to find canonical isomorphisms between the Picard groups of these different models. So we take f and f' to be two rational curves $\mathbf{P}^1 \to \mathscr{B}$ of equal multidegree $\delta_{\mathscr{B}}$. They induce pull-backs $\mathscr{T}_f$, $\mathscr{T}_{f'}$ and $\mathscr{X}_f$, $\mathscr{X}_{f'}$. Since $\mathscr{T}_f$ and $\mathscr{T}_{f'}$ have equal types and classes, $\mathscr{X}_f$ and $\mathscr{X}_{f'}$ are isomorphic as G-varieties. We get a commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{X}^*(G) \arrow[r, "{(\iota, -\eta_{\mathscr{T}})}"'] \arrow[d, equal] & \mathrm{Pic}^G(X) \oplus \mathrm{Pic}(\mathscr{B}) \arrow[r, "\vartheta \otimes \pi^*"'] \arrow[d, "{(\mathrm{id}, \delta_{\mathscr{B}})}"'] & \mathrm{Pic}(\mathscr{X}) \arrow[r] \arrow[d, "f^*_{\mathscr{X}}"] \arrow[dd, bend left=60, "{f'_{\mathscr{X}}}^*"] & 0 \\ 0 \arrow[r] & \mathcal{X}^*(G) \arrow[r, "{\left(\iota, -\eta_{\mathscr{T}_f}\right)}"'] \arrow[d, equal] & \mathrm{Pic}^G(X) \oplus \mathrm{Pic}(\mathbf{P}^1_k) \arrow[r, "\vartheta_f \otimes \pi_f^*"'] \arrow[d, equal] & \mathrm{Pic}(\mathscr{X}_f) \arrow[r] \arrow[d, dashed, "\exists !"', "\simeq"] & 0 \\ 0 \arrow[r] & \mathcal{X}^*(G) \arrow[r, "{\left(\iota, -\eta_{\mathscr{T}_{f'}}\right)}"'] & \mathrm{Pic}^G(X) \oplus \mathrm{Pic}(\mathbf{P}^1_k) \arrow[r, "\vartheta_{f'} \otimes \pi_{f'}^*"'] & \mathrm{Pic}(\mathscr{X}_{f'}) \arrow[r] & 0 \end{tikzcd}$$

providing a canonical isomorphism $\mathrm{Pic}(\mathscr{X}_f) \simeq \mathrm{Pic}(\mathscr{X}_{f'})$.

5.2.2.2. *Multidegrees of sections of $\mathscr{X}_f \to \mathbf{P}^1_k$.* Assume now that $f : \mathbf{P}^1_k \to \mathscr{B}$ comes from a morphism $g : \mathbf{P}^1_k \to \mathscr{X}$, that is to say, $f = \pi \circ g$. Then we obtain the following Cartesian square

$$\begin{tikzcd} \mathscr{X}_f \arrow[r, "f_{\mathscr{X}}"] \arrow[d, shift right, "\pi_f"'] & \mathscr{X} \arrow[d, "\pi"] \\ \mathbf{P}^1_k \arrow[r, "f"] \arrow[u, dashed, shift right, "\sigma"'] \arrow[ur, "g"] & \mathscr{B} \end{tikzcd}$$

in which g induces a unique section $\sigma : \mathbf{P}^1_k \to \mathscr{X}_f$ such that $g = f_{\mathscr{X}} \circ \sigma$. We deduce from this square the relations on degree maps

$$\begin{aligned} \mathbf{deg}(g) &= \mathbf{deg}(\sigma) \circ f_{\mathscr{X}}^* \\ \mathbf{deg}(f) &= \mathbf{deg}(g) \circ \pi^* \\ &= \mathbf{deg}(\sigma) \circ f_{\mathscr{X}}^* \circ \pi^* \\ \mathrm{id}_{\mathrm{Pic}(\mathbf{P}^1_k)} &= \mathbf{deg}(\sigma) \circ \pi_f^*. \end{aligned}$$

Setting 5.2.4. Let $L_1, ..., L_{r_X}$ be a family of line bundles on X whose classes form a $\mathbf{Z}$-basis of $\mathrm{Pic}(X)$. We fix a section $s : \mathrm{Pic}(X) \to \mathrm{Pic}^G(X)$ by choosing a G-linearisation on each L_i.

From (5.2.2.48) and (5.2.2.49) one deduces the commutative diagram of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Pic}(\mathscr{B}) \arrow[r, "\pi^*"] \arrow[d, "\mathbf{deg}(f)"] & \mathrm{Pic}(\mathscr{X}) \arrow[r, "\widetilde{\varpi}"] \arrow[d, "f_{\mathscr{X}}^*"] & \mathrm{Pic}^G(X)/\iota(\mathcal{X}^*(G)) \simeq \mathrm{Pic}(X) \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \mathrm{Pic}(\mathbf{P}^1_k) \arrow[r, "\pi_f^*"] & \mathrm{Pic}(\mathscr{X}_f) \arrow[r, "\widetilde{\varpi_f}"] & \mathrm{Pic}^G(X)/\iota(\mathcal{X}^*(G)) \simeq \mathrm{Pic}(X) \arrow[r] & 0 \end{tikzcd}$$

where the arrow $\widetilde{\varpi_f} : \mathrm{Pic}(\mathscr{X}_f) \to \mathrm{Pic}(X) \simeq \mathbf{Z}$ is obtained from $\widetilde{\varpi}$ by replacing ϑ by ϑ_f and π by π_f.

Furthermore $\mathbf{deg}(\sigma) : \mathrm{Pic}(\mathscr{X}_f) \to \mathrm{Pic}(\mathbf{P}^1_k)$ induces by composition with $s : \mathrm{Pic}(X) \to \mathrm{Pic}^G(X)$ and $\vartheta_f : \mathrm{Pic}^G(X) \to \mathrm{Pic}(\mathscr{X}_f)$ a multidegree

$$\delta_X(\sigma) : \mathrm{Pic}(X) \to \mathrm{Pic}(\mathbf{P}^1_k) \simeq \mathbf{Z}$$

sending the class of a line bundle L on X to

$$\langle \mathbf{deg}(\sigma) \mid \vartheta_f \circ s([L]) \rangle = \langle \mathbf{deg}(g) \mid (\vartheta \circ s([L]) \rangle.$$

If σ' is another section obtained in this way, that is to say from another g' and another f' such that $\pi \circ g' = f'$ and $\mathbf{deg}(g) = \mathbf{deg}(g')$, then $\mathbf{deg}(f) = \mathbf{deg}(f') = \delta_{\mathscr{B}}$, and $\mathscr{X}_f \simeq \mathscr{X}_{f'}$. The following commutative diagram summarises the situation and shows that $\delta_X(\sigma) = \delta_X(\sigma') = \deg(g) \circ \vartheta \circ s$.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Pic}(\mathscr{B}) \arrow[r, "\pi^*"] \arrow[d, "\delta_{\mathscr{B}}"'] & \mathrm{Pic}(\mathscr{X}) \arrow[r] \arrow[d, "f_{\mathscr{X}}^*"] \arrow[dl, "\mathbf{deg}(g)"'] \arrow[dd, bend left, "{f'_{\mathscr{X}}}^*"] & \mathrm{Pic}(X) \arrow[r] \arrow[l, bend right, "\vartheta\circ s"'] \arrow[dl, "\upsilon_f\circ s"] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}(\mathbf{P}^1_k) \arrow[r, "\pi_f^*"] \arrow[d, equal] & \mathrm{Pic}(\mathscr{X}_f) \arrow[r, dashed] \arrow[d, "\simeq"] \arrow[l, bend left, "\mathbf{deg}(\sigma)"] & \mathrm{Pic}(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Pic}(\mathbf{P}^1_k) \arrow[r, "\pi_{f'}^*"] & \mathrm{Pic}(\mathscr{X}_{f'}) \arrow[r] \arrow[l, bend left, "\mathbf{deg}(\sigma')"] & \mathrm{Pic}(X) \arrow[r] & 0
\end{tikzcd}$$

By duality, from (5.2.2.48) we get an exact sequence

$$0 \longrightarrow \mathrm{Pic}(\mathscr{X})^\vee \xrightarrow{(\vartheta^\vee, \pi_*)} \mathrm{Pic}^G(X)^\vee \oplus \mathrm{Pic}(\mathscr{B})^\vee \longrightarrow \mathcal{X}_*(G) \longrightarrow 0$$

which allows us to decompose a multidegree on $\mathscr{X}$.

Lemma 5.2.5. *Let* $\mathbf{deg}(g) = (\delta_X^G(g), \delta_{\mathscr{B}}(g))$ *viewed in* $\mathrm{Pic}^G(X)^\vee \oplus \mathrm{Pic}(\mathscr{B})^\vee$. *Then*

$$[L] \in \mathrm{Pic}^G(X) \longmapsto \langle \delta_X^G(g) \mid \vartheta([L]) \rangle - \langle \delta_X(g) \mid \varpi([L]) \rangle$$

defines a cocharacter of G given by

$$\chi \in \mathcal{X}^*(G) \longmapsto \langle \delta_{\mathscr{B}}(g) \mid \eta_{\mathscr{T}} \circ \iota(\chi) \rangle.$$

Proof. We use again our favorite exact sequences to get the following diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & \mathcal{X}^*(G) \arrow[r, equal] \arrow[d, "{(\iota, -\eta_{\mathscr{T}})}"] & \mathcal{X}^*(G) \arrow[d, "\iota"] & \\
 & & \mathrm{Pic}^G(X) \oplus \mathrm{Pic}(\mathscr{B}) \arrow[r] \arrow[d, "\vartheta\otimes\pi^*"'] \arrow[dl] & \mathrm{Pic}^G(X) \arrow[d, shift left, "\varpi"] \arrow[dl, "\vartheta"'] \arrow[ddll, bend left] & \\
0 \arrow[r] & \mathrm{Pic}(\mathscr{B}) \arrow[r, "\pi^*"] \arrow[d, "\delta_{\mathscr{B}}(g)"'] & \mathrm{Pic}(\mathscr{X}) \arrow[r] \arrow[d, dashed] \arrow[dl, "\mathbf{deg}(g)"'] & \mathrm{Pic}(X) \arrow[r] \arrow[d] \arrow[u, bend left, shift left, "s"] \arrow[dll, bend left, "\delta_X(g)"] & 0 \\
0 \arrow[r] & \mathrm{Pic}(\mathbf{P}^1_k) & \delta_X^G(g)\ 0 & 0 &
\end{tikzcd}$$

from which one reads $\delta_X(g) = \delta_X^G(g) \circ s$.

Let L be a G-linearised line bundle on X. Since s is a section of ϖ, there exists a unique character $\chi \in \mathcal{X}^*(G)$ such that

$$[L] = (s \circ \varpi)([L]) + \iota(\chi)$$

in $\mathrm{Pic}^G(X)$. Taking intersection degrees, we get

$$\begin{aligned}\delta_X^G(g)\cdot\vartheta(L) &= \delta_X(g)\cdot\varpi(L)+\mathbf{deg}(g)\cdot(\vartheta\circ\iota\,(\chi))\\ &= \delta_X(g)\cdot\varpi(L)+\mathbf{deg}(g)\cdot(\pi^*\circ\eta_{\mathscr{T}}\,(\chi))\\ &= \delta_X(g)\cdot\varpi(L)+\delta_{\mathscr{B}}(g)\cdot\eta_{\mathscr{T}}(\chi).\end{aligned}$$

Hence

$$(\delta_X^G(g)\cdot\vartheta-\delta_X(g)\cdot\varpi):[L]\longmapsto\delta_{\mathscr{B}}\cdot\eta_{\mathscr{T}}([L]-s\circ\varpi[L])$$

lies in $\mathcal{X}_*(G)$. □

We reformulate these remarks in terms of moduli spaces of rational curves in the following paragraph.

5.2.3. Moduli spaces of morphisms and sections. In what follows $\mathscr{U}$ is a dense open subset of $\mathscr{B}$, U a dense open subset of X which is table under the action of G, and $\widetilde{\mathscr{U}}$ is the intersection of the preimage of $\mathscr{U}$ together with $\mathscr{T}\times^G U$ in $\mathscr{X}$.

Let k' be an extension of k and $f:\mathbf{P}^1_{k'}\to\mathscr{B}_{k'}$ be a k'-morphism given by a k'-point x of $\mathrm{Hom}(\mathbf{P}^1_k,\mathscr{B})_{\mathscr{U}}$. As before, the morphism f defines pull-backs $\mathscr{T}_f=\mathscr{T}\times_f\mathbf{P}^1_{k'}$ and $\mathscr{X}_f=\mathbf{P}^1_{k'}\times_f\mathscr{X}_{k'}$ over $\mathbf{P}^1_{k'}$. We fix once and for all a representative $\mathscr{T}_{\delta_{\mathscr{B}}}$ of the isomorphism class of $\mathscr{T}_f$, whenever $\delta_{\mathscr{B}}=\mathbf{deg}(f):\mathrm{Pic}(\mathscr{B}_{k'})\to\mathrm{Pic}(\mathbf{P}^1_{k'})$, as well as a corresponding twisted product $\mathscr{X}_{\delta_{\mathscr{B}}}$. We have canonical isomorphisms $\mathscr{T}_{\delta_{\mathscr{B}}}\simeq\mathscr{T}_f$, $\mathscr{X}_{\delta_{\mathscr{B}}}\simeq\mathscr{X}_f$ and $\mathrm{Pic}(\mathscr{X}_{\delta_{\mathscr{B}}})\simeq\mathrm{Pic}(\mathscr{X}_f)$.

Lemma 5.2.6. *The schematic fibre M_x of*

$$\pi_*:\mathrm{Hom}(\mathbf{P}^1_k,\mathscr{X})_{\widetilde{\mathscr{U}}}\to\mathrm{Hom}(\mathbf{P}^1_k,\mathscr{B})_{\mathscr{U}}$$

over the k'-point x corresponding to f is canonically isomorphic to

$$\mathrm{Hom}_{\mathbf{P}^1_{k'}}(\mathbf{P}^1_{k'},\mathscr{X}_f)_U\simeq\mathrm{Hom}_{\mathbf{P}^1_{k'}}(\mathbf{P}^1_{k'},\mathscr{X}_{\delta_{\mathscr{B}}})_U.$$

PROOF. On one hand, if we consider T-points of $M_{k'}$, where T is a scheme over k', we get T-morphisms $g:T\times_{k'}\mathbf{P}^1_{k'}\to T\times_{k'}\mathscr{X}_{k'}$ and a commutative diagram

$$\begin{tikzcd}
\mathscr{X}_T\times_{\mathscr{B}_T}\mathbf{P}^1_T \arrow[r, "\mathrm{pr}_{\mathscr{X}_T}"] \arrow[d, "\mathrm{pr}_{\mathbf{P}^1_T}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathscr{X}_T \arrow[d, "\pi_T"] \\
\mathbf{P}^1_T \arrow[ur, "g"] \arrow[r, "(\mathrm{id}_T\times f)"] & \mathscr{B}_T.
\end{tikzcd}$$

The product $\mathscr{X}_T\times_{\mathscr{B}_T}\mathbf{P}^1_T$ is nothing else but the extension of scalars of $\mathscr{X}_f$ to T and the previous square is nothing else but

$$\begin{tikzcd}
T\times_{k'}\mathscr{X}_f \arrow[r] \arrow[d, shift right, "\mathrm{pr}_{\mathbf{P}^1_T}"'] \arrow[dr, phantom, "\ulcorner", very near start] & T\times_{k'}\mathscr{X}_{k'} \arrow[d, "(\mathrm{id}\times\pi_{k'})"] \\
T\times_{k'}\mathbf{P}^1_{k'} \arrow[u, dashed, shift right, "\exists!"'] \arrow[ur, "g"] \arrow[r, "(\mathrm{id}_T\times f)"] & T\times_{k'}\mathscr{B}_{k'}
\end{tikzcd} \tag{5.2.3.50}$$

giving the existence of a unique T-section $\sigma:\mathbf{P}^1_T\to T\times_{k'}\mathscr{X}_f$. On the other hand, such a T-section defines an unique T-morphism $g:\mathbf{P}_{T'}\to T'\times\mathscr{X}_{k'}$ making the bottom right triangle commutative, that is, a unique T-point of $M_{k'}$. Thus the schematic fibre of π_* at the k'-point x corresponding to $f:\mathbf{P}^1_{k'}\to\mathscr{B}_{k'}$ is canonically isomorphic to $\mathrm{Hom}_{\mathbf{P}^1_{k'}}(\mathbf{P}^1_{k'},\mathscr{X}_f)$ as a k'-scheme. □

In particular, the previous argument shows that for every k'-scheme T there is a map of sets

$$\Sigma_\delta(T) : \mathrm{Hom}_k^\delta(\mathbf{P}^1_k, \mathscr{X})(T) \to \mathrm{Hom}_{\mathbf{P}^1_k}^{\delta_X^G \circ s}(\mathbf{P}^1_k, \mathscr{X}_{\delta_{\mathscr{B}}})(T)$$

sending a T-point $g : \mathbf{P}^1_T \to \mathscr{X} \times_k T$ of multidegree $\delta \in \mathrm{Pic}(\mathscr{X})^\vee$ to the unique T-point σ of $\mathrm{Hom}_{\mathbf{P}^1_k}(\mathbf{P}^1_k, \mathscr{X}_f) \simeq \mathrm{Hom}_{\mathbf{P}^1_k}(\mathbf{P}^1_k, \mathscr{X}_{\delta_{\mathscr{B}}})$ given by the dashed arrow in (5.2.3.50). Note that this construction is functorial in T, leading to a morphism of schemes

$$\Sigma_\delta : \mathrm{Hom}_k^\delta(\mathbf{P}^1_k, \mathscr{X}) \to \mathrm{Hom}_{\mathbf{P}^1_k}^{\delta_X^G \circ s}(\mathbf{P}^1_k, \mathscr{X}_{\delta_{\mathscr{B}}}).$$

From Proposition 2.3.30 page 59 we deduce:

Proposition 5.2.7. *Let $s : \mathrm{Pic}(X) \to \mathrm{Pic}^G(X)$ be a section of the forgetful morphism ϖ. Then for any class $\delta = (\delta_X^G, \delta_{\mathscr{B}})$ we have*

$$\begin{aligned}&\left[\mathrm{Hom}_k^{\delta_{\mathscr{X}}}(\mathbf{P}^1_k, \mathscr{X})_{\widetilde{\mathscr{U}}}\right] \\ &= \left[\mathrm{Hom}_k^{\delta_{\mathscr{B}}}(\mathbf{P}^1_k, \mathscr{B})_{\mathscr{U}}\right]\left[\mathrm{Hom}_{\mathbf{P}^1_k}^{\delta_X^G \circ s}(\mathbf{P}^1_k, \mathscr{X}_{\delta_{\mathscr{B}}})_U\right]\end{aligned}$$

in $K_0\mathbf{Var}_k$.

5.2.4. Asymptotic behaviour. We assume that the motivic constants $\tau_{\mathbf{P}^1_k}(X)$ and $\tau_{\mathbf{P}^1_k}(\mathscr{B})$ are well-defined in $\widehat{\mathscr{M}_k} = \widehat{\mathscr{M}_k}^w$ or $\widehat{\mathscr{M}_k} = \widehat{\mathscr{M}_k}^{\dim}$.

Lemma 5.2.8. *The symbol $\tau_{\mathbf{P}^1_k}(\mathscr{X})$ is well-defined and one has*

$$\tau_{\mathbf{P}^1_k}(\mathscr{X}) = \tau_{\mathbf{P}^1_k}(X)\tau_{\mathbf{P}^1_k}(\mathscr{B})$$

in $\widehat{\mathscr{M}_k}$.

Proof. As abstract series, the equality $\tau_{\mathbf{P}^1_k}(\mathscr{X}) = \tau_{\mathbf{P}^1_k}(X)\tau_{\mathbf{P}^1_k}(\mathscr{B})$ is a consequence of the multiplicative property of the motivic Euler product given by Proposition 2.4.12. Indeed, by local triviality of the fibration, one has the relation $[\mathscr{X} \times_k \mathbf{P}^1_k] = [X \times_k \mathbf{P}^1_k][\mathscr{B} \times_k \mathbf{P}^1_k]$ in $\mathscr{M}_{\mathbf{P}^1_k}$. Since $\tau_{\mathbf{P}^1_k}(X)$ and $\tau_{\mathbf{P}^1_k}(\mathscr{B})$ both converge in $\widehat{\mathscr{M}_k}$, so does $\tau_{\mathbf{P}^1_k}(\mathscr{X})$. □

Theorem 5.2.9. *Let X and $\mathscr{B}$ be two Fano-like varieties*[1] *defined over the base field k. Assume that G is an $H90$-multiplicative group*[2] *acting on X and that every line bundle on X admits a G-linearisation. Let U and $\mathscr{U}$ be dense open subsets respectively of X and $\mathscr{U}$, with U stable under the action of G.*

Let $\mathscr{T}$ be a G-torsor over $\mathscr{B}$ and

$$\mathscr{X} = \mathscr{T} \times^G X$$

the twisted product[3] *of X and $\mathscr{T}$ over $\mathscr{B}$. Let $\widetilde{\mathscr{U}}$ be the intersection of the preimage of $\mathscr{U}$ together with $\mathscr{T} \times^G U$ in $\mathscr{X}$.*

Assume that the motivic Batyrev-Manin-Peyre principle for rational curves[4] *holds both for X and $\mathscr{B}$, for curves generically intersecting U and $\mathscr{U}$ respectively, which means*

1. See Définition A, page 17.
2. See Definition 5.2.1, page 167. For example, take G to be a split torus.
3. See Section 5.2.1, page 165.
4. See Question 1, page 17, and more generally Question 2, page 78.

that

$$\left[\mathrm{Hom}_k^{\delta_X}(\mathbf{P}^1_k, X)_U\right] \mathbf{L}^{-\delta_X \cdot \omega_X^{-1}} \longrightarrow \tau(X)$$
$$\left[\mathrm{Hom}_k^{\delta_{\mathscr{B}}}(\mathbf{P}^1_k, \mathscr{B})_{\mathscr{U}_{\mathscr{B}}}\right] \mathbf{L}^{-\delta_{\mathscr{B}} \cdot \omega_{\mathscr{B}}^{-1}} \longrightarrow \tau(\mathscr{B})$$

when $d(\delta_X, \partial\mathrm{Eff}(X)^\vee)$ and $d(\delta_{\mathscr{B}}, \partial\mathrm{Eff}(\mathscr{B})^\vee)$ both tend to infinity, in $\widehat{\mathscr{M}_k} = \widehat{\mathscr{M}_k}^w$ or $\widehat{\mathscr{M}_k}^{\dim}$.
Assume furthermore that equidistribution of rational curves[5] *holds for X.*

Then for $\delta \in \mathrm{Eff}(\mathscr{X})^\vee_{\mathbf{Z}}$ the normalized class

$$\left[\mathrm{Hom}_k^{\delta}(\mathbf{P}^1_k, \mathscr{X})_{\widetilde{\mathscr{U}}}\right] \mathbf{L}^{-\delta \cdot \omega_{\mathscr{X}}^{-1}}$$

tends to the non-zero effective element

$$\tau(\mathscr{X}) = \tau(X)\tau(\mathscr{B}) \in \widehat{\mathscr{M}_k}$$

when the distance $d(\delta, \partial\mathrm{Eff}(\mathscr{X})^\vee)$ goes to infinity.

Together with Theorem 5.1.7, we get the following.

Corollary 5.2.10. *Let X be a smooth projective split toric variety with open orbit $U \simeq \mathbf{G}_m^n$ and $\mathscr{T} \to \mathscr{B}$ a $\mathbf{G}_m^n$-torsor above a Fano-like variety $\mathscr{B}$. Assume that the Batyrev-Manin-Peyre principle holds for rational curves on $\mathscr{B}$. Then it holds as well for rational curves on the twisted product $\mathscr{X} = \mathscr{T} \times^G X$.*

Proof of Theorem 5.2.9. Let $\delta_{\mathscr{X}} = (\delta_X^G, \delta_{\mathscr{B}}) \in \mathrm{Pic}(\mathscr{X})^\vee$ viewed in $\mathrm{Pic}^G(X)^\vee \oplus \mathrm{Pic}(\mathscr{B})^\vee$. Fix a section

$$s : \mathrm{Pic}(X) \to \mathrm{Pic}^G(X)$$

of the forgetful morphism ϖ as in Setting 5.2.4. That is to say, fix line bundles $L_1, ..., L_{r_X}$ forming a basis of $\mathrm{Pic}(X)$ together with a G-linearisation on each of them.

Given a curve $f : \mathbf{P}^1_k \to \mathscr{B}$, we know from the previous sections that the isomorphism class (as a scheme over $\mathbf{P}^1_k$) of the pull-back $\mathscr{X}_f$ only depends on its multidegree $\delta_{\mathscr{B}}$, and that it is a twisted model of X over $\mathbf{P}^1_k$. In the beggining of Section 5.2.3, we chose once and for all a representative $\mathscr{X}_{\delta_{\mathscr{B}}}$ of its isomorphism class.

$$\begin{tikzcd} \mathscr{X}_{\delta_{\mathscr{B}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathscr{X} \arrow[d, "\pi"] \\ \mathbf{P}^1_k \arrow[r] & \mathscr{B} \end{tikzcd}$$

This model comes with functors $\vartheta_{\delta_{\mathscr{B}}}$ so that $\vartheta_{\delta_{\mathscr{B}}}(L_i)$ is a twisted model of L_i on $\mathscr{X}_{\delta_{\mathscr{B}}}$, and $\vartheta(s \circ \varpi(\omega_X^{-1}))$ a model of ω_X^{-1}. We fix $\delta_{\mathscr{B}}$ and consider sections σ of $\mathscr{X}_{\delta_{\mathscr{B}}}$ of corresponding multidegree

$$\delta_X^G \circ s : [L] \mapsto \delta_X^G \cdot s([L]) = \delta \cdot \vartheta_{\delta_{\mathscr{B}}}(s([L])).$$

By Theorem 3.2.6,

$$\left[\mathrm{Hom}_{\mathbf{P}^1_k}^{\delta_X^G \circ s}(\mathbf{P}^1_k, \mathscr{X}_{\delta_{\mathscr{B}}})_U\right] \mathbf{L}^{-(\delta_X^G \circ s) \cdot \omega_V^{-1}}$$

5. See Definition 3.2.3, page 81.

tends to $\tau_{\vartheta(s\circ\varpi(\omega_X^{-1}))}(\mathscr{X}_\delta)$ as $d(\delta_X^G\circ s,\partial\operatorname{Eff}(X)^\vee)\to\infty$. Note that doing this way we obtain a motivic Tamagawa number with respect to the model $\vartheta(s\circ\varpi(\omega_X^{-1}))$ of ω_X^{-1} and that we can apply Lemma 5.2.5 to get the relation

$$\tau_{\vartheta(s\circ\varpi(\omega_X^{-1}))}(\mathscr{X}_{\delta_{\mathscr{B}}}) = \mathbf{L}_k^{\delta_{\mathscr{B}}\cdot\eta_{\mathscr{T}}([\omega_V^{-1}]-s\circ\varpi[\omega_V^{-1}])}\tau_{\vartheta(\omega_X^{-1})}(\mathscr{X}_{\delta_{\mathscr{B}}}) = \mathbf{L}_k^{\delta_{\mathscr{B}}\cdot\eta_{\mathscr{T}}([\omega_V^{-1}]-s\circ\varpi[\omega_V^{-1}])}\tau(X).$$

By Proposition 5.2.7 we have the equality of classes

$$\begin{aligned}&\left[\operatorname{Hom}_k^\delta(\mathbf{P}^1_k,\mathscr{X})_{\widetilde{\mathscr{U}}}\right]\\ &=\left[\operatorname{Hom}_k^{\delta_{\mathscr{B}}}(\mathbf{P}^1_k,\mathscr{B})_{\mathscr{U}}\right]\left[\operatorname{Hom}_{\mathbf{P}^1_k}^{\delta_X^G\circ s}(\mathbf{P}^1_k,\mathscr{X}_{\delta_{\mathscr{B}}})_U\right].\end{aligned}$$

Moreover, the expression

$$\omega_{\mathscr{X}}=\vartheta(\omega_X)\otimes\pi^*(\omega_{\mathscr{B}})$$

and the projection formula provide the decomposition of anticanonical degrees

$$\begin{aligned}\delta\cdot\omega_{\mathscr{X}}^{-1}&=\delta_X^G\cdot\omega_X^{-1}+\delta_{\mathscr{B}}\cdot\omega_{\mathscr{B}}^{-1}\\ &=(\delta_X^G\circ s)\cdot\varpi[\omega_V^{-1}]+\delta_{\mathscr{B}}\cdot\eta_{\mathscr{T}}([\omega_V^{-1}]-s\circ\varpi[\omega_V^{-1}])+\delta_{\mathscr{B}}\cdot\omega_{\mathscr{B}}^{-1}\end{aligned}$$

so that the normalised class

$$\left[\operatorname{Hom}_k^\delta(\mathbf{P}^1_k,\mathscr{X})_{\widetilde{\mathscr{U}}}\right]\mathbf{L}^{-\delta\cdot\omega_{\mathscr{X}}^{-1}}$$

is the product

$$\left[\operatorname{Hom}_k^{\delta_{\mathscr{B}}}(\mathbf{P}^1_k,\mathscr{B})_{\mathscr{U}}\right]\mathbf{L}^{-\delta_{\mathscr{B}}\cdot\omega_{\mathscr{B}}^{-1}}\times\left[\operatorname{Hom}_{\mathbf{P}^1_k}^{\delta_X^G\circ s}(\mathbf{P}^1_k,\mathscr{X}_{\delta_{\mathscr{B}}})_U\right]\mathbf{L}^{-(\delta_X^G\circ s)\cdot\omega_V^{-1}}\mathbf{L}^{-\delta_{\mathscr{B}}\cdot\eta_{\mathscr{T}}([\omega_V^{-1}]-s\circ\varpi[\omega_V^{-1}])}$$

of well-normalised classes, as expected.

To conclude the proof, we use [**CLT01b**, Théorème 2.2.9] ensuring that under our assumptions we have

$$\operatorname{Eff}(\mathscr{X})^\vee_{\mathbf{Z}}=\vartheta(\operatorname{Eff}^G(X))^\vee_{\mathbf{Z}}\oplus\pi^*(\operatorname{Eff}(\mathscr{B}))^\vee_{\mathbf{Z}}$$

in $\operatorname{Pic}^G(X)^\vee\oplus\operatorname{Pic}(\mathscr{B})^\vee$. Hence the condition $d(\delta,\partial\operatorname{Eff}(\mathscr{X})^\vee)\to\infty$ means

$$d(\delta_X^G,\partial\operatorname{Eff}^G(X)^\vee)\to\infty\text{ and }d(\delta_{\mathscr{B}},\partial\operatorname{Eff}(X)^\vee)\to\infty,$$

the first of these two conditions implying $d(\delta_X^G\circ s,\partial\operatorname{Eff}(X)^\vee)\to\infty$ by Lemma 5.2.5 for $\delta_{\mathscr{B}}$ fixed. The result follows by continuity of the multiplication. $\square$

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