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présentée par

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Algèbres planaires et sous-algèbres maximales abéliennes dans les algèbres de von Neumann

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dirigée par Vaughan Jones et Andrzej Zuk

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Résumé

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Cette thèse présente des résultats sur les algèbres planaires et les sous-algèbres maximales abéliennes dans des algèbres de von Neumann.

Les deux premiers chapitres portent sur une construction qui, à une algèbre planaire d'un sous-facteur, associe un facteur II_1.

Dans le premier chapitre, on définit une classe d'algèbres planaires, qualifiées de non coloriées, qui est adaptée à la théorie des probabilités libres. De plus cette classe contient la classe des algèbres planaires d'un sous-facteur. On montre qu'à toute algèbre planaire non coloriée on peut associer une algèbre de von Neumann. Le résultat principal est que cette algèbre de von Neumann est un facteur II_1.

Dans le deuxième chapitre, on considère le facteur II_1 construit à partir d'une algèbre planaire d'un sous-facteur. On considère une sous-algèbre maximale abélienne génériquement associée à l'algèbre planaire. Le résultat principal est que cette sous-algèbre maximale abélienne est maximale hyperfinie.

Dans le troisième chapitre, on considère un invariant introduit par Takesaki pour des sous-algèbres maximales abéliennes. Le résultat principal est de montrer que cet invariant est obtenu par l'action du normalisateur. En particulier, on répond à une question de Takesaki en montrant que toute sous-algèbre maximale abélienne singulière est simple.

Mots-clefs

Algèbres d'opérateurs, algèbres de von Neumann, algèbres planaires, sous-algèbres maximales abéliennes, invariant de Takesaki, théorie spectrale, théorie ergodique.

Abstract

This thesis presents some results on planar algebras and maximal abelian subalgebras in von Neumann algebras.

The first and second chapter consider a construction that associates a II_1 factor to a subfactor planar algebra.

In the first chapter, we define a class of planar algebras, called unshaded, that is well adapted to the theory of free probability. Furthermore, this class contains the class of subfactor planar algebras. We show that we can associate a von Neumann algebra to an unshaded planar algebra. The main result is that this von Neumann algebra is a II_1 factor.

In the second chapter, we consider a subfactor planar algebra and a II_1 factor associate to it. There is a maximal abelian subalgebra that is generically associates to this planar algebra. The main result is that this subalgebra is maximal hyperfinite.

In the third chapter, we consider an invariant for maximal abelian subalgebras due to Takesaki. The main result is to show that this invariant is coming from the action of the normalizer. In particular, we answer to a question of Takesaki by showing that a singular maximal abelian subaglebra is simple.

Keywords

Operator algebras, von Neumann algebras, planar algebras, maximal abelian subalgebras, spectral theory, ergodic theory, Takesaki invariant.

Table des matières

Introduction

Cette introduction a pour but de fixer le cadre mathématique de cette thèse. Je vais introduire les principales notions et notations. Pour une présentation plus précise, voir Dixmier [10, 11], Takesaki [61] ou Connes [6].

0.1 Algèbres de von Neumann

On considère un espace de Hilbert séparable sur le corps des complexes $\mathcal{H}$ et l'ensemble des opérateurs linéaires bornés de $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$. On note $\langle \cdot,\cdot \rangle$ le produit scalaire, linéaire à gauche, de $\mathcal{H}$. Soit $*$ l'adjonction de $\mathcal{B}(\mathcal{H})$ définie telle que $\langle a\eta,\xi\rangle = \langle \eta, a^*\xi\rangle$, pour tout opérateur $a \in \mathcal{B}(\mathcal{H})$ et tous vecteurs $\eta,\xi \in \mathcal{H}$. Si $S \subset \mathcal{B}(\mathcal{H})$ est un sous-ensemble, on note S' le commutant de S et S'' son bicomutant. Une algèbre de von Neumann [64, 37] est une sous-$*$-algèbre $M \subset \mathcal{B}$ qui est égale à son bicommutant.

On considère une topologie localement convexe sur $\mathcal{B}(\mathcal{H})$ qui est la topologie la plus grossière qui rende continues les applications suivantes : $a \in \mathcal{B}(\mathcal{H}) \longmapsto \sum_n \langle a(\xi_n),\eta_n\rangle$ où $\xi_n,\eta_n \in \mathcal{H}$ sont des suites bornées de l'espace de Hilbert $\mathcal{H}$. On appelle cette topologie la topologie ultrafaible. Un morphisme d'algèbres de von Neumann est un morphisme de $*$-algèbres qui est continu pour la topologie ultrafaible. On rappelle qu'un morphisme d'algèbres de von Neumann ϕ est ultrafaiblement continu si et seulement si il est normal, i.e. si $(x_i)_i$ est une suite généralisée d'opérateurs positifs croissante, alors $\sup_i \phi(x_i) = \phi(\sup_i x_i)$. Le théorème du bicommutant de von Neumann nous dit que M est une algèbre de von Neumann si et seulement si c'est une $*$-algèbre ultrafaiblement fermée et unitaire. En particulier, M est fermée pour la norme d'opérateur et est donc une C^*-algèbre.

On peut montrer qu'une algèbre de von Neumann est une C^*-algèbre unitaire telle qu'il existe un espace de Banach E tel que M est égale au dual topologique de E. Cet espace de Banach E est unique à isomorphisme près ; on le note M_*. Rappelons qu'une algèbre de von Neumann admet un prédual séparable M_* si et seulement si elle admet une représentation fidèle sur un espace de Hilbert séparable. Ainsi dans notre définition, toute algèbre de von Neumann a un prédual séparable.

On appelle facteur une algèbre de von Neumann ayant un centre égal aux homothéties, i.e. $Z(M) = M \cap M' = \mathbb{C}.1$ où 1 est l'unité de M. Plus généralement, on note $1_{\mathcal{H}}$ l'unité de $\mathcal{B}(\mathcal{H})$ ou simplement 1 si le contexte est clair.

0.1.1 Classification en types, facteurs

Soit M une algèbre de von Neumann. Soit $p,q \in M$ des projecteurs, c'est à dire que $p = p^2 = p^*$. On a deux relations d'ordre sur l'ensemble des projecteurs de M. On note $p \leqslant q$ lorsqu'on a l'inclusion $Imp \subset Imq$, où $Imp = \{p(\xi), \xi \in \mathcal{H}\}$ désigne l'image d'un opérateur. La deuxième relation d'ordre est dite de Murray et von Neumann ; on note $p \preceq q$ s'il existe une isométrie partielle $u \in M$ telle que $uu^* = p$ et $u^*u \leqslant q$. Une projection q

est infinie s'il existe $p \in M$ telle que $p \preceq q$ et $q < p$. C'est à dire que Imq est strictement inclus dans Imp. Une projection est finie si elle n'est pas infinie. Une projection $p \in M$ est minimale s'il n'existe pas de projection non nulle $q \in M$ telle que $q < p$. L'algèbre M est diffuse si elle n'admet pas de projections minimales et elle est finie si elle n'a que des projections finies.

Définition 0.1.1. Soit M un facteur. Les facteurs se classifient en trois types :
Un facteur est de type I s'il admet des projections minimales.
Il est de type II s'il est diffus et s'il admet des projections finies non nulles.
Il est de type III si toutes ses projections non nulles sont infinies.

Une algèbre de von Neumann M est finie si et seulement si il existe une trace normale fidèle $tr : M \longrightarrow \mathbb{C}$. Si M est un facteur fini, alors cette trace est unique (à multiplication par un scalaire près). On appelle facteur II_1 un facteur II fini.

On appelle hyperfinie une algèbre de von Neumann qui contient une chaîne de sous-algèbres de von Neumann de dimension finie dont l'union est ultrafaiblement dense. Murray et von Neumann [37] ont montré qu'il n'existe qu'un seul facteur II_1 hyperfini à isomorphisme près. Rappelons un résultat de Connes [5] qui montre que la notion d'algèbres de von Neumann hyperfinies est équivalente à la moyennabilité et à l'injectivité. On peut effectuer des sommes directes d'algèbres de von Neumann mais aussi des produits tensoriels :

Définition 0.1.2. Soit $M_i \subset \mathcal{B}(\mathcal{H}_i), i = 1{,}2$ deux algèbres de von Neumann. Considérons le produit tensoriel algébrique $M_1 \otimes M_2$ qui agit sur le produit tensoriel d'espaces de Hilbert $\mathcal{H}_1 \otimes \mathcal{H}_2$. On définit le produit tensoriel de M_1 et M_2 comme étant le bicommutant du produit tensoriel algébrique $M_1 \otimes M_2$ dans $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. On continue à le noter $M_1 \otimes M_2$.

0.1.2 Modules et bimodules

Soit P,Q des algèbres de von Neumann et $\mathcal{H}$ un espace de Hilbert. Supposons qu'il existe une représentation normale $\pi : P \longmapsto \mathcal{B}(\mathcal{H})$ et une représentation normale de l'algèbre opposée $\rho : Q^{op} \longrightarrow \mathcal{B}(\mathcal{H})$ telle que $\pi(P)$ commute avec $\rho(Q)$. On notera parfois $p.\xi.q$ au lieu de $\pi(p)\rho(q)(\xi)$, où $p \in P$, $q \in Q$ et $\xi \in \mathcal{H}$. L'espace de Hilbert muni de ces deux actions est un $P-Q$-bimodule, ou simplement bimodule si le contexte est clair ; on le note ${}_P\mathcal{H}_Q$. Si $Q = \mathbb{C}$, alors $\mathcal{H}$ est un P-module à gauche, et on le note ${}_P\mathcal{H}$. De même, si $P = \mathbb{C}$, on a un Q-module à droite noté $\mathcal{H}_Q$.

Dans cette thèse, on aura souvent $P = Q = A$, où A est une algèbre de von Neumann abélienne. Ainsi, ρ est une représentation de A, car $A = A^{op}$.

Supposons que tr est une trace normale fidèle sur A ; on note $L^2(A)$ l'espace de Hilbert associé à la construction GNS pour cette trace. L'espace de Hilbert $L^2(A)$ est muni de sa structure de $A-A$-bimodule classique provenant de la multiplication de A. Soit $\mathcal{K}$ un espace de Hilbert ; on a une structure de bimodule sur l'espace de Hilbert

$$L^2(A) \otimes \mathcal{K} \otimes L^2(A)$$

qui est la suivante

$$a_1.(b_1 \otimes \xi \otimes b_2).a_2 = (a_1.a_2) \otimes \xi \otimes (b_2.a_2).$$

On note

$${}_A L^2(A) \otimes \mathcal{K} \otimes L^2(A)_A$$

ce bimodule et

$${}_A L^2(A) \otimes L^2(A)_A$$

si $\mathcal{K} = \mathbb{C}$.

0.1.3 Réduction, intégrale directe

La théorie de décomposition et réduction a été élaborée par von Neumann [65] ; on peut trouver un exposé très détaillé dans [11] et [61].

Soit $(Y,\mathcal{D},\nu)$ un espace de Borel standard avec une mesure σ-finie. Considérons un champ mesurable d'espaces de Hilbert $\{\mathcal{H}_t, t \in Y\}$; on note l'intégrale directe correspondante

$$\mathcal{H} = \int_Y^{\oplus} \mathcal{H}_t d\nu(t).$$

Ses vecteurs sont les champs mesurables de vecteurs $\xi_t \in \mathcal{H}_t$ de carré intégrable, que l'on note

$$\xi = \int_Y^{\oplus} \xi_t d\nu(t).$$

Le produit scalaire de $\mathcal{H}$ est le suivant :

$$\langle \eta,\xi \rangle = \int_Y \langle \eta_t,\xi_t \rangle d\nu(t).$$

On appelle algèbre diagonale, la famille d'opérateurs qui agit par homothétie sur chacune des fibres ; c'est une algèbre de von Neumann commutative isomorphe à l'algèbre des fonctions mesurables bornées de Y dans $\mathbb{C}$, $L^\infty(Y,\nu)$.

Rappelons que si l'on a une représentation $\rho : L^\infty(Y,\nu) \longrightarrow \mathcal{B}(\mathcal{H})$, alors il existe un champ mesurable d'espaces de Hilbert $\{\mathcal{H}_t, t \in Y\}$ et une transformation unitaire

$$\phi : \mathcal{H} \longrightarrow \int_Y^{\oplus} \mathcal{H}_t d\nu(t),$$

tel que $\phi\rho(L^\infty(Y,\nu))\phi^*$ soit égale à l'algèbre diagonale.

Son commutant est l'algèbre des opérateurs décomposables, c'est à dire l'ensemble des opérateurs b tel qu'il existe un champ mesurable d'opérateurs $b_t \in \mathcal{B}(\mathcal{H}_t)$ tel que

$$b(\xi) = \int_Y^{\oplus} b_t(\xi_t) d\nu(t).$$

On note l'intégrale directe de ce champ d'opérateurs par

$$b = \int_Y^{\oplus} b_t d\nu(t).$$

Soit B une C^*-algèbre et $\{\pi_t, t \in Y\}$ un champ mesurable de représentations

$$\pi_t : B \longrightarrow \mathcal{B}(\mathcal{H}_t) \ ;$$

alors on peut définir la représentation égale à l'intégrale directe

$$\pi = \int_Y^{\oplus} \pi_t d\nu(t)$$

telle que

$$\pi(b)(\xi) = \int_Y^{\oplus} \pi_t(b)(\xi_t) d\nu(t).$$

Théorème 0.1.3. *Soit B une C^*-algèbre séparable et $\pi : B \longrightarrow \mathcal{B}(\mathcal{H})$ une représentation telle que $\pi(B)$ soit dans le commutant de l'algèbre diagonale. Alors il existe un champ mesurable de représentations $\{\pi_t, t \in Y\}$ tel que*

$$\pi = \int_Y^{\oplus} \pi_t d\nu(t).$$

On rappelle un célèbre résultat de von Neumann qui est à l'origine de la théorie de réduction. Soit M une algèbre de von Neumann ; alors il existe un espace de Borel standard $(Y,\mathcal{D},\nu)$, un champ mesurable d'espaces de Hilbert $\{\mathcal{H}_t, t \in Y\}$, un champ mesurable de facteurs $\{M_t \subset \mathcal{B}(\mathcal{H}_t), t \in Y\}$ tel que M soit isomorphe à l'intégrale directe

$$\int_Y^{\oplus} M_t d\nu(t).$$

De plus, l'algèbre diagonale est conjuguée avec le centre de M, i.e. il existe une transformation unitaire ϕ telle que $\phi Z(M)\phi^*$ est l'algèbre diagonale.

Algèbres des suites centrales, ultraproduits

On introduit la notion de suites centrales et d'ultraproduits. Murray et von Neumann [38] ont été les premiers à regarder les suites centrales pour distinguer des facteurs. Ces notions ont été axiomatisées à l'aide d'ultraproduits par McDuff [35].

Considérons un facteur II_1 M muni de son unique trace tr. On note $L^2(M)$ l'espace de Hilbert issu de la construction GNS et $\|\cdot\|_2$ la norme hilbertienne associée.

Définition 0.1.4. Soit ω un ultrafiltre non principal sur les entiers naturels $\mathbb{N}$. Considérons l'algèbre des suites bornées de M, c'est à dire le produit tensoriel d'algèbres de von Neumann $M \otimes \ell^\infty(\mathbb{N})$, où $\ell^\infty(\mathbb{N})$ désigne les suites bornées à valeurs complexes. Soit $\mathcal{I}_\omega$ l'idéal bilatère engendré par les suites $(x_n)_n$ tel que $\|x_n\|_2$ converge vers 0 selon l'ultrafiltre ω. Alors le quotient de $M \otimes \ell^\infty(\mathbb{N})$ par $\mathcal{I}_\omega$ est appelé ultraproduit de M selon ω. On continue à noter $(x_n)_n$ un élément de M^ω. C'est un facteur II_1 muni de la trace $tr_\omega((x_n)_n) = \lim_{n\to\omega} tr(x_n)$, où $\lim_{n\to\omega}$ désigne la limite selon l'ultrafiltre ω. L'algèbre des suites centrales est $M^\omega \cap M'$, que l'on note M_ω. Le facteur M a la propriété Γ si M_ω n'est pas réduit aux homothéties.

On appelle facteur de McDuff un facteur II_1 M tel que M soit isomorphe à $M \otimes R$, où R est le facteur II_1 hyperfini. McDuff prouve que M est un facteur de McDuff si et seulement si l'algèbre des suites centrales M_ω n'est pas commutative. Rappelons un résultat de Dixmier [12] qui montre que si M_ω n'est pas une algèbre triviale alors elle est diffuse.

0.1.4 Sous-algèbres maximales abéliennes

La théorie des sous-algèbres maximales abéliennes (MASAs) débute avec l'article [9] dans lequel Dixmier définit le normalisateur et l'algèbre de von Neumann qu'il engendre. Considérons une MASA $A \subset M$; son normalisateur est le sous-groupe des unitaires $u \in M$ tels que $uAu^* = A$. Il est noté $N_M(A)$. On regarde la sous-algèbre de von Neumann de M engendrée par le groupe $N_M(A)$; lorsque cette algèbre est égale à A, la MASA est dite singulière et, lorsqu'elle est égale à M, elle est dite régulière. Si M est un facteur, on dit que $A \subset M$ est semi-régulière lorsque $N_M(A)'' \subset M$ est un sous-facteur non trivial. Remarquons qu'il existe des MASAs dans des facteurs qui ne rentrent pas dans cette classification. Considérons deux algèbres de von Neumann finies munies d'une trace (M,τ_M) et (N,τ_N). On note $M * N$ le produit libre de ces deux algèbres de von Neumann par rapport aux traces, au sens de Voiculescu [39]. On peut montrer que si $A \subset N$ est une MASA diffuse dans une algèbre de von Neumann finie, alors le normalisateur de A dans $N * M$ est égal au normalisateur de A dans N. Considérons la sous-algèbre de Cartan du facteur II_1 hyperfini $C \subset R$ et l'algèbre de von Neumann abélienne des fonctions mesurables bornées du segment $[0;1]$ dans $\mathbb{C}$, $L^\infty([0;1])$. Alors $(C \otimes L^\infty([0;1])) \subset (R \otimes L^\infty([0;1])) * R$ est une MASA dans un facteur II_1. D'après les travaux de Dykema [13], $(R \otimes L^\infty([0;1])) * R$

est isomorphe au facteur du groupe libre à deux générateurs. De plus, le normalisateur de la sous-algèbre $(C \otimes L^\infty([0;1])) \subset (R \otimes L^\infty([0;1])) * R$ engendre l'algèbre de von Neumann $R \otimes L^\infty([0;1])$. On sort donc de la classification de Dixmier pour une MASA dans un facteur II_1.

Rappelons qu'un morphisme de MASAs $\phi \in Hom(A_1 \subset M_1, A_2 \subset M_2)$ est un morphisme d'algèbres de von Neumann $\phi : M_1 \longrightarrow M_2$ tel que $\phi(A_1) \subset A_2$.

Takesaki définit un deuxième invariant [60] qui est une classe d'équivalence d'une relation d'équivalence mesurable sur une désintégration d'une représentation de C^*-algèbre. Plus précisément, voici la définition de l'invariant de Takesaki pour une MASA $A \subset M$ dans un facteur II_1. Soit $(Y,\mathcal{D},\nu)$ un espace de probabilité standard tel que A est isomorphe à l'algèbre des fonctions mesurables bornées $L^\infty(Y,\nu)$. Soit π et ρ les actions gauche et droite de M sur l'espace de Hilbert $L^2(M)$. Considérons un champ mesurable d'espaces de Hilbert $\{\mathcal{H}_t, t \in Y\}$ tel que $L^2(M)$ s'identifie à l'intégrale directe

$$\int_Y^\oplus \mathcal{H}_t d\nu(t),$$

de telle sorte que $\rho(A)$ devient l'algèbre diagonale. Soit $B \subset M$ une sous-C^*-algèbre séparable et ultrafaiblement dense. Considérons un champ mesurable de représentations de B $\{\pi_t, t \in Y\}$ tel que la restriction de π à B, $\pi|_B$, soit égale à l'intégrale directe

$$\int_Y^\oplus \pi_t d\nu(t).$$

Définition 0.1.5. Soit $\mathcal{R} \subset Y \times Y$ la relation d'équivalence telle que $(s,t) \in \mathcal{R}$ si π_s est unitairement équivalente à π_t. On appelle $\mathcal{R}$ la relation d'équivalence de Takesaki.

Définissons une relation d'ordre sur les sous-parties de $Y \times Y$. Ceci nous permettra de définir plus rigoureusement ce que l'on entend par relation d'équivalence de Takesaki.

Définition 0.1.6. Soit $E,F \subset Y \times Y$; alors $E \prec F$ si et seulement si il existe un espace négligeable $N \subset Y$ tel que $E \backslash N^2 \subset F \backslash N^2$. C'est clairement une relation d'ordre. On définit une relation notée $\equiv$ définie telle que $E \equiv F$ si et seulement si $E \prec F$ et $F \prec E$. Cette relation est une relation d'équivalence sur les sous-ensembles de $Y \times Y$.

Définition 0.1.7. On note $\widehat{\mathcal{R}}$ la classe d'équivalence de $\mathcal{R}$ pour "$\equiv$" ; c'est l'invariant de Takesaki.

En particulier, la classe de $\mathcal{R}$ ne dépend pas du choix de la C^*-algèbre B.

La MASA est dite simple lorsque $\mathcal{R} \equiv \Delta Y$, où ΔY est la diagonale. Takesaki relie son invariant, dans [60], au normalisateur en montrant qu'une MASA simple est singulière. Dans le chapitre 3 on démontre la réciproque.

Feldman et Moore [15, 16] appellent sous-algèbre de Cartan une MASA régulière ayant une espérance conditionnelle sur la sous-algèbre abélienne. Ils les caractérisent par une relation d'équivalence et un 2-cocycle. On donne cette construction ci-dessous. Notons que dans notre cadre d'algèbres de von Neumann finies il y a toujours une espérance conditionnelle.

Soit $(Y,\mathcal{D},\nu)$ un espace de probabilité standard et $\mathcal{R} \subset Y \times Y$ une relation d'équivalence quasi-invariante, qui est un borélien avec des orbites dénombrables. Soit σ un 2-cocycle sur la relation d'équivalence $\mathcal{R}$. Considérons l'algèbre D des fonctions mesurables $f : \mathcal{R} \longrightarrow \mathbb{C}$ telles qu'il existe $n \geqslant 0$ tel que les ensembles $\{s, f(s,t) \neq 0\}$ et $\{t, f(s,t) \neq 0\}$ soient finis quels que soient $s,t \in Y$. On munit l'espace D du produit suivant :

$$(f * g)(s,t) = \sum_u f(s,u)g(u,t)\sigma(s,u,t).$$

Soit $\tilde{\nu}$ la mesure sur $\mathcal{R}$ qui vérifie pour toute fonction mesurable positive $f : \mathcal{R} \longrightarrow \mathbb{R}_+$,

$$\int_{\mathcal{R}} f d\tilde{\nu} = \int_Y \sum_t f(s,t) d\nu(s).$$

On considère l'espace de Hilbert des fonctions mesurables de carrés intégrables $L^2(\mathcal{R},\tilde{\nu})$, et la représentation

$$\pi : D \longrightarrow \mathcal{B}(L^2(\mathcal{R},\tilde{\nu}))$$

telle que $\pi(f)(g) = f * g$. On note le bicommutant de $\pi(D)$ par $L^\infty(\mathcal{R},\sigma)$; il admet une sous-algèbre de von Neumann abélienne égale aux fonctions dont le support est inclus dans la diagonale de $\mathcal{R}$. Ceci est la construction de Feldman et Moore ; toute sous-algèbre de Cartan est de cette forme. On sait que toute algèbre de von Neumann n'est pas issue de cette construction ; en effet, Voiculescu a montré [63] qu'un facteur de groupe libre ne contenait pas de sous-algèbre de Cartan.
Notons que ce résultat a été amélioré par Ozawa et Popa [41]. En effet, ils démontrent qu'un facteur de groupe libre est fortement solide, i.e. une sous-algèbre moyennable a un normalisateur qui engendre une sous-algèbre moyennable.

Un invariant, capable de distinguer des MASAs singulières apparaît dans les travaux de Pukanszky [53]. Considérons le $A-A$-bimodule $L^2(M) \ominus L^2(A)$, c'est à dire l'orthogonale de A dans l'espace de Hilbert $L^2(M)$. L'algèbre $\mathcal{A} = \{A,JAJ\}''$ générée par les actions gauche et droite de A, où J est l'opérateur de conjugaison de Tomita, agit sur ce bimodule. Il existe un espace de probabilité (Y^2,μ) tel que $\mathcal{A} \simeq L^\infty(Y^2,\mu)$.
Considérons une désintégration de $L^2(M) \ominus L^2(A)$ par rapport à l'action de $\mathcal{A}$, i.e.

$$L^2(M) \ominus L^2(A) \simeq \int_{Y^2}^{\oplus} \mathcal{H}_{s,t} d\mu(s,t).$$

La fonction dimension $d(s,t) = \dim \mathcal{H}_{s,t}$ est mesurable.

Définition 0.1.8. L'invariant de Pukanszky est l'image de la fonction d ; il est noté $Puk(A \subset M)$ ou plus simplement $Puk(A)$.

Popa [45] a montré des relations entre l'invariant de Pukanszky et le normalisateur ; en effet si $1 \notin Puk(A)$ alors la MASA est singulière. Si $A \subset M$ est régulière, alors $Puk(A) = \{1\}$. En revanche les réciproques sont fausses [66].

Certaines MASAs ont la propriété étonnante d'être maximales hyperfinies. Une des questions de Kadison dans [31] était de savoir si tout opérateur auto-adjoint est inclus dans une copie du facteur II_1 hyperfini. Soit $\mathbb{F}_n = \langle a_1, \cdots ,a_n\rangle$ le groupe libre à n générateurs et $L(\mathbb{Z}) \subset L(\mathbb{F}_n)$ la MASA engendrée par l'élément a_1. Popa répond par la négative en montrant que la MASA $L(\mathbb{Z}) \subset L(\mathbb{F}_n)$ est maximale hyperfinie. Pour cela il introduit la propriété d'orthogonalité asymptotique, que l'on note en abrégé AOP.

Définition 0.1.9. Soit $A \subset M$ une sous-algèbre de von Neumann d'une algèbre de von Neumann finie. Cette sous-algèbre a la AOP si pour tout élément $x \in M^\omega \cap A'$ tel que $E_{A^\omega}(x) = 0$ et pour tout élément $y \in M$ tel que $E_A(y) = 0$, xy est orthogonal à yx.

On note $E_A : M \longrightarrow A$ l'espérance conditionnelle suivant la trace de M et $E_{A^\omega} : M^\omega \longrightarrow A^\omega$ l'espérance conditionnelle suivant la trace tr_ω. On identifie M et son image dans M^ω. Popa montre qu'une MASA singulière avec la AOP est maximale hyperfinie. D'autres exemples de sous-algèbres abéliennes maximales hyperfinies ont été donnés plus récemment. Soit

$$h = \sum_{i=0}^{n} (a_i + a_i^*)$$

et $A \subset L(\mathbb{F}_n)$ l'algèbre de von Neumann engendrée par h. C'est une MASA dans le facteur du groupe libre à n générateurs. On l'appelle la sous-algèbre radiale. En utilisant la base de Radulescu [54], les auteurs de [3] montrent que la sous-algèbre radiale a la AOP. Ils en déduisent qu'elle est maximale hyperfinie. Jolissaint [23] et Shen [57] ont exhibé d'autres exemples, toujours en utilisant la AOP de Popa.

On peut remarquer que la sous-algèbre radiale et la sous-algèbre engendrée par un générateur du groupe libre $L(\mathbb{Z}) \subset L(\mathbb{F}_2)$ sont singulières, d'invariant de Pukanszky égal à $\{\infty\}$ et sont maximales hyperfinies. C'est une question ouverte de savoir si elles sont isomorphes ou non.

0.1.5 Sous-facteurs et algèbres planaires

Théorie des sous-facteurs

On appelle sous-facteur, une inclusion unitaire de facteurs de type II_1. Voir [30, 18] pour plus de précisions. La théorie moderne des sous-facteurs commence en 1983 avec Jones [24]. Dans cet article, Jones introduit l'indice $[M : N]$ d'un sous-facteur $N \subset M$ qui est la dimension de von Neumann $\dim_N L^2(M)$ où $L^2(M)$ est l'espace de Hilbert issu de la construction GNS associée à l'unique trace de M. Jones montre que cet indice ne prend que les valeurs suivantes $\{4cos^2\frac{\pi}{n}, n \geqslant 3\} \cup [4;\infty]$. Il définit la tour de Jones et l'ensemble des commutants relatifs qui constitue l'invariant standard. Cette construction donnera lieu, entre autres, au polynôme de Jones [25, 26, 27] qui est un invariant pour les noeuds. L'invariant standard connaît plusieurs axiomatisations comme le système de Popa [46, 47], le paragroupe d'Ocneanu ou l'algèbre planaire [28]. Il peut se voir comme une paire de graphes bi-partis où les sommets représentent des bimodules irréductibles et les arcs des morphismes de bimodules. Rappelons que l'invariant standard n'est pas un invariant total ; voir Bisch et al. [1].

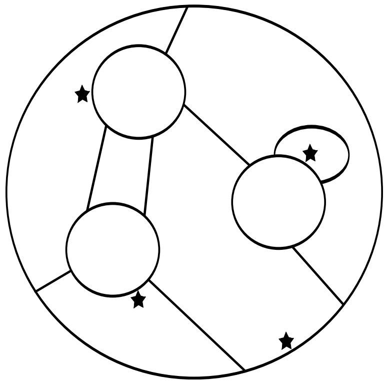
Algèbres planaires

Les algèbres planaires ont été introduites par Jones en 1999 [28] pour axiomatiser l'invariant standard. On donne une brève définition d'une algèbre planaire d'un sous-facteur ; pour plus de détails voir Jones [28] ou Peters [42].

Pour parler d'algèbres planaires, on doit définir ce qu'est un enchevêtrement. On donne une définition d'un enchevêtrement colorié utilisé dans le formalisme des algèbres planaires d'un sous-facteur. On fait le choix de ne considérer que les enchevêtrements dont les étoiles sont placées dans les parties non coloriées, ce qui simplifie légèrement les notations.

Définition 0.1.10. Un enchevêtrement est constitué d'un disque extérieur D_0, d'une famille de sous-disques $D_1, \cdots, D_k$, d'un ensemble fini de cordes entre les disques qui ne se coupent pas. Si elles ne partent pas d'un disque, alors elles forment des boucles fermées. On demande que chaque disque D_i ait un nombre pair d'intersections avec les cordes. Chaque disque a un segment repéré par une étoile $\star$. On colorie la moitié des parties connexes de l'enchevêtrement privé des sous-disques et des cordes. On demande que chaque étoile soit dans une zone non coloriée.

Voici un exemple d'enchevêtrement :



On identifie deux enchevêtrements si l'on peut passer de l'un à l'autre par un homéomorphisme qui préserve l'orientation.

Les enchevêtrements se composent entre eux en plaçant un enchevêtrement dans un disque intérieur d'un autre. On efface alors le disque intérieur et relie les cordes. Pour effectuer cette opération, il faut que les étoiles $\star$ soient sur le même segment, qu'on ait le même nombre de cordes et que le coloriage corresponde. Munis de cette opération, les enchevêtrements ont une structure d'opérade planaire. On a une involution sur les enchevêtrements $E \longmapsto E^*$ qui associe la réflexion d'un enchevêtrement.

Passons maintenant à la définition d'une algèbre planaire d'un sous-facteur.

Définition 0.1.11. Soit $\mathcal{P} = (\mathcal{P}_n)_{n \geqslant 0}$ une famille dénombrable d'espaces vectoriels complexes munis d'une involution antilinéaire $*$. L'ensemble $\mathcal{P}$ est une algèbre planaire d'un sous-facteur si l'on a une action des enchevêtrements sur $\mathcal{P}$ qui respecte la structure d'opérade planaire.

L'action des enchevêtrements commute avec les involutions : i.e. $E(x_1, \cdots ,x_k)^* = E^*(x_1^*, \cdots ,x_k^*)$ où E est un enchevêtrement, $x_i \in \mathcal{P}_{n_i}$. On demande que les $\mathcal{P}_n$ soient de dimension finie et que $\mathcal{P}_0 = \mathbb{C}$. On demande que l'action des enchevêtrements soit sphériquement invariante et que le module de l'algèbre planaire $\delta > 1$.

0.1.6 Construction d'un facteur II$_1$ à partir d'une algèbre planaire d'un sous-facteur

Dans [46, 47, 49], Popa montre que tout système de Popa peut être réalisé par un sous-facteur ; Popa et Shlyakhtenko ont démontré dans [52] que ce sous-facteur peut se réaliser dans le facteur du groupe à une infinité de générateurs $L(\mathbb{F}_\infty)$. Ces résulats ont été redémontrés en utilisant le formalisme des algèbres planaires de Jones. Dans l'article [20], Guionnet et al. associent à toute algèbre planaire d'un sous-facteur $\mathcal{P}$ un sous-facteur $N \subset M$ tel que l'algèbre planaire associée à ce sous-facteur $N \subset M$ soit égale à $\mathcal{P}$. Notons que dans le cas de profondeur finie, le sous-facteur se réalise dans un facteur de groupe libre interpolé $L(\mathbb{F}_t)$ [55, 13] où t est une fonction de l'indice de Jones et de l'indice global ; voir [21]. Une partie de ces résultats a été démontrée indépendamment dans [33, 34] par Kodiyalam et Sunder.

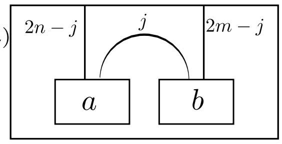
Une construction plus simple a été donnée dans l'article [29] par Jones et al.. C'est cette construction que l'on considèrera dans la suite.

Dans les articles cités, on construit une tour d'algèbres de von Neumann ; dans notre contexte nous ne considérerons que la première d'entre elles.

On prend la convention de représenter les enchevêtrements comme des rectangles contenant des sous-rectangles tel que l'étoile est placée en haut à gauche des rectangles sauf indication

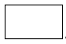
contraire.
Soit $\mathcal{P}$ une algèbre planaire d'un sous-facteur, et $Gr(\mathcal{P}) = \bigoplus_{n\geqslant 0} \mathcal{P}_n$ la somme algébrique des $\mathcal{P}_n$. On équipe $Gr(\mathcal{P})$ du produit $\star$ défini tel que :

$$a \star b = \sum_{j=0}^{min(2n,2m)}$$

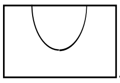


où $a \in \mathcal{P}_n$ et $b \in \mathcal{P}_m$. On étend l'involution de l'algèbre planaire $\mathcal{P}$ sur $Gr(\mathcal{P})$ et l'on obtient une algèbre involutive unitaire $(Gr(\mathcal{P}), \star, *)$ où l'unité est le diagramme vide :

$$1 =$$



On étend la forme sesquilinéaire $\langle \cdot,\cdot \rangle$ provenant de la trace de l'algèbre planaire $\mathcal{P}$ sur $Gr(\mathcal{P})$; la forme est définie positive. Elle définit donc un produit scalaire ; on note $\mathcal{H}$ la complétion de $Gr(\mathcal{P})$ pour celui-ci. On remarque que $\mathcal{H}$ est égale à la somme hilbertienne des $\mathcal{P}_n$. La multiplication $\star$ est bornée sur $\mathcal{H}$, ce qui nous donne une représentation fidèle de $Gr(\mathcal{P})$ dans $\mathcal{H}$. On note $M_{\mathcal{P}}$ son bicommutant, qui se trouve être un facteur II_1 d'après [29].
Définition 0.1.12. On appelle cup l'élément suivant de $\mathcal{P}_1$:



On désigne cet élément par le symbol $\cup$. C'est un élément auto adjoint de $M_{\mathcal{P}}$ qui génère une MASA notée $A \subset M_{\mathcal{P}}$. On appelle cette MASA, l'algèbre générée par cup.

0.1.7 Notations, définitions

On rappelle des notations utilisées tout au long de cette thèse. On note $\mathcal{P}$ une algèbre planaire, coloriée ou non et $M_{\mathcal{P}}$ l'algèbre de von Neumann finie obtenue en effectuant la construction de la section 0.1.6. La sous-algèbre générée par cup est notée $A \subset M_{\mathcal{P}}$. La multiplication dans $M_{\mathcal{P}}$ est notée $\star$ en référence à la multiplication de l'algèbre graduée $Gr(\mathcal{P})$. On note δ le module de l'algèbre planaire $\mathcal{P}$. On note ω un ultrafiltre non principal sur les entiers naturels $\mathbb{N}$ et M^ω l'ultraproduit associé à une algèbre de von Neumann finie M. Si M est une algèbre de von Neumann finie munie d'une trace tr, on note $L^2(M)$ l'espace de Hilbert issu de la construction GNS associée à la trace tr. On note π la représentation standard de M dans $L^2(M)$, qui est l'action à gauche. On note ρ l'action à droite de M dans $L^2(M)$. Une MASA sera notée $A \subset M$. On prend la convention française selon laquelle 0 appartient à l'ensemble des entiers naturels $\mathbb{N}$. L'espace de Hilbert des suites complexes de carré sommable est noté $\ell^2(\mathbb{N})$, sa base orthonormée standard est notée $\{e_n, n \in \mathbb{N}\}$. Enfin, l'opérateur de décalage à droite agissant sur $\ell^2(\mathbb{N})$ est noté s, i.e. $s(e_n) = e_{n+1}$. On identifiera dans le chapitre 3 l'algèbre A et l'algèbre des fonctions mesurables bornées $L^\infty(Y,\nu)$, où $(Y,\mathcal{D},\nu)$ est une espace de probabilité standard.

0.2 Contenu de la thèse

Cette thèse contient trois chapitres rédigés en anglais. Le premier généralise la construction de la section 0.1.6 pour une collection d'algèbres planaires plus large que les algèbres planaires d'un sous-facteur. On montre que l'algèbre de von Neumann issue de cette construction est un facteur II_1 en observant l'algèbre de von Neumann engendrée par l'élément cup. Le deuxième s'intéresse à la notion de propriété d'orthogonalité asymptotique, et à la construction de la section 0.1.6 pour une algèbre planaire d'un sous-facteur. On montre que l'algèbre générée par l'élément cup est maximale hyperfinie. Le troisième s'intéresse à l'invariant de Takesaki pour des MASAs dans des facteurs de type II_1. On répond à la conjecture de Takesaki de 1963 comme quoi son invariant est issu de l'action du normalisateur et donc qu'une MASA est singulière si et seulement si elle est simple.

0.2.1 Algèbres planaires non coloriées et construction d'un facteur II_1

L'objectif de ce chapitre est de généraliser la construction de la section 0.1.6, pour une classe d'algèbres planaires qui contient les algèbres planaires d'un sous-facteur. Puis de montrer que l'on obtient toujours des facteur II_1 à l'issue de cette construction.

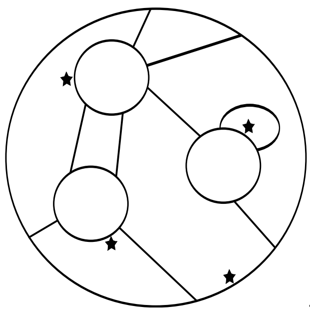
Avant d'expliquer plus précisément ce chapitre, on présente un exemple qui a motivé ce travail de généralisation.

Considérons l'algèbre des polynômes non commutatifs de degré pair en l indéterminés sur le corps des complexes $\mathbb{C}_{pair}\langle X_1, \cdots ,X_l\rangle$. Elle a une structure d'algèbre planaire (lois tensorielles) ; voir [28][exemple 2.6].

On veut avoir une structure d'algèbre planaire sur tous les polynômes $\mathbb{C}\langle X_1, \cdots ,X_l\rangle$; pour cela on a besoin de plus d'enchevêtrements ; on considère alors l'ensemble des enchevêtrements non coloriés, c'est à dire ayant un nombre quelconque de cordes (pair ou impair) qui partent des disques. On les appelle non coloriés car dès qu'un sous-disque a un nombre impair de cordes sur sa frontière, il n'est pas possible de colorier la moitié des composantes connexes de l'enchevêtrement privé des sous-disques et des cordes.

Définition 0.2.1. Un enchevêtrement non colorié est un disque avec un nombre fini de sous-disques et de cordes qui ne se croisent pas. Une corde peut être une boucle ou un arc entre deux disques. On considère un segment sur chaque disque (une partie connexe de la frontière d'un disque privé des extrémités des cordes) que l'on marque par une étoile $\star$; enfin on ordonne les disques intérieurs.

Voici un exemple d'un tel enchevêtrement :



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On munit $\mathbb{C}\langle X_1, \cdots ,X_l\rangle$ d'une action des enchevêtrements non coloriés. Pour ce faire on prolonge de façon évidente les lois d'algèbres planaires des polynômes de degré pair. On peut effectuer une construction analogue à celle de [29] pour cette algèbre de polynômes en munissant $\mathbb{C}\langle X_1, \cdots ,X_l\rangle$ de la multiplication suivante :

$$\begin{aligned} X_{i_1}\cdots X_{i_n} \star X_{j_1}\cdots X_{j_m} &= X_{i_1}\cdots X_{i_n}X_{j_1}\cdots X_{j_m} + \delta_{i_n,j_1}X_{i_1}\cdots X_{i_{n-1}}X_{j_2}\cdots X_{j_m} \\ &+ \delta_{i_n,j_1}\delta_{i_{n-1},j_2}X_{i_1}\cdots X_{i_{n-2}}X_{j_3}\cdots X_{j_m} + \cdots , \end{aligned}$$

où δ_{ij} est le symbole de Kronecker. Par exemple,

$$X_1X_2X_3 \star X_3X_1 = X_1X_2X_3^2X_1 + X_1X_2X_1.$$

L'algèbre de von Neumann finie associée à cette construction pour $\mathbb{C}\langle X_1, \cdots, X_l\rangle$ est le facteur du groupe libre à l générateurs $L(\mathbb{F}_l)$. Cette construction est identique à celle donnée par Voiculescu dans [62] dans laquelle il associe à un espace de Hilbert réel $\mathcal{H}_{\mathbb{R}}$ le facteur du groupe libre ayant $dim\mathcal{H}_{\mathbb{R}}$ générateurs $L(\mathbb{F}_{dim\mathcal{H}_{\mathbb{R}}})$. On remarque que l'image des indéterminées $\{X_1, \cdots, X_l\}$ dans l'algèbre de von Neumann finie est une famille libre d'éléments semi-circulaires.

L'objet principal de ce chapitre est de donner une généralisation naturelle des algèbres planaires d'un sous-facteur sur laquelle agissent les enchevêtrements non coloriés et pour laquelle on peut toujours effectuer la construction décrite dans la section 0.1.6. On commence par définir de telles algèbres planaires que l'on qualifie de non coloriées, on montre que l'on peut généraliser cette construction et que l'on obtient un facteur II_1. Pour ce faire, nous allons suivre la même démarche que celle proposée dans l'article [29] où les auteurs démontrent que l'on obtient des facteurs II_1 dans le cas d'algèbres planaires d'un sous-facteur.

Ainsi ce travail généralise l'espace des polynômes non commutatifs dans le cadre des algèbres planaires. C'est donc un cadre naturel pour la théorie des probabilités libres. La construction étudiée dans ce chapitre peut être vue comme une généralisation du foncteur gaussien libre de Voiculescu.

Ce travail a aussi été motivé par l'étude de la sous-algèbre abélienne engendrée par cup en vue d'avancer sur la conjecture qui demande si la sous-algèbre radiale est isomorphe à la sous-algèbre abélienne engendrée par un générateur du groupe libre. Une discussion sur ce sujet se trouve dans l'appendice de ce chapitre.

Voici un plan détaillé du chapitre.

Dans la section 1.1.1, on présente les axiomes d'une algèbre planaire non-coloriée.

Définition 0.2.2. Soit $\mathcal{P} = \{\mathcal{P}_n, n \geqslant 0\}$ une famille d'espaces vectoriels complexes de dimension finie. On suppose que pour chaque $n \geqslant 0$, il existe une involution anti-linéaire $* : \mathcal{P}_n \longrightarrow \mathcal{P}_n$ et un produit scalaire $\langle \cdot, \cdot \rangle$. On suppose que $\mathcal{P}_0 = \mathbb{C}$ et que l'on a une action des enchevêtrements non coloriés sur $\mathcal{P}$ qui est compatible avec la composition des enchevêtrements, avec les involutions $*$ et qui est sphériquement invariante.

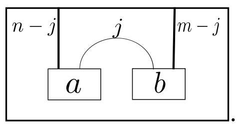
On prend la convention de représenter graphiquement un enchevêtrement par un rectangle avec des sous-rectangles que l'on appelle boîtes. Un élément de $\mathcal{P}_n$ peut être vu comme une boîte munie d'un label qui a n cordes sur sa bordure (au lieu de $2n$ cordes dans la convention classique d'une algèbre planaire d'un sous-facteur).

Dans la section 1.1, on construit une algèbre de von Neumann $M_{\mathcal{P}}$ à partir d'une algèbre planaire d'un sous-facteur $\mathcal{P}$. On considère la somme directe algébrique des espaces vectoriels $\mathcal{P}_n$ qui est notée

$$Gr(\mathcal{P}) = \bigoplus_{n \geqslant 0} \mathcal{P}_n.$$

C'est un espace vectoriel muni d'un produit scalaire $\langle \cdot, \cdot \rangle$ provenant des produits scalaires dont sont munis chacun des espaces $\mathcal{P}_n$. On note $\mathcal{H}$ la complétion de $Gr(\mathcal{P})$ pour cette structure hilbertienne. On munit l'espace $Gr(\mathcal{P})$ de la multiplication suivante : Si $a \in \mathcal{P}_n$ et $b \in \mathcal{P}_m$,

$$a \star b = \sum_{j=0}^{min(n,m)}$$



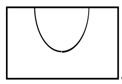
Ceci confère à l'espace $Gr(\mathcal{P})$ une structure de $*$-algèbre associative. Soit $a \in Gr(\mathcal{P})$; on peut définir l'opérateur de $Gr(\mathcal{P})$ dans lui-même qui, à b, associe $a \star b$. La principale difficulté dans la construction est de montrer que cet opérateur est borné pour la structure hilbertienne de $Gr(\mathcal{P})$. Ceci est démontré dans la proposition 1.1.4 ; ainsi on a une représentation fidèle π de la $*$-algèbre $Gr(\mathcal{P})$ dans l'espace de Hilbert $\mathcal{H}$. On note $M_{\mathcal{P}}$ l'algèbre de von Neumann égale au bicommutant de $\pi(Gr(\mathcal{P}))$. On identifie $Gr(\mathcal{P})$ avec son image dans $M_{\mathcal{P}}$ et continue de noter la multiplication de l'algèbre de von Neumann $M_{\mathcal{P}}$ par $\star$. Soit $1 \in M_{\mathcal{P}}$, l'unité de l'algèbre de von Neumann $M_{\mathcal{P}}$ qui est égale au diagramme vide de $\mathcal{P}_0$. La fonctionnelle

$$a \in M_{\mathcal{P}} \longrightarrow \langle a,1 \rangle$$

est une trace fidèle sur $M_{\mathcal{P}}$ que l'on note tr. L'algèbre $M_{\mathcal{P}}$ est donc une algèbre de von Neumann finie. Soit $L^2(M_{\mathcal{P}})$ l'espace de Hilbert issu de la construction GNS associée à la trace tr. On remarque que la représentation standard de $M_{\mathcal{P}}$ sur $L^2(M_{\mathcal{P}})$ est conjuguée à l'action de $M_{\mathcal{P}}$ sur l'espace de Hilbert $\mathcal{H}$ donné par la multiplication $\star$.

Dans la section 1.2, on considère une sous-algèbre abélienne de $M_{\mathcal{P}}$:

Définition 0.2.3. Considérons l'élément suivant de l'espace $\mathcal{P}_2$:

.

que l'on appelle cup et note $\cup$. Soit $A \subset M_{\mathcal{P}}$ la sous-algèbre de von Neumann engendrée par $\cup$; c'est une algèbre de von Neumann abélienne que l'on appelle l'algèbre générée par cup. On note $L^2(A)$ le sous-espace de Hilbert de $L^2(M_{\mathcal{P}})$ égal à la fermeture de A.

L'étude de cette sous-algèbre va nous donner des informations sur la structure de $M_{\mathcal{P}}$. On va étudier la structure de $A-A$-bimodule de l'espace de Hilbert $L^2(M_{\mathcal{P}})$. Mais d'abord, fixons une terminologie propre à ce chapitre.

Définition 0.2.4. On appelle bimodule un espace de Hilbert $\mathcal{H}$ muni d'une structure de $A-A$-bimodule. Considérons l'espace de Hilbert $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ et s l'opérateur de décalage unilatéral sur l'espace de Hilbert $\ell^2(\mathbb{N})$, i.e. $s(e_n) = e_{n+1}$ où

$$\{e_n, n \in \mathbb{N}\}$$

est la base hilbertienne standard de $\ell^2(\mathbb{N})$. L'application

$$\frac{\cup - 1}{\delta^{\frac{1}{2}}} \longmapsto s + s^*$$

définit une représentation normale de l'algèbre de von Neumann générée par cup A dans l'espace de Hilbert $\ell^2(\mathbb{N})$. Considérons la structure de bimodule sur l'espace de Hilbert $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ suivante :

$$\frac{\cup - 1}{\delta^{\frac{1}{2}}}.\xi_1 \otimes \xi_2 = (s + s^*)(\xi_1) \otimes \xi_2 \quad \text{et}$$
$$\xi_1 \otimes \xi_2.\frac{\cup - 1}{\delta^{\frac{1}{2}}} = \xi_1 \otimes (s + s^*)(\xi_2),$$

pour tout $\xi_1,\xi_2 \in \ell^2(\mathbb{N})$. On appelle bimodule grossier l'espace de Hilbert $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ muni de cette structure de bimodule.

Soit $\pi : M_{\mathcal{P}} \longrightarrow L^2(M_{\mathcal{P}})$ la représentation standard de $M_{\mathcal{P}}$, i.e. l'action régulière gauche de $M_{\mathcal{P}}$ sur l'espace de Hilbert $L^2(M_{\mathcal{P}})$. Soit $\rho : M_{\mathcal{P}}^{op} \longrightarrow L^2(M_{\mathcal{P}})$ la représentation standard de l'algèbre opposée $M_{\mathcal{P}}^{op}$, i.e. l'action droite de $M_{\mathcal{P}}$ sur l'espace de Hilbert $L^2(M_{\mathcal{P}})$. On munit l'espace de Hilbert $L^2(M_{\mathcal{P}})$ d'une structure de bimodule en restreignant les représentations π et ρ à l'algèbre de von Neumann A. Plus généralement, tout sous-espace de Hilbert $\mathcal{K} \subset L^2(M_{\mathcal{P}})$ qui est stable par l'action de A à gauche et à droite sera muni de cette structure de bimodule.

On veut montrer que l'algèbre de von Neumann $M_{\mathcal{P}}$ est un facteur II_1. Pour ce faire, on va montrer que le bimodule égal à l'orthogonale de $L^2(A)$, $L^2(M_{\mathcal{P}}) \ominus L^2(A)$, est isomorphe à une somme directe du bimodule grossier. Puis on montrera que cela implique que $M_{\mathcal{P}}$ est un facteur II_1 et que l'algèbre générée par cup est maximale abélienne. Ce sera notre stratégie.

On commence par écrire ${}_AL^2(M_{\mathcal{P}})_A$ comme une somme directe de trois bimodules. En effet, dans la section 1.2.1, on considère un sous-espace de Hilbert $V \subset L^2(M_{\mathcal{P}})$ et le sous-bimodule engendré par V dans ${}_AL^2(M_{\mathcal{P}})_A$ que l'on note ${}_A\overline{V}_A$. C'est à dire que ${}_A\overline{V}_A$ est le plus petit bimodule inclus dans $L^2(M_{\mathcal{P}})$ qui contient V. De même, on considère le sous-bimodule de ${}_AL^2(M_{\mathcal{P}})_A$ engendré par l'espace $\mathcal{P}_1$, vu comme sous-espace de Hilbert de $L^2(M_{\mathcal{P}})$, que l'on note ${}_A\overline{\mathcal{P}_1}_A$. On prouve dans la proposition 1.2.1 que le bimodule ${}_AL^2(M_{\mathcal{P}})_A$ est égal à la somme directe

$${}_AL^2(A)_A \oplus {}_A\overline{\mathcal{P}_1}_A \oplus {}_A\overline{V}_A.$$

On étudie chacun de ces bimodules. La généralisation de la démonstration de [29][Theorem 4.9] à nos algèbres planaires non coloriées est évidente et nous donne le résultat que le bimodule ${}_A\overline{V}_A$ est isomorphe à une somme directe de bimodules grossiers. La principale difficulté de ce chapitre est l'étude du bimodule ${}_A\overline{\mathcal{P}_1}_A$. Dans la section 1.2.2, on montre que le bimodule ${}_A\overline{\mathcal{P}_1}_A$ est isomorphe à une somme directe de bimodules grossiers.

Expliquons de façon précise cette étude de ${}_A\overline{\mathcal{P}_1}_A$.

On suppose que l'espace $\mathcal{P}_1$ est non nul ; prenons un vecteur non nul $b \in \mathcal{P}_1$ et considérons le bimodule engendré par b que l'on note ${}_A\overline{b}_A$. On veut montrer que le bimodule ${}_A\overline{b}_A$ est isomorphe au bimodule grossier. Pour ce faire, on donne explicitement une base hilbertienne de ${}_A\overline{b}_A$ dans le lemme 1.2.3. Puis, on construit, dans la proposition 1.2.4, une transformation unitaire

$$\eta_b : {}_A\overline{b}_A \longrightarrow \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}).$$

Cette transformation vérifie les égalités suivantes:

$$\eta_b\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\eta_b^* = \alpha + (s + s^*) \otimes 1 \quad \text{et}$$
$$\eta_b\rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\eta_b^* = 1 \otimes (s + s^*),$$

où α est un opérateur donné qui agit sur $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. Ceci implique que le bimodule ${}_A\overline{b}_A$ est isomorphe au bimodule grossier si et seulement si il existe un unitaire u qui agit sur l'espace de Hilbert $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ tel que u commute avec l'opérateur $1_{\ell^2(\mathbb{N})} \otimes (s + s^*)$ et tel que

$$u(\alpha + (s + s^*) \otimes 1_{\ell^2(\mathbb{N})})u^* = (s + s^*) \otimes 1_{\ell^2(\mathbb{N})}.$$

On va chercher un tel unitaire u.

Voici une explication détaillée sur la preuve de l'existence de cet unitaire. On considère l'algèbre de von Neumann engendrée par l'opérateur $1 \otimes (s + s^*)$ qui est

$$D = \{1 \otimes (s + s^*)\}'' \subset \mathcal{B}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})).$$

On va diagonaliser l'espace de Hilbert $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ de telle façon que D devienne l'algèbre diagonale. Puis on va chercher un unitaire décomposable qui conjugue l'opérateur $\alpha + (s + s^*) \otimes 1$ avec l'opérateur $(s + s^*) \otimes 1$. On décompose l'espace de Hilbert $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ en produit tensoriel de telle manière que l'algèbre de von Neumann D devienne l'algèbre des opérateurs diagonaux. Pour ce faire, on calcule la distribution de l'opérateur $s + s^*$

dans l'espace $\mathcal{B}(\ell^2(\mathbb{N}))$ muni de la forme linéaire $x \in \mathcal{B}(\ell^2(\mathbb{N})) \longmapsto \langle xe_0,e_0\rangle$. Ceci signifie que l'on considère la forme linéaire $f \in \mathcal{C}(\mathbb{R}) \longmapsto \langle f(s+s^*)e_0,e_0\rangle$ où $\mathcal{C}(\mathbb{R})$ est l'espace des fonctions continues de la droite réelle à valeurs dans le plan complexe. Le symbole $f(s+s^*)$ désigne l'opérateur de $\mathcal{B}(\ell^2(\mathbb{N}))$ obtenu par calcul fonctionnel. La distribution de $s+s^*$ est la mesure de Radon ν de support $[-2;2]$ qui est absolument continue par rapport à la mesure de Lebesgue dt et égale à

$$d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi}dt.$$

C'est la mesure semi-circulaire à multiplication par un scalaire près. On note

$$\eta_\nu : \ell^2(\mathbb{N}) \longrightarrow L^2([-2;2],\nu)$$

la transformation unitaire associée ; elle vérifie que $\eta_\nu(s+s^*)\eta_n u^*$ est l'opérateur de multiplication par la fonction identité. On considère alors la transformation unitaire

$$1 \otimes \eta_\nu : \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$$

qui conjugue l'algèbre de von Neumann D avec l'algèbre de von Neumann des opérateurs diagonaux de l'espace de Hilbert $\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$. On considère l'opérateur suivant :

$$c = (1 \otimes \eta_\nu) \circ (\alpha + (s+s^*) \otimes 1) \circ (1 \otimes \eta_\nu)^* \in \mathcal{B}(\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)).$$

Il commute avec l'algèbre diagonale D ; c'est donc un opérateur décomposable. Dans la proposition 1.2.6, on donne un champ mesurable d'opérateurs $\{c_t, t \in [-2;2]\}$ par rapport à l'espace mesuré $([-2;2],\nu)$ tel que c s'écrive comme l'intégrale directe

$$c = \int_{[-2;2]}^{\oplus} c_t d\nu(t).$$

On remarque que pour tout $t \in [-2;2]$, l'opérateur c_t est une perturbation compacte de l'opérateur $s+s^*$. On prouve dans le lemme 1.2.8 que c_t est unitairement équivalent à l'opérateur $s+s^*$, et ce, quel que soit $t \in [-2;2]$. Pour ce faire on utilise deux théorèmes de perturbation, le théorème de Kato-Rosenblum et le théorème de Weyl-von Neumann. Un résultat de Dixmier [11][Lemme 2,Chap.II,§2] nous assure qu'il existe un champ mesurable d'unitaires $\{u_t, t \in [-2;2]\}$ tel que pour tout t, $u_t c_t u_t^* = s+s^*$. Il en résulte que l'opérateur égal à l'intégrale directe

$$u = \int_{[-2;2]}^{\oplus} u_t d\nu(t)$$

conjugue c et $(s+s^*)\otimes 1$. De plus, il commute avec l'algèbre diagonale car est décomposable. Nous avons donc démontré que le bimodule ${}_A\overline{b}_A$ est isomorphe au bimodule grossier.

Ceci implique que le bimodule engendré par l'espace $\mathcal{P}_1$ est isomorphe à une somme directe de bimodules grossiers. Ainsi, le bimodule $L^2(M_\mathcal{P}) \ominus L^2(A)$ est isomorphe à une somme directe de bimodules grossiers. On conclut avec le théorème 1.2.11 en montrant que $M_\mathcal{P}$ est un facteur II_1 et que $A \subset M_\mathcal{P}$ est une sous-algèbre maximale abélienne (MASA).

Remarque 0.2.5. On discute dans l'appendice de ce chapitre d'une deuxième motivation de ce travail qui est une conjecture sur les MASAs. La question est la suivante ; est-ce que l'algèbre radiale est isomorphe à l'algèbre engendrée par un générateur dans un facteur de groupe libre ? On montre ici que l'algèbre générée par cup a les mêmes invariants que ces deux MASAs. Pour le choix de l'algèbre planaire d'un sous-facteur égale aux polynômes non commutatifs de degré pair, cup est isomorphe à l'algèbre générée par un générateur. En revanche, on ne sait pas si, dans le cas de l'algèbre planaire non-coloriée de tous les polynômes non commutatifs, on a toujours cette isomorphie. De plus, cette MASA est générée par le polynôme $\sum_i X_i^2$, ce qui est très proche du générateur de l'algèbre radiale.

0.2.2 Une algèbre planaire d'un sous-facteur donne une sous-algèbre abélienne maximale hyperfinie

On étudie la construction de la section 0.1.6 dans le cas classique d'algèbres planaires d'un sous-facteur. Ce chapitre propose de démontrer le théorème suivant :

Théorème 0.2.6. *Soit $\mathcal{P}$ une algèbre planaire d'un sous-facteur ; soit $M_{\mathcal{P}}$ le facteur II_1 associé, (voir section 0.1.6). Considérons $A \subset M_{\mathcal{P}}$ la sous-algèbre maximale abélienne générée par l'élément cup, (voir définition 0.1.12).*
Alors $A \subset M_{\mathcal{P}}$ est maximale hyperfinie.

On rappelle un résultat de Popa [43] qui affirme qu'une MASA singulière avec la AOP (voir définition 0.1.9) dans un facteur II_1 est maximale hyperfinie et redonne sa démonstration. Puis, on montre que la sous-algèbre générée par cup $A \subset M_{\mathcal{P}}$ est une MASA singulière qui vérifie la AOP. Ainsi, cette sous-algèbre est maximale hyperfinie.

Voici un plan détaillé de ce chapitre.

Dans la section 2.1, on considère une MASA singulière dans un facteur II_1 $B \subset M$ et suppose que cette MASA vérifie la AOP. Cette section rappelle la démonstration de Popa qui montre que $B \subset M$ est maximale hyperfinie. On suppose qu'il existe une algèbre de von Neumann intermédiaire $B \subset L \subset M$. Soit p la projection centrale de L tel que pL est une algèbre de von Neumann de type I et $(1-p)L$ une algèbre de von Neumann de type II_1. On considère alors chacune des inclusions $pB \subset pL$ et $(1-p)B \subset (1-p)L$. Ce sont des MASAs singulières qui vérifient la AOP. On rappelle qu'une MASA dans une algèbre de von Neumann type I finie est régulière. Ainsi, $pB = pL$. On observe l'inclusion $(1-p)B \subset (1-p)M$; on utilise le fait que M est une intégrale directe du facteur hyperfini R. On suppose que $p \neq 1$. En utilisant le lemme d'entrelacement de Popa [50], on montre qu'on peut inclure une algèbre de von Neumann de type II dans une algèbre de von Neumann de type I finie. On arrive à une contradiction ; ainsi $p = 1$ et donc $B = L$.

Dans la section 2.2, on rappelle la construction de la section 0.1.6 qui, à une algèbre planaire d'un sous-facteur $\mathcal{P}$, associe un facteur II_1. On note $(\pi, L^2(M_{\mathcal{P}}))$ la représentation standard de l'algèbre de von Neumann $M_{\mathcal{P}}$. L'action droite de $M_{\mathcal{P}}$ est notée ρ ; on note δ le module de l'algèbre planaire d'un sous-facteur $\mathcal{P}$. L'espace de Hilbert des suites complexes de carré sommable est noté $\ell^2(\mathbb{N})$, sa base orthonormée standard $\{e_n, n \geqslant 0\}$. On note l'opérateur de décalage à droite sur $\ell^2(\mathbb{N})$ par s, i.e. $s(e_n) = e_{n+1}$.

Remarquons que dans ce chapitre on travaille avec une algèbre planaire d'un sous-facteur, ainsi l'espace noté $\mathcal{P}_n$ dans ce chapitre correspond à l'espace noté $\mathcal{P}_{2n}$ dans le chapitre 1.

On étudie la sous-algèbre de von Neumann générée par l'élément cup $A \subset M_{\mathcal{P}}$. On définit un sous-espace $V \subset L^2(M_{\mathcal{P}})$ et rappelle la structure de bimodule de l'espace de Hilbert $L^2(M_{\mathcal{P}})$ dans la proposition 2.2.1. On donne une transformation unitaire

$$\psi : L^2(M_{\mathcal{P}}) \longrightarrow \ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})),$$

qui vérifie :

$$(\psi\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\psi^*)(x \otimes v \otimes y) = (s + s^*)(x) \otimes v \otimes y \tag{1}$$

$$(\psi\rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\psi^*)(x \otimes v \otimes y) = x \otimes v \otimes (s + s^*)(y), \tag{2}$$

pour tout $x,y \in \ell^2(\mathbb{N})$ et $v \in V$. De plus, $\psi(L^2(A)) = \ell^2(\mathbb{N})$. Ceci entraîne que $A \subset M_{\mathcal{P}}$ est une MASA singulière.

Dans la section 2.3, on montre que $A \subset M_{\mathcal{P}}$ a la AOP et donc que cette sous-algèbre est maximale hyperfinie d'après la section 2.1. C'est le résultat le plus long à démontrer de ce chapitre.

Pour ce faire on fixe un ultrafiltre non principal ω sur les entiers naturels $\mathbb{N}$. On considère l'ultraproduit $M_{\mathcal{P}}^{\omega}$ muni de sa trace tr_{ω} qui prolonge la trace de $M_{\mathcal{P}}$. On identifie $M_{\mathcal{P}}$ et son image dans l'ultraproduit donné par l'injection diagonale. On note $E_{A^{\omega}} : M_{\mathcal{P}}^{\omega} \longrightarrow A^{\omega}$ l'espérance conditionnelle par rapport à la trace tr_{ω} et on note $E_A : M \longrightarrow A$ l'espérance conditionnelle par rapport à la trace tr de $M_{\mathcal{P}}$. Soit $x \in M_{\mathcal{P}}^{\omega} \cap A'$ un élément du commutant relatif tel que $E_{A^{\omega}}(x) = 0$. Soit $(x_n)_n$ une suite bornée d'éléments de M qui représente x ; on la choisit de telle sorte que, pour tout n, $E_A(x^{(n)}) = 0$. On choisit un élément $y \in M_{\mathcal{P}}$ tel que $E_A(y) = 0$. Par approximation, on peut supposer qu'il existe un entier $J \geqslant 0$ tel que

$$y \in \bigoplus_{i<J} \mathcal{P}_i.$$

A chaque entier naturel $J \geqslant 0$ on associe un sous-espace $Z_J \subset L^2(M_{\mathcal{P}})$ égal à la fermeture de l'espace vectoriel linéairement engendré par les vecteurs

$$\psi^*(e_k \otimes v \otimes e_r)$$

tel que $\min(k,r) \leqslant J$ et $v \in V$. On note Q_J la projection orthogonale d'image Z_J. L'objectif est de montrer que

$$\lim_{n \to \omega} Q_J(x^{(n)}) = 0.$$

On commence par transposer le problème via la transformation unitaire ψ et considérer la suite de vecteurs $\psi(x^{(n)})$. Pour tout n, $\psi(x^{(n)}) \in \ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})$ car $E_A(x^{(n)}) = 0$ et, par l'équation 1, on a

$$\lim_{n \to \omega} ((s + s^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} - 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (s + s^*))(\psi(x^{(n)}) = 0.$$

Dans un premier temps, on oublie l'espace de Hilbert V et travaille avec une suite $\eta^{(n)} \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ telle que

$$\lim_{n \to \omega} ((s + s^*) \otimes 1_{\ell^2(\mathbb{N})} - 1_{\ell^2(\mathbb{N})} \otimes (s + s^*))(\eta^{(n)}) = 0.$$

On prouve dans la proposition 2.3.7 que pour tout $i \geqslant 0$,

$$\lim_{n \to \omega} (q_{e_i} \otimes 1)(\eta^{(n)}) = 0$$

et

$$\lim_{n \to \omega} (1 \otimes q_{e_i})(\eta^{(n)}) = 0,$$

où q_{e_i} est la projection orthogonale sur la droite vectorielle $\mathbb{C}e_i$. En utilisant le lemme 2.3.6, ceci implique que

$$\lim_{n \to \omega} (q_{e_i} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})})(\psi(x^{(n)}) = 0$$

et

$$\lim_{n \to \omega} (1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes q_{e_i})(\psi(x^{(n)}) = 0.$$

On remarque que la projection Q_J est égale à la somme finie des projections

$$\sum_{i=0}^{J} (q_{e_i} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} + 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes q_{e_i}).$$

Ainsi, cela implique que

$$\lim_{n\to\omega} Q_J(x^{(n)}) = 0.$$

Jusqu'à maintenant, on a seulement exploité le comportement de l'opérateur $s+s^*$. On a juste utilisé la structure de bimodule de ${}_AL^2(M_{\mathcal{P}})_A$; regardons maintenant le produit de deux éléments de $M_{\mathcal{P}}$. Le lemme 2.3.8 implique la chose suivante : si $z,w \in M_{\mathcal{P}}$ tels que $E_A(z) = E_A(w) = 0$, que z est orthogonal à Z_{J-1} et que w est dans la somme directe $\bigoplus_{j<J} \mathcal{P}_i$, alors $z \star w$ est orthogonal à $w \star z$.

Ce lemme, conjugué au fait que

$$\lim_{n\to\omega} Q_J(x^{(n)}) = 0,$$

implique que $x \star y$ est orthogonal à $y \star x$ et donc que la sous-algèbre générée par cup $A \subset M_{\mathcal{P}}$ a la AOP. On a démontré que $A \subset M_{\mathcal{P}}$ est une MASA singulière qui vérifie la AOP ; ainsi, d'après la section 2.1, elle est maximale hyperfinie.

0.2.3 L'invariant de Takesaki est la relation d'équivalence engendrée par le normalisateur

Dans ce chapitre, on considère une MASA $A \subset M$ dans un facteur II_1 et l'on fixe un espace compact avec une mesure de probabilité de Borel $(Y,\mathcal{D},\nu)$ tel que A est isomorphe à l'algèbre des fonctions mesurables et bornées $L^\infty(Y,\nu)$. On identifie A et $L^\infty(Y,\nu)$ et l'on considère un représentant de l'invariant de Takesaki qui est une relation d'équivalence $\mathcal{R} \subset Y \times Y$.

Considérons l'action du normalisateur $N_M(A)$ sur l'espace mesuré $(Y,\mathcal{D},\nu)$. On a un morphisme de groupes

$$\Theta : N_M(A) \longrightarrow \mathfrak{I}(Y,\nu)$$

où $\mathfrak{I}(Y,\nu)$ est le groupe des bijections bimesurables de l'espace de Borel Y qui préservent la mesure ν. A tout sous-groupe dénombrable $G < N_M(A)$ on peut associer sa relation d'équivalence orbitale

$$\mathcal{N}_G = \{(\Theta_u(t),t), t \in Y, u \in G\}.$$

Takesaki a montré que pour tout sous-groupe dénombrable $G < N_M(A)$, il résulte que $\mathcal{N}_G \prec \mathcal{R}$, c'est à dire qu'il existe un espace de mesure nulle $N \subset Y$ tel que $\mathcal{N}_G \backslash N^2 \subset \mathcal{R}$ (voir la définition 0.1.6 de la relation d'ordre $\prec$). Il a laissé la réciproque ouverte. L'objectif principal de ce chapitre est de montrer que la réciproque est vraie, c'est à dire qu'il existe un sous-groupe dénombrable $G < N_M(A)$ tel que les deux relations d'équivalence $\mathcal{N}_G$ et $\mathcal{R}$ sont équivalentes, i.e. $\mathcal{N}_G \equiv \mathcal{R}$. Pour ce faire, on introduit deux sous-ensembles de Y^2 que l'on appelle la relation d'équivalence faible de Takesaki $\mathcal{WR}$ et l'ensemble des atomes $\mathcal{Y}$. On peut montrer que leur classe d'équivalence pour la relation "$\equiv$" sont des invariants pour la MASA $A \subset M$ et qu'ils proviennent de la structure de $A-A$-bimodule de $L^2(M)$. On montre qu'il existe un sous-groupe dénombrable $G_{\max} < N_M(A)$ tel que

$$\{G_{\max} \cup A\}'' = N_M(A)''$$

en tant que sous-algèbres de von Neumann de M. Ceci implique que pour tout sous-groupe dénombrable $H < N_M(A)$, on a $\mathcal{N}_H \prec \mathcal{N}_{G_{\max}}$. On considère et l'on fixe un tel sous-groupe $G_{\max} < N_M(A)$ et on montre que

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}_{G_{\max}}.$$

On montre de plus que le groupoïde $\mathcal{R}_{FM}$ introduit par Feldman et Moore [15, 16] est équivalent, suivant la relation "$\equiv$", à la relation d'équivalence de Takesaki.

On considère des produits tensoriels de MASAs $A \subset M$, où

$$A = \bigotimes_l A_l$$

et

$$M = \bigotimes_l M_l.$$

On calcule la relation d'équivalence de Takesaki pour $A \subset M$ en fonction des relations d'équivalence de Takesaki des $A_l \subset M_l$. On utilise le théorème principal de ce chapitre en disant que la relation d'équivalence de Takesaki est équivalente à l'ensemble des atomes. Ceci implique que l'algèbre de von Neumann générée par le normalisateur $N_M(A)'' \subset M$ est égale au produit tensoriel des algèbres de von Neumann générées par les normalisateurs $\bigotimes_l N_{M_l}(A_l)'' \subset M$. C'est une nouvelle preuve d'un résultat de Chifan [4].

Dans la dernière partie, on regarde une inclusion de groupes dénombrables et discrets $H < G$ telle que l'inclusion des algèbres de von Neumann issue des groupes $L(H) \subset L(G)$ est une MASA. On montre alors que la relation d'équivalence de Takesaki $\mathcal{R}$ est équivalente à la relation d'équivalence orbitale donnée par l'action du normalisateur

$$N_G(H) = \{k \in G, kH = Hk\}.$$

Ceci implique comme corollaire que l'algèbre de von Neumann générée par $N_G(H)$ est égale à l'algèbre de von Neumann générée par le normalisateur des algèbres de von Neumann $N_{L(G)}(L(H))$. C'est une nouvelle preuve d'un résultat de Fang et al. [14].

Voici un plan plus détaillé de ce chapitre. On considère une sous-C^*-algèbre $B \subset M$ qui est séparable et dense pour la topologie ultrafaible ; on considère la C^*-algèbre des fonctions continues de Y dans $\mathbb{C}$, $\mathcal{C}(Y)$. On suppose que B contient cette algèbre, on a donc le carré d'inclusions suivant :

$$\begin{array}{ccc} L^\infty(Y,\nu) & \subset & M \\ \cup & & \cup \\ \mathcal{C}(Y) & \subset & B \end{array}.$$

Pour démontrer le théorème principal de ce chapitre, on va introduire deux sous-ensembles de Y^2. Le premier d'entre eux s'appelle l'ensemble des atomes $\mathcal{Y} \subset Y^2$ qui est introduit dans la section 3.1. En voici la construction : soit $\mathcal{A} = \{\pi(A),\rho(A)\}'' \subset \mathcal{B}(L^2(M))$ la sous-algèbre de von Neumann générée par les actions gauche et droite de A sur l'espace de Hilbert $L^2(M)$. Il existe une probabilité de Borel μ sur l'espace Y^2 qui est invariante sous l'action du flip $\theta(s,t) = (t,s)$. Considérons l'inclusion $A \subset \mathcal{A}$ donnée par $a \in A \longmapsto \pi(a) \in \mathcal{A}$. Elle induit la projection $p_2(s,t) = t$ de Y^2 sur Y et il en résulte que la mesure projetée $p_{2*}\mu$ est équivalente à ν. Par un résultat classique de décomposition de mesure, voir [2][Chap. 6,§3], il existe une famille de mesures $\{\mu_t, t \in Y\}$ telle que, pour tout $t \in Y$, μ_t est une probabilité de Borel sur l'espace Y^2 qui est supportée sur l'image réciproque de t par p_2, c'est à dire sur $Y \times \{t\}$. Ces mesures vérifient que, pour toute fonction mesurable et positive $f : Y^2 \longrightarrow \mathbb{R}_+$, l'application $t \longmapsto \mu_t(f)$ est mesurable et que

$$\mu(f) = \int_Y \mu_t(f) d\nu(t).$$

On assimile ces mesures à des mesures sur Y au lieu de mesures sur Y^2 supportées sur $Y \times \{t\}$. On note

$$\mu = \int_Y \mu_t d\nu(t).$$

Définition 0.2.7. L'ensemble

$$\mathcal{Y} = \{(s,t) \in Y^2, \mu_t(\{s\}) > 0, \mu_s(\{t\}) > 0\}$$

est appelé l'ensemble des atomes de la MASA $A \subset M$.

Cet ensemble a des propriétés intéressantes ; chacune de ses sections

$$\mathcal{Y}(t) = \{s \in Y, (s,t) \in \mathcal{Y}\}$$

est dénombrable. De plus on montre dans la proposition 3.1.2 que si $h : X \longrightarrow Z$ est une fonction mesurable où $X,Z \subset Y$ telle que son graphe Γ_h est inclus dans l'ensemble des atomes $\mathcal{Y}$, alors h préserve la classe de la mesure. C'est à dire que si $N \subset X$ est mesurable, alors il est de mesure nulle si et seulement si son image par h est un ensemble négligeable, c'est à dire inclus dans un borélien de mesure nulle.

En utilisant un théorème de Feldman et Moore [15] et la proposition 3.1.2, on montre qu'il existe une famille dénombrable d'automorphismes $\{h_n, n \geqslant 0\} \subset \mathfrak{I}(Y,\nu)$ telle que l'ensemble des atomes est équivalent à l'union de leurs graphes, i.e.

$$\mathcal{Y} \equiv \bigcup_{n \geqslant 0} \Gamma_{h_n}.$$

Dans la section 3.2, on étudie l'action gauche des fonctions continues sur Y, $\mathcal{C}(Y)$, sur l'espace de Hilbert $L^2(M)$ muni de sa structure de A-module à droite. On commence par donner une forme adaptée du A-module à droite $L^2(M)_A$ dans la section 3.2.1. Premièrement, on considère une décomposition abstraite de l'espace de Hilbert $L^2(M)$ par rapport à l'algèbre de von Neumann abélienne $\mathcal{A}$. On a donc un champ mesurable d'espaces de Hilbert $\{\mathcal{H}_{s,t}, (s,t) \in Y^2\}$ sur l'espace mesuré (Y^2,μ) tel que $L^2(M)$ soit unitairement équivalent à l'intégrale directe

$$\mathcal{H} = \int_{Y^2}^{\oplus} \mathcal{H}_{s,t} d\mu(s,t)$$

via la transformation unitaire ψ. De plus, pour toutes fonctions $f,g \in A$, et tout vecteur

$$\xi = \int_{Y^2}^{\oplus} \xi_{s,t} d\mu(s,t) \in \mathcal{H},$$

$$\psi\pi(f)\rho(g)\psi^*\xi = \int_{Y^2}^{\oplus} f(s)g(t)\xi_{s,t} d\mu(s,t).$$

Deuxièmement, on utilise la décompostion de μ en intégrale de mesure :

$$\mu = \int_Y \mu_t d\nu(t).$$

On considère pour tout $t \in Y$, l'espace de Hilbert $\mathcal{K}_t$ qui est égal à l'intégrale directe des

$$\{\mathcal{H}_{s,t}, s \in Y\}$$

par rapport à l'espace mesuré (Y,μ_t), i.e.

$$\mathcal{K}_t = \int_Y^{\oplus} \mathcal{H}_{s,t} d\mu_t(s).$$

Il se trouve que la famille $\{\mathcal{K}_t, t \in Y\}$ est un champ mesurable d'espaces de Hilbert par rapport à l'espace mesuré (Y,ν). On note cette intégrale directe

$$\mathcal{K} = \int_Y^{\oplus} \mathcal{K}_t d\nu(t).$$

D'après un article de Guichardet [19], la transformation évidente de $\mathcal{H}$ vers $\mathcal{K}$ qui, à un vecteur

$$\xi = \int_{Y^2}^{\oplus} \xi_{s,t} d\mu(s,t) \in \mathcal{H},$$

associe le vecteur

$$\zeta = \int_Y^{\oplus} \zeta_t d\nu(t) \in \mathcal{K},$$

où

$$\zeta_t = \int_Y^{\oplus} \xi_{s,t} d\mu_t(s) \in \mathcal{K}_t,$$

est une transformation unitaire. C'est un analogue du théorème de Fubini. On note φ cette transformation. Soit $\phi = \varphi \circ \psi$ la transformation unitaire de $L^2(M)$ sur $\mathcal{K}$. Il résulte que pour tout $f \in A$ et $\zeta \in \mathcal{K}$,

$$\phi\rho(f)\phi^*\zeta = \int_Y^{\oplus} f(t)\zeta_t d\nu(t).$$

C'est cette décomposition en intégrale directe de $L^2(M)$ que l'on considère dans le chapitre.
Soit $\pi|_B$ la restriction à B de l'action régulière gauche sur l'espace de Hilbert $L^2(M)$. D'après le théorème 0.1.3, elle se décompose en une intégrale directe de représentation de la C^*-algèbre B, $\pi_t : B \longrightarrow \mathcal{B}(\mathcal{K}_t)$ telle que pour tout $b \in B$,

$$\phi\pi(b)\phi^* = \int_Y^{\oplus} \pi_t(b) d\nu(t).$$

On définit la relation d'équivalence de Takesaki comme étant

$$\mathcal{R} = \{(s,t) \in Y^2, \pi_s \simeq \pi_t\}$$

pour cette désintégration spécifique de la représentation π. On définit une deuxième relation d'équivalence que l'on appelle la relation d'équivalence faible de Takesaki qui est

$$\mathcal{WR} = \{(s,t) \in Y^2, \pi_s|_{\mathcal{C}(Y)} \simeq \pi_t|_{\mathcal{C}(Y)}\}.$$

Ce sont les couples $(s,t) \in Y^2$ tels que les restrictions des représentations π_s et π_t à la sous-C^*-algèbre $\mathcal{C}(Y) \subset B$, sont unitairement équivalents.

Dans la section 3.2.2, on utilise la décompostion donnée en section 3.2.1 pour expliciter l'action gauche de $\mathcal{C}(Y)$. Ceci va nous donner des informations précieuses sur la relation d'équivalence faible de Takesaki. On considère l'opérateur

$$\begin{aligned} f_t : \mathcal{K}_t &\longrightarrow \mathcal{K}_t \\ \eta &\longmapsto \int_Y^{\oplus} f(s)\eta_s d\mu_t(s). \end{aligned}$$

L'intégrale directe des opérateurs f_t est égale, modulo une conjugaison par ϕ, à $\pi(f)$, i.e.

$$\pi(f) = \phi \int_Y^{\oplus} f_t \nu(t) \phi^*.$$

De plus, l'application

$$\begin{aligned} r_t : \mathcal{C}(Y) &\longrightarrow \mathcal{B}(\mathcal{K}_t) \\ f &\longmapsto f_t \end{aligned}$$

est une représentation de la C^*-algèbre $\mathcal{C}(Y)$. La famille $\{r_t, t \in Y\}$ est un champ mesurable de représentations ; il résulte alors que la restriction de l'action gauche sur $L^2(M)$ à l'algèbre des fonctions continues $\mathcal{C}(Y)$ est égale à

$$\pi|_{\mathcal{C}(Y)} = \phi \int_Y^{\oplus} r_t d\nu(t) \phi^*.$$

Par unicité de la décomposition de représentations, on montre que pour tout $f \in \mathcal{C}(Y)$, le scalaire $f(t)$ est une valeur propre de l'opérateur $\pi_t(f)$ ν-presque partout.

Dans la section 3.3, on considère l'action du normalisateur $N_M(A)$ sur l'espace de probabilité $(Y,\mathcal{D},\nu)$ via le morphisme Θ. On veut considérer la relation d'équivalence orbitale, mais le groupe $N_M(A)$ n'est pas dénombrable. On s'intéresse alors à ses sous-groupes dénombrables, $G < N_M(A)$, et l'on note $\mathcal{N}_G$ sa relation d'équivalence orbitale associée, i.e.

$$\mathcal{N}_G = \{(\Theta_u(t),t), t \in Y, u \in Y\}.$$

On prouve dans la proposition 3.3.2 que la relation d'ordre $\prec$ sur les sous-ensembles de Y^2 est compatible avec la relation d'inclusions d'algèbres de von Neumann engendrées par les groupes. C'est à dire que si $H,G < N_M(A)$ sont des sous-groupes dénombrables, alors on a

$$\{H \cup A\}'' \subset \{G \cup A\}''$$

si et seulement si on a $\mathcal{N}_H \prec \mathcal{N}_G$. On démontre dans la proposition 3.3.3, qu'il existe un sous-groupe dénombrable $G_{\max} < N_M(A)$ tel que

$$\{G_{\max} \cup A\}'' = N_M(A)''.$$

Ceci implique que pour tout sous-groupe dénombrable $H < N_M(A)$, $\mathcal{N}_H \prec \mathcal{N}_{G_{\max}}$. On fixe un tel sous-groupe $G_{\max} < N_M(A)$ et l'on note sa relation d'équivalence orbitale $\mathbb{N}_{G_{\max}}$ par $\mathcal{N}$.

Dans la section 3.4, on démontre le résultat principal de ce chapitre. Le théorème 3.4.1 prouve que la relation d'équivalence de Takesaki $\mathcal{R}$, la relation d'équivalence faible de Takesaki $\mathcal{WR}$, l'ensemble des atomes $\mathcal{Y}$ et la relation d'équivalence orbitale $\mathcal{N}$ sont tous équivalents, i.e.

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}.$$

Pour démontrer ce théorème, on va montrer la chaîne d'inclusion

$$\mathcal{R} \prec \mathcal{WR} \prec \mathcal{Y}.$$

Puis on va construire un sous-groupe dénombrable $G < N_M(A)$ tel que $\mathcal{Y} \prec \mathcal{N}_G$, et donc $\mathcal{Y} \prec \mathcal{N}$. On montre alors que tout sous-groupe dénombrable $H < N_M(A)$ vérifie que $\mathcal{N}_H \prec \mathcal{R}$ et en particulier $\mathcal{N} \prec \mathcal{R}$. Ceci implique l'équivalence

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}.$$

Voici un plan plus précis de la preuve. L'inclusion $\mathcal{R} \subset \mathcal{WR}$ est évidente par définition. Pour montrer que $\mathcal{WR} \prec \mathcal{Y}$, on considère une fonction continue et injective $f \in \mathcal{C}(Y)$.

D'après la proposition 3.2.1, on sait que le scalaire $f(t)$ est une valeur propre de l'opérateur $\pi_t(f)$ pour tout t en dehors d'un espace de mesure nul N. On considère un couple (s,t) de $\mathcal{WR}$ tel que $s,t \in Y \backslash N$. Les opérateurs $\pi_s(f)$ et $\pi_t(f)$ sont unitairement conjugués ; il résulte que $f(s)$ est une valeur propre de l'opérateur $\pi_t(f)$. En utilisant la décomposition de $\mathcal{K}_t$ donnée dans la section 3.2.1, on montre dans le lemme 3.4.2 que s est un atome de la mesure μ_t. En faisant le même raisonnement mais en changeant le rôle de s et t, on montre que t est un atome de la mesure μ_s. Ainsi, $\mathcal{WR} \prec \mathcal{Y}$.
L'inclusion $\mathcal{Y} \prec \mathcal{N}$ est plus technique ; on peut regarder différents travaux sur le sujet [15], [51], [36]. On considère un automorphisme $h \in \mathfrak{I}(Y,\nu)$ tel que son graphe est inclus dans l'ensemble des atomes $\mathcal{Y}$. On prouve dans le lemme 3.4.3 qu'il existe un unitaire $u \in N_M(A)$ tel que $\Theta_u = h$ presque partout. Ainsi, le graphe de h, $\Gamma_h \prec \mathcal{N}_{\langle u \rangle}$, où $\langle u \rangle$ est le sous-groupe de $N_M(A)$ engendré par u. D'après le théorème 3.1.3, il existe une famille dénombrable d'automorphismes $\{h_n, n \geqslant 0\}$ dans $\mathfrak{I}(Y,\nu)$ telle que

$$\mathcal{Y} \equiv \bigcup_n \Gamma_{h_n}.$$

Il existe alors, pour tout n, un élément du normalisateur $u_n \in N_M(A)$ tel que $h_n = \Theta_{u_n}$ ν-presque partout. On pose G le sous-groupe dénombrable de $N_M(A)$ généré par les u_n ; alors $\mathcal{Y} \prec \mathcal{N}_G$. Par maximalité de la relation d'équivalence $\mathcal{N}$, on a $\mathcal{N}_G \prec \mathcal{N}$ et donc $\mathcal{Y} \prec \mathcal{N}$.

Il reste à montrer que $\mathcal{N} \prec \mathcal{R}$. C'est ce que démontre Takesaki avec [60][Theorem 4.1]. On a donc démontré que nos quatre ensembles sont équivalents pour la relation "$\equiv$". Ainsi,

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}.$$

En particulier, on montre qu'une MASA est simple si et seulement si elle est singulière. On a alors beaucoup de régularités sur la relation d'équivalence de Takesaki. Elle a des orbites dénombrables presque partout, elle préserve la mesure. Elle est décrite par la relation d'équivalence orbitale d'un sous-groupe du normalisateur. Il est étonnant de voir qu'on peut la définir à l'aide de $\mathcal{WR}$ qui est une relation issue de la structure de $A-A$-bimodule de $L^2(M)$. Remarquons que la clé de la preuve réside dans l'introduction de l'ensemble des atomes $\mathcal{Y}$ qui nous donne toutes les régularités recherchées pour la relation d'équivalence de Takesaki $\mathcal{R}$.

On va donner un corollaire du résultat principal en considérant le groupe plein de la relation d'équivalence de Takesaki. Mais d'abord, voici quelques définitions.

Définition 0.2.8. Soit $\mathcal{R}_0 \subset Y \times Y$ une relation d'équivalence de l'espace (Y,ν). On considère l'espace des automorphismes boréliens qui préservent la mesure $\mathfrak{I}(Y,\nu)$. On définit une relation d'équivalence "$\doteq$" sur $\mathfrak{I}(Y,\nu)$ telle que $h_1 \doteq h_2$ si et seulement si $h_1(t) = h_2(t)$ presque partout. On note $[\mathcal{R}_0]$ l'ensemble des classes d'équivalences d'éléments $h \in \mathfrak{I}(Y,\nu)$ telles que $\Gamma_h \prec \mathcal{R}_0$. L'ensemble $[\mathcal{R}_0]$ est un groupe que l'on appelle le groupe plein de la relation d'équivalence $\mathcal{R}_0$.

Définition 0.2.9. Soit $N_M(A)$ le normalisateur de $A \subset M$ et $U(A)$ le groupe des unitaires de A. Le groupe quotient de $N_M(A)$ par $U(A)$ est appelé le groupe de Weyl de $A \subset M$ et il est noté $W_M(A)$.

On a le corollaire suivant de l'équivalence $\mathcal{R} \equiv \mathcal{N}$: l'application $\Theta : N_M(A) \longrightarrow \mathfrak{I}(Y,\nu)$ induit un isomorphisme de groupes entre le groupe de Weyl $W_M(A)$ et le groupe plein de la relation d'équivalence de Takesaki $[\mathcal{R}]$.

Dans la section 3.4.1, on considère la relation d'équivalence qui apparaît dans les travaux de Feldman et Moore [16]. Soit $\mathcal{R}_{FM} \subset Y \times Y$ un sous-ensemble de Borel qui est une relation d'équivalence quasi-invariante et à orbites dénombrables. On la munit d'un 2-cocycle σ à valeur dans le cercle complexe $\mathbb{T}$. Dire que la relation d'équivalence $\mathcal{R}_{FM}$ est quasi-invariante signifie que la saturation d'un espace de mesure nulle est de mesure nulle. On peut alors associer au couple $(\mathcal{R}_{FM},\sigma)$ une sous-algèbre de Cartan dans une algèbre de von Neumann $L^\infty(Y,\nu) \subset M(\mathcal{R}_{FM},\sigma)$. Pour rester dans le cadre de ce chapitre, on suppose que $M(\mathcal{R}_{FM},\sigma)$ est un facteur II_1. On montre dans le corollaire 3.4.8, que la relation d'équivalence de Takesaki $\mathcal{R}$ pour la MASA $L^\infty(Y,\nu) \subset M(\mathcal{R}_{FM},\sigma)$ est équivalente à $\mathcal{R}_{FM}$, i.e. $\mathcal{R} \equiv \mathcal{R}_{FM}$. La preuve part de $\mathcal{R} \equiv \mathcal{N}_G$, puis c'est juste une traduction du formalisme de Feldman et Moore.

Dans la section 3.5, on considère une famille dénombrable de MASAs dans des facteurs II_1, $\{A_l \subset M_l, l \in \Lambda\}$ et l'inclusion $A \subset M$ où $A = \bigotimes_l A_l$ et $M = \bigotimes M_l$. C'est une MASA dans un facteur II_1. On calcule la relation d'équivalence de Takesaki $\mathcal{R}$ pour $A \subset M$ en fonction des relations d'équivalence de Takesaki $\mathcal{R}_l$ pour les $A_l \subset M_l$. Pour ce faire, on utilise le fait que la relation $\mathcal{R}$ est équivalente à l'ensemble des atomes $\mathcal{Y}$. Puis on veut montrer dans le corollaire 3.5.2 que le normalisateur $N_M(A)$ engendre la même algèbre de von Neumann que le produit tensoriel des algèbres de von Neumann engendrées par les $N_{M_l}(A_l)$. La démonstration est la suivante. D'après le théorème 3.4.1, il existe des sous-groupes dénombrables $G_l < N_{M_l}(A_l)$ tel que $\mathcal{R}_l \equiv \mathcal{N}_{G_l}$. On considère les unitaires $u = \otimes_l u_l$ tels que $u_l \in G_l$ et $u_l = 1$ hormis pour un nombre fini de l. Ces unitaires forment un sous-groupe dénombrable du normalisateur $N_M(A)$ que l'on note G. On démontre que $\mathcal{R} \equiv \mathcal{N}_G$; ceci implique, par la remarque 3.4.4, que $\{G \cup A\}'' = N_M(A)''$.
Or, par définition de G, on obtient

$$\{G \cup A\}'' = \bigotimes_l \{G_l \cup A_l\}''.$$

De plus $\mathcal{R}_l \equiv \mathcal{N}_{G_l}$, d'où $\{G_l \cup A_l\}'' = N_{M_l}(A_l)''$. Ceci nous permet de conclure.

Dans la section 3.6, on considère le cas particulier de MASAs provenant d'une inclusion de groupes discrets dénombrables $H < G$. On observe donc la MASA $L(H) \subset L(G)$ des algèbres de von Neumann de groupes. Notre espace de probablité standard $(Y,\mathcal{D},\nu)$ est remplacé par le groupe dual de H égal aux caractères $t : H \longrightarrow \mathbb{T}$ que l'on note $\widehat{H}$ muni de sa tribu borélienne $\mathcal{D}$ et de sa mesure de Haar invariante à gauche ν. La sous-algèbre séparable et ultrafaiblement dense $B \subset M$ est remplacée par la C^*-algèbre réduite du groupe G, $C^*_r(G) \subset L(G)$. L'algèbre des fonctions continues sur Y est remplacée par la C^*-algèbre du groupe abélien H, $C^*(H)$. Remarquons que H est abélien et donc moyennable, ce qui entraîne que sa C^*-algèbre réduite correspond à sa C^*-algèbre universelle. On continue à noter $\mathcal{R}$ et $\mathcal{WR}$ les relations d'équivalence de Takesaki et de Takesaki faible sur l'espace $\widehat{H}$. Considérons le groupe des normalisateurs du sous-groupe $H < G$ qui est

$$N_G(H) = \{k \in G, kHk^{-1} = H\}.$$

C'est un sous-groupe dénombrable du normalisateur des algèbres $N_{L(G)}(L(H))$. L'action de $N_G(H)$ sur l'espace $\widehat{H}$ est donnée explicitement par le morphisme :

$$\begin{aligned} ad : N_G(H) &\longrightarrow Aut(\widehat{H},\nu) \\ k &\longmapsto ad_k \end{aligned}$$

tel que $ad_k(t)(h) = t(khk^{-1})$ où $k \in N_G(H)$, $t \in \widehat{H}$ et $h \in H$. Cette action correspond à la restriction au sous-groupe $N_G(H) < N_{L(G)}(L(H))$ du morphisme Θ considéré dans les sections précédentes. On note $\mathcal{N}$ la relation d'équivalence orbitale associée à cette action.

Dans le théorème 3.6.6, on montre de façon élémentaire que

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{N}.$$

En particulier, ceci induit par la remarque 3.4.4 que l'algèbre de von Neumann générée par le groupe $N_G(H)$ est égale à l'algèbre de von Neumann générée par le groupe $N_{L(G)}(L(H))$, i.e.

$$N_G(H)'' = N_{L(G)}(L(H))''.$$

0.2.4 Questions ouvertes, pistes de recherche

Chapitre 1

Quelle est la classe d'isomorphie du facteur II_1 $M_{\mathcal{P}}$ pour une algèbre planaire non coloriée ? Connaît-on deux algèbres planaires $\mathcal{P}_1$,$\mathcal{P}_2$ tel que $M_{\mathcal{P}_1}$ soit isomorphe à $M_{\mathcal{P}_2}$ et tel que les sous-algèbres générées par cup soient non isomorphes ?
Dans le cas où l'algèbre $\mathcal{P}$ est l'algèbre des polynômes non commutatifs en l variables $\mathbb{C}\langle X_1, \cdots, X_l\rangle$, est-ce que la sous-algèbre générée par cup est isomorphe à la sous-algèbre radiale du facteur du groupe libre $L(\mathbb{F}_l)$?
Même question pour la sous-algèbre maximale abélienne engendrée par un générateur $L(\mathbb{Z}) \subset L(\mathbb{F}_l)$?

Chapitre 2

Existe-t-il une classe naturelle de facteur II_1 qui admette une sous-algèbre abélienne avec la AOP ? Même question avec une sous-algèbre abélienne maximale hyperfinie.
Que se passe-t-il si l'on change la trace de l'algèbre graduée $Gr(\mathcal{P})$? Dans quels cas peut-on construire une algèbre de von Neumann ? Dans quels cas obtient-on un facteur II_1 ? A-t-on des algèbres de von Neumann avec la propriété de Haagerup ?
Considérons une algèbre planaire non-coloriée $\mathcal{P}$. Est-ce que la sous-algèbre générée par cup $A_{\mathcal{P}} \subset M_{\mathcal{P}}$ est toujours maximale hyperfinie ?

Chapitre 3

A-t-on les même résultats pour des facteurs de type III ?
Même question lorsqu'il y a une espérance conditionnelle sur la MASA.
Voir si l'on peut montrer des résultats de solidité forte en utilisant la relation d'équivalence de Takesaki. Par exemple Jesse Peterson a posé cette question qui est de savoir s'il y a une démonstration simple qui montre que la relation d'équivalence de Takesaki d'une MASA dans un facteur de groupe libre est hyperfinie.
Exploiter en théorie des représentations le fait que la relation d'équivalence de Takesaki est équivalente à la relation d'équivalence faible de Takesaki.

Chapitre 1

Unshaded planar algebras and a construction of a II_1 factor

Introduction

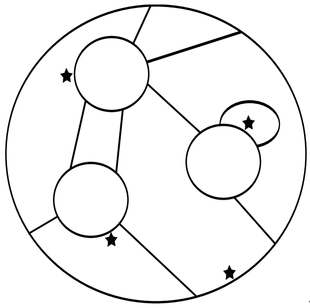
The modern theory of subfactors has been initiated by Jones [24]. He introduced the standard invariant that has been formalized as a λ-lattice by Popa [47] and as a subfactor planar algebra by Jones [28], (see also Peters [42] for an introduction to planar algebras). Popa [46, 47, 49] proved that any standard invariant comes from a subfactor, Popa and Shlyakhtenko proved [52] that the subfactor can be realized in the infinite free group factor $L(\mathbb{F}_\infty)$. Guionnet et al. [20, 21] gave a planar proof of this result and show, in the finite depth case, that the subfactor can be realized in an interpolated free group factor $L(\mathbb{F}_t)$, see the definition of an interpolated free group factor by Dykema and Radulescu [13, 55], where t is a function of the Jones index and the global index. Note that those results have been partially proved independently by Sunder and Kodiyalam in [33, 34].

In [29], the authors associate a tower of II_1 factors $\{M_k, k \geqslant 0\}$ to any subfactor planar algebra $\mathcal{P}$. We will consider the construction of the first von Neumann algebra M_0. The aim of this paper is to give a generalization of a subfactor planar algebra and provide an analogue of the construction of [29] for them. Then we prove that we still obtain a II_1 factor as in the subfactor planar algebra's case.

This generalization of the notion of subfactor planar algebra has been motivated by the following example: Consider the space of non commutative complex polynomials over l variables and of even degree $\mathbb{C}_{even}\langle X_1, \cdots ,X_l\rangle$. It has a planar algebra structure by following the tensor rules, see Jones [28][Example 2.6]. We want to extend this planar algebra structure for all the polynomials $\mathbb{C}\langle X_1, \cdots ,X_l\rangle$. In order to do that we need a larger set of planar tangles that act on the planar algebra. We define an unshaded planar tangle that can have an odd or even number of strings coming from a disk. If there is an odd number of boundary points on a disk, there is no hope in trying to shade it. For this reason, we call those planar tangles the unshaded planar tangles. We give here a definition similar to the one given by Peters [42] for the shaded case:

Definition 1.0.10. An unshaded planar tangle has an outer disk, a finite number of inner disks, and a finite number of non-intersecting strings. A string can be either a closed loop or an edge with endpoints on boundary circles. We require that there be a marked point (denoted by $\star$) on the boundary of each disk, and that the inner disks are ordered.

Here is an example of an unshaded planar tangle:



We will suppose that two tangles are equal if they are isotopic. We can compose them by placing a tangle inside an interior disk of another lining up the marked points, and connecting endpoints of strands. To do this we need to have the same number of endpoints, and we match the distinguished intervals.

By considering the action of the planar tangles on the subfactor planar algebra $\mathbb{C}_{even}\langle X_1, \cdots, X_l\rangle$, one can define in an obvious way the action of the class of unshaded planar tangles on the algebra $\mathbb{C}\langle X_1, \cdots, X_l\rangle$.

By mimicking the construction given in [29], we defined a multiplication $\star$ on $\mathbb{C}\langle X_1, \cdots, X_l\rangle$ which is:

$$\begin{aligned} X_{i_1} \cdots X_{i_n} \star X_{j_1} \cdots X_{j_m} &= X_{i_1} \cdots X_{i_n} X_{j_1} \cdots X_{j_m} \\ &+ \delta_{i_n, j_1} X_{i_1} \cdots X_{i_{n-1}} X_{j_2} \cdots X_{j_m} \\ &+ \delta_{i_n, j_1} \delta_{i_{n-1}, j_2} X_{i_1} \cdots X_{i_{n-2}} X_{j_3} \cdots X_{j_m} + \cdots, \end{aligned}$$

where δ_{ij} is the Kronecker symbol. For example,

$$X_1 X_2 X_3 \star X_3 X_2 = X_1 X_2 X_3^2 X_2 + X_1 X_2^2 + X_1.$$

Be aware that $\star$ is used in two different contexts, for a multiplication and to indicate a particular segment on a disk in a tangle.

Then, using a GNS construction, we provide a von Neumann algebra which is isomorphic to the free group factor $L(\mathbb{F}_l)$ with l generators, and $\{X_1, \cdots, X_l\}$ is a free semicircular family in the sense of Voiculescu [39].

Notation 1.0.11. If G is a countable discrete group, the group von Neumann algebra associated to G is denoted by $L(G)$.

Note that this construction corresponds to the free gaussian functor of Voiculescu [62] that sends a real Hilbert space $\mathcal{H}_{\mathbb{R}}$ to the free group factor with $\dim \mathcal{H}_{\mathbb{R}}$ generators: $L(\mathbb{F}_{\dim \mathcal{H}_{\mathbb{R}}})$.

In this paper, our objective is to show that there exists a natural generalization of subfactor planar algebras on which the collection of unshaded planar tangles acts. We call them the unshaded planar algebras. We show that we can still construct a von Neumann algebra for them and obtain a II_1 factor as in the subfactor planar algebra's case.

Note that an unshaded planar algebra can be seen as a generalization of the space $\mathbb{C}\langle X_1, \cdots, X_l\rangle$ from a free probability point of view. Furthermore, the construction of von Neumann algebra from an unshaded planar algebra is a generalization of the free gaussian functor.

Here is a precise description of the paper.
In section 1.1, we give a definition of an unshaded planar algebra $\mathcal{P}$. It is a countable family

of finite dimensional complex vector spaces $\{\mathcal{P}_n, n \geqslant 0\}$ on which the set of unshaded planar tangles is acting and following a few extra axioms. We choose the axioms in such a way that any subfactor planar algebra is an unshaded planar algebra, where for any $k \geqslant 0$, the space $\mathcal{P}_{2k+1}$ is equal to the null vector space. The algebra of non commutative polynomials is an example of an unshaded planar algebra.

We prove that we can associate a von Neumann algebra $M_{\mathcal{P}}$ to an unshaded planar algebra $\mathcal{P}$.

The von Neumann algebra M_0 in the article of Jones et al. [29] corresponds to the von Neumann algebra $M_{\mathcal{P}}$.

The principal difficulty is to show that the multiplication is bounded, see proposition 1.1.4. It is done by doing a graphical proof. One can embed the algebraic direct sum of the vector spaces $\mathcal{P}_n$, which we denote by

$$Gr(\mathcal{P}) = \bigoplus_{n \geqslant 0} \mathcal{P}_n,$$

in the von Neumann algebra $M_{\mathcal{P}}$. We identify this algebraic direct sum and its image in $M_{\mathcal{P}}$. We consider the trace of the unshaded planar algebra that can be extend as a faithful normal state tr on $M_{\mathcal{P}}$. In particular $M_{\mathcal{P}}$ is a finite von Neumann algebra. We denote the product of this von Neumann algebra by $\star$ in reference to the multiplication of the algebra $Gr(\mathcal{P})$.

In the classical case, where $\mathcal{P}$ is a subfactor planar algebra, it is proved in [29] that $M_{\mathcal{P}}$ is a II_1 factor. The rest of this article is devoted to prove that $M_{\mathcal{P}}$ is a II_1 factor for any unshaded planar algebra $\mathcal{P}$.

To do this, we proceed as in [29], we look at the cup subalgebra $A \subset M_{\mathcal{P}}$ introduced in section 1.2. The cup subalgebra $A \subset M_{\mathcal{P}}$ is an abelian von Neumann subalgebra generated by a self-adjoint element of $\mathcal{P}_2$. We call this element cup and denote it by the symbol $\cup$.

The main theorem of this paper is:

Theorem 1.0.12. *Let $\mathcal{P}$ be an unshaded planar algebra, then $M_{\mathcal{P}}$ is a II_1 factor and $A \subset M_{\mathcal{P}}$ is a maximal abelian subalgebra.*

Let us explain how we prove it, but first we recall what we mean by a bimodule.

Definition 1.0.13. In all this paper, a bimodule is a Hilbert space $\mathcal{K}$ with two normal representations of the von Neumann algebra A, $\tilde{\pi}$,$\tilde{\rho} : A \longrightarrow \mathcal{B}(\mathcal{K})$ where $\mathcal{B}(\mathcal{K})$ is the von Neumann algebra of bounded linear operator on $\mathcal{K}$. We require that $\tilde{\pi}(A)$ commutes with $\tilde{\rho}(A)$. If the context is clear, we will write $a.\xi.b$ instead of $\tilde{\pi}(a)\tilde{\rho}(b)(\xi)$ for a,$b \in A$ and $\xi \in \mathcal{K}$. We will write ${}_A\mathcal{K}_A$ to indicate that we consider $\mathcal{K}$ as a bimodule.

Two bimodules ${}_A\mathcal{K}_A$ and ${}_A\mathcal{H}_A$ are isomorphic if there exists a unitary transformation $u : \mathcal{K} \longrightarrow \mathcal{H}$, such that for any vector $\xi \in \mathcal{K}$ and elements a_1,$a_2 \in A$, we have that $u(a_1.\xi.a_2) = a_1.u(\xi).a_2$.

Consider the GNS Hilbert space $L^2(M_{\mathcal{P}})$ coming from the faithful trace tr. We identify the von Neumann algebra $M_{\mathcal{P}}$ and its image in $L^2(M_{\mathcal{P}})$. We denote the left and right action of $M_{\mathcal{P}}$ on this Hilbert space by π and ρ, hence

$$\pi(x)\rho(y)(z) = x \star z \star y,$$

for x,y,$z \in M_{\mathcal{P}}$. By restricting the action π and ρ to the von Neumann algebra A, we get a structure of bimodule on $L^2(M_{\mathcal{P}})$. If $S \subset L^2(M_{\mathcal{P}})$ is a subset, we define the bimodule generated by S. It is the smallest sub-Hilbert space of $L^2(M_{\mathcal{P}})$ that contains S and is stable by the actions of A. We equip this Hilbert space with a structure of bimodule by restricting the actions of A on this Hilbert space. We denote this bimodule by ${}_A\overline{S}_A$. For example, let $L^2(A) \subset L^2(M_{\mathcal{P}})$ be the closure of A in $L^2(M_{\mathcal{P}})$, it is a Hilbert space that is stable by the action of A. Hence it is a bimodule.

Consider the Hilbert space of complex sequences that are square summable $\ell^2(\mathbb{N})$. Note that we take the convention that $0 \in \mathbb{N}$. Let s be the shift operator on $\ell^2(\mathbb{N})$, meaning that $s(e_n) = e_{n+1}$, where $\{e_n, n \in \mathbb{N}\}$ is the standard basis of the Hilbert space $\ell^2(\mathbb{N})$. Consider the tensor product of Hilbert spaces $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$, it has a structure of bimodule which is:

$$\frac{\cup - 1}{\delta^{\frac{1}{2}}}.(\xi_1 \otimes \xi_2) = (s + s^*)(\xi_1) \otimes \xi_2$$

and

$$(\xi_1 \otimes \xi_2).\frac{\cup - 1}{\delta^{\frac{1}{2}}} = \xi_1 \otimes (s + s^*)(\xi_2),$$

where ξ_1,$\xi_2 \in \ell^2(\mathbb{N})$. In proposition 1.2.2, we will show that this bimodule is well defined. We call this bimodule the coarse bimodule.

Here is the strategy to prove the main result of this paper that is that $M_{\mathcal{P}}$ is a II_1 factor. We consider the Hilbert subspace of $L^2(M_{\mathcal{P}})$ equal to the orthogonal complement of $L^2(A)$, i.e. $L^2(M_{\mathcal{P}}) \ominus L^2(A)$. It is stable by the actions of A, hence is a bimodule. We will show that this bimodule is isomorphic to a direct sum of the coarse bimodule. Then, in theorem 1.3.4 we show that this implies that $M_{\mathcal{P}}$ is a II_1 factor and the cup subalgebra is maximal abelian. Here is a more detailed explanation of the proof.

We decompose the bimodule ${}_AL^2(M_{\mathcal{P}})_A$. We consider a subspace $V \subset L^2(M_{\mathcal{P}})$ defined in section 1.2.1 and denote by ${}_A\overline{V}_A$ the bimodule generated by V. Let ${}_A\overline{\mathcal{P}_1}_A$ be the bimodule generated by the 1-box space. We prove in proposition 1.2.1 that the bimodule ${}_AL^2(M_{\mathcal{P}})_A$ is equal to the direct sum

$${}_AL^2(A)_A \oplus {}_A\overline{\mathcal{P}_1}_A \oplus {}_A\overline{V}_A.$$

Following the proof of [29][theorem 4.9.] we have that ${}_A\overline{V}_A$ is isomorphic to a direct sum of coarse bimodules, see proposition 1.2.2.

Hence, to conclude, we need to show that ${}_A\overline{\mathcal{P}_1}_A$ is isomorphic to a direct sum of coarse bimodules, this is the main difficulty of this paper.

It is done in section 1.2.2. To do this we suppose there exists a non null vector $b \in \mathcal{P}_1$ and look at the bimodule generated by b that we denote by ${}_A\overline{b}_A$. We want to prove that ${}_A\overline{b}_A$ is isomorphic to the coarse bimodule. We begin by giving an orthonormal basis of the Hilbert space ${}_A\overline{b}_A$ in lemma 1.2.3. Then we construct a unitary transformation

$$\eta_b : {}_A\overline{b}_A \longrightarrow \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$$

in proposition 1.2.4.

Notation 1.0.14. If $\mathcal{K}$ is a Hilbert space, we denote the identity function on $\mathcal{K}$ by $1_{\mathcal{K}}$, or simply 1 if the context is clear.

We compute the action of A on the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ and get that

$$\eta_b \pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\eta_b^* = (s + s^*) \otimes 1 + \alpha$$

and

$$\eta_b \rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\eta_b^* = 1 \otimes (s + s^*),$$

where α is a given operator on the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. Therefore, the bimodule ${}_A\overline{b}_A$ is isomorphic to the coarse bimodule if and only if there exists a unitary

$$u \in \mathcal{B}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$$

that commute with the operator $1 \otimes (s+s^*)$ and satisfies that

$$u(\alpha + (s+s^*) \otimes 1)u^* = (s+s^*) \otimes 1.$$

Here is an explanation of how we proceed to prove that there exists such a unitary.

Notation 1.0.15. Let $\mathcal{K}$ be a Hilbert space and $\mathcal{B}(\mathcal{K})$ the space of bounded operators. If $S \subset \mathcal{B}(\mathcal{K})$ is a subset, we denote its commutant in the algebra $\mathcal{B}(\mathcal{K})$ by S' and its bicommutant by S''.

We consider the abelian von Neumann algebra generated by the operator $1 \otimes (s+s^*)$ which is $D = \{1 \otimes (s+s^*)\}'' \subset \mathcal{B}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$. We decompose the Hilbert space in tensor product such that D is the diagonal algebra. Let us recall some definitions of functional calculus and distribution:

Definition 1.0.16. Let $a \in \mathcal{B}(\ell^2(\mathbb{N}))$ be a self-adjoint operator acting on the Hilbert space $\ell^2(\mathbb{N})$, and $f \in \mathcal{C}(\mathbb{R})$ a continuous function from the real line to the complex plane. We denote the operator of $\mathcal{B}(\ell^2(\mathbb{N}))$ obtain by functional calculus by $f(a)$. Furthermore, we have a linear functional

$$\begin{aligned} \mathcal{C}(\mathbb{R}) &\longrightarrow \mathbb{C} \\ f &\longmapsto \langle f(a)(e_0), e_0 \rangle, \end{aligned}$$

which defines a measure on $\mathbb{R}$ supported on the spectrum of a. We call it the distribution of a.

Note that this corresponds to the notion of distribution given in [39], where the non commutative probability space is $(\mathcal{B}(\ell^2(\mathbb{N})), \psi)$ with the linear functional $\psi(a) = \langle a(e_0), e_0 \rangle$.

We compute the distribution of the operator $s+s^*$ in proposition 1.2.5, this is already known because $\frac{s+s^*}{2}$ is a semicircular element in the sense of Voiculescu [39]. We find that the spectrum of $s+s^*$ is equal to $[-2;2]$ and denote its distribution by ν, which is absolutely continuous with respect to the Lebesgue measure and equal to

$$d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi} dt.$$

We define a unitary transformation

$$\eta_\nu : \ell^2(\mathbb{N}) \longrightarrow L^2([-2;2], \nu),$$

such that $\eta_\nu (s+s^*) \eta^*$ acts as the multiplication by the identity function on $[-2;2]$. We conjugate the von Neumann algebra generated by the operator $1_{\ell^2(\mathbb{N})} \otimes (s+s^*)$ with the diagonal algebra acting on the Hilbert space $\ell^2(\mathbb{N}) \otimes L^2([-2;2], \nu)$ via the unitary transformation $1_{\ell^2(\mathbb{N})} \otimes \eta_\nu$.

The operator

$$c := (1_{\ell^2(\mathbb{N})} \otimes \eta_\nu) \circ (\alpha + (s+s^*) \otimes 1_{\ell^2(\mathbb{N})}) \circ (1_{\ell^2(\mathbb{N})} \otimes \eta_\nu)^*$$

is commuting with the diagonal algebra. Hence, it can be decomposed in a measurable operator field $\{c_t, t \in [-2;2]\}$, where $c_t \in \mathcal{B}(\ell^2(\mathbb{N}))$. We give an explicit decomposition of c in proposition 1.2.6. Then we can write c as a direct integral of operators:

$$c = \int_{[-2;2]}^{\oplus} c_t d\nu(t).$$

It means that if we identify the Hilbert space $\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$ with the Hilbert space of measurable functions

$$\begin{aligned} \xi : [-2;2] &\longrightarrow \ell^2(\mathbb{N}) \\ t &\longmapsto \xi_t \end{aligned}$$

such that

$$\int_{[-2;2]} \|\xi_t\|^2 d\nu(t) < \infty,$$

we have that

$$c(\xi) = \int_{[-2;2]}^{\oplus} c_t(\xi_t) d\nu(t).$$

We notice that c_t is a finite rank perturbation of the operator $s+s^*$, for any $t \in [-2;2]$. We prove in lemma 1.2.8 that $s+s^*$ and c_t are unitarily equivalent. To do this, we use two spectral analysis results, the so called, Weyl-von Neumann theorem and the Kato-Rosenblum theorem. Hence, by a result of Dixmier [11][Lemma 2,Chap.II,§2], one can find a <u>measurable</u> operator field of unitaries $\{u_t, t \in [-2;2]\}$ such that the decomposable unitary

$$u = \int_{[-2;2]}^{\oplus} u_t d\nu(t)$$

conjugates the operators c and $(s+s^*) \otimes 1$. Hence, the unitary u is the one that we were looking for. This unitary commutes with the von Neumann algebra D, because it is decomposable. Furthermore, the unitary u conjugates the two operators $(s+s^*) \otimes 1$ and $\alpha + (s+s^*) \otimes 1$. Hence, the bimodule ${}_A\overline{b}_A$ is isomorphic to the coarse bimodule.

This implies that the bimodule ${}_A\overline{\mathcal{P}_1}_A$ is isomorphic to a direct sum of coarse bimodules. Therefore, the bimodule $L^2(M_{\mathcal{P}}) \ominus L^2(A)$ is isomorphic to a direct sum of coarse bimodules. We conclude in theorem 1.2.11 that $M_{\mathcal{P}}$ is a II_1 factor and the cup subalgebra $A \subset M_{\mathcal{P}}$ is a maximal abelian subalgebra.

In the appendix, we compute the tower of von Neumann algebras associated to an unshaded planar algebra $\mathcal{P}$. We get that the standard invariant given by this tower is equal to the subfactor planar algebra $\tilde{\mathcal{P}}$, where $\tilde{\mathcal{P}} = \{\mathcal{P}_{2n}, n \in \mathbb{N}\}$.

Then, we look at the theory of maximal abelian subalgebras (MASAs). We discuss about a conjecture on MASAs in the free group factor. We explain why the cup subalgebra is involved in this conjecture.

1.1 A von Neumann algebra associated to an unshaded planar algebra

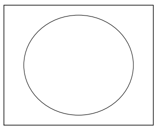
1.1.1 Definition of an unshaded planar algebra

Definition 1.1.1. An unshaded planar algebra $\mathcal{P}$ is a family of finite dimensional complex vector spaces $\{\mathcal{P}_n\}_{n \geqslant 0}$, called the n-box spaces. For any $n \geqslant 0$, there is an anti-linear involution $* : \mathcal{P}_n \longrightarrow \mathcal{P}_n$. Any unshaded planar tangle defines a linear map $\mathcal{P}_{n_1} \otimes ... \otimes \mathcal{P}_{n_k} \to \mathcal{P}_{n_0}$. The natural number n_i corresponds to the number of endpoints on the ith interior disk and n_0 to number of endpoints on the exterior disk. The action of the collection of unshaded planar tangle is compatible with the composition of tangles.

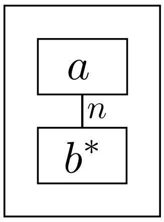
We require that $\mathcal{P}$ is spherically invariant and the action of the tangles is compatible with the anti-linear involutions $*$ of the $\mathcal{P}_n$, i.e.

$$T(b_{n_1}, \ldots, b_{n_i})^* = T^*({b_{n_1}}^*, ..., {b_{n_i}}^*)$$

for any vectors $b_{n_i} \in \mathcal{P}_{n_i}$ and any tangle T, where T^* is the reflection of T for any line in the plane. We denote the modulus of the planar algebra $\mathcal{P}$ by δ, which is the value of a close loop

$\delta =$ 

and suppose that $\delta > 1$. We assume that $\mathcal{P}$ is non degenerate, i.e. for any $n \geqslant 0$, the sesquilinear form $\langle \cdot, \cdot \rangle$ defined on each $\mathcal{P}_n$ by

$\langle a, b \rangle =$ 

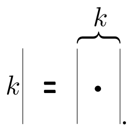
is an inner product of $\mathcal{P}_n$.

Remark 1.1.2. A subfactor planar algebra is an unshaded planar algebra where all the odd box spaces $\mathcal{P}_{2k+1} = \{0\}$.

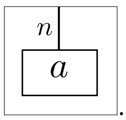
In all the paper, a planar algebra will denote an unshaded planar algebra.

1.1.2 Setup

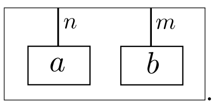
We follow the setup of [29]. Let $\mathcal{P} = (\mathcal{P}_n)_{n \geqslant 0}$ be an unshaded planar algebra. Let $Gr(\mathcal{P})$ be the graded vector space equal to the algebraic direct sum of the vector spaces $\mathcal{P}_n$, i.e. $Gr(\mathcal{P}) = \bigoplus_{n \geqslant 0} \mathcal{P}_n$. We extend the inner product of each $\mathcal{P}_n$ on $Gr(\mathcal{P})$ making it an orthogonal direct sum. We still write $\mathcal{P}_n$ when it is considered as the n-graded part of $Gr(\mathcal{P})$. To simplify the pictures, as in the article of Kodiyalam and Sunder [33] we decorate strands in a planar tangle with non-negative integers to represent cabling of that strand. For example :



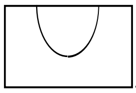
An element $a \in \mathcal{P}_n$ will be represent as a box:



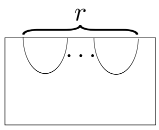
We assume that the distinguished first interval is at the top left of the box. If not we will denote this distinguished first interval by $\star$. We define a multiplication on $Gr(\mathcal{P})$ by requiring that if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$, then $a \bullet b \in \mathcal{P}_{n+m}$ is given by

$a \bullet b =$ 

Consider the element of $\mathcal{P}_2$:

.

We call it cup and denote it by the symbol $\cup$. The element

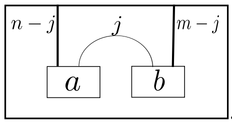


is denoted by the symbol $\cup^{\bullet r}$. For example, $\cup^{\bullet 2} = \cup \bullet \cup$. We use the convention that $0 = \cup^{\bullet k}$ for $k < 0$ and $1 = \cup^{\bullet 0}$. In particular, $a \bullet \cup^{\bullet r} = 0$ if $r \leqslant -1$, for any a.

1.1.3 Construction of a finite von Neumann algebra

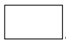
We present a generalization of the construction described in [29].

We equip the graded vector space $Gr(\mathcal{P})$ with the product $\star$ described by the following planar tangle:

$$a \star b = \sum_{j=0}^{min(n,m)}$$ ,

where $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. The $*$-structure on $Gr(\mathcal{P})$ is the involution coming from the planar algebra.

Proposition 1.1.3. *The algebraic structure $(Gr(\mathcal{P}), \star, *)$ is a unital involutive complex algebra. The unity of $Gr(\mathcal{P})$ is the empty diagram:*

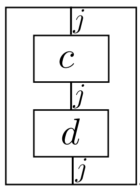
$$1 =$$ .

Proof. The proof is the same that in the subfactor planar algebra's case, see [29][proposition 3.2.]. ☐

We define a trace on $Gr(\mathcal{P})$ by the formula $tr(a) = \langle a,1 \rangle$ so that the trace of an element is its zero-graded piece. The inner product $\langle a,b \rangle$ is clearly equal to $tr(ab^*)$ and is positive definite by definition of an unshaded planar algebra. Let $\mathcal{H}$ be the Hilbert space equal to the completion of $Gr(\mathcal{P})$ for the inner product $\langle \cdot,\cdot \rangle$. We denote its trace by $\| \cdot \|_{\mathcal{H}}$. We remark that the Hilbert space $\mathcal{H}$ is equal to the orthogonal direct sum $\bigoplus_{n=0}^{\infty} \mathcal{P}_n$. We prove in the next proposition that the left multiplication by elements of the graded vector space $Gr(\mathcal{P})$ is bounded for the prehilbert space structure.

Proposition 1.1.4. *If $a \in Gr(\mathcal{P})$, there exists a positive constant $C > 0$ such that, for any $b \in Gr(\mathcal{P})$, $\|a \star b\|_{\mathcal{H}} \leqslant C\|b\|_{\mathcal{H}}$.*

Proof. Let $j \geqslant 0$, and consider the vector space $\mathcal{P}_{2j}$. We equip $\mathcal{P}_{2j}$ with the product $\times$ defined as follows:

$$c \times d =$$ ,

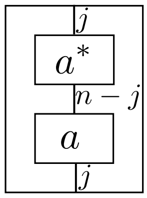
where $c,d \in \mathcal{P}_{2j}$. We equip $\mathcal{P}_{2j}$ with the involution $*$ coming from the planar algebra structure of $\mathcal{P}$. The tangle acts with respect to the $*$-structure of $\mathcal{P}_{2j}$ by definition of an unshaded planar algebra. Hence, for any $c,d \in \mathcal{P}_{2j}$ we have that $(c \times d)^* = d^* \times c^*$. Let $\| \cdot \|_{\mathcal{P}_{2j}}$ be the norm:

$$\|a\|_{\mathcal{P}_{2j}} = \sup\{\|a \times d\|_{\mathcal{H}}, d \in \mathcal{P}_{2j}, \|d\|_{\mathcal{H}} = 1\},$$

where $a \in \mathcal{P}_{2j}$. The $*$-algebra $(\mathcal{P}_{2j}, \times, *)$ with the norm $\| \cdot \|_{\mathcal{P}_{2j}}$ is a (finite dimensional) C^*-algebra.

Let $n \geqslant 0$ and $j \leqslant n$, consider an element $a \in \mathcal{P}_n$ and

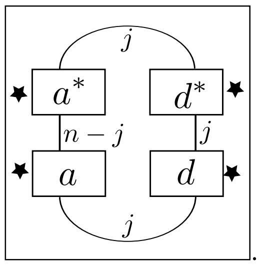
$\alpha_j =$



in $\mathcal{P}_{2j}$.

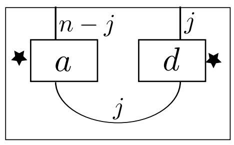
Let us show that α_j is a positive element of the C^*-algebra $\mathcal{P}_{2j}$. For this, we prove that for any $d \in \mathcal{P}_{2j}$, $\langle \alpha_j \times d, d \rangle \geqslant 0$.
We have that

$\langle \alpha_j \times d, d \rangle =$



If we denote

$\gamma_j =$



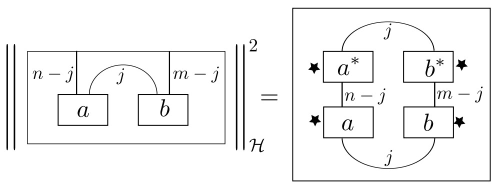
$\in \mathcal{P}_n$,

we have that

$$\langle \alpha_j \times d, d \rangle = \|\gamma_j\|_{\mathcal{H}}^2 \geqslant 0,$$

this tells us that α_j is positive.

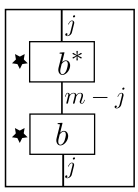
Let us show that there exists a positive constant $C > 0$, such that for any $m \geqslant 0$ and any vector $b \in \mathcal{P}_m$, we have that $\|a \times b\|_{\mathcal{H}} \leqslant C\|b\|_{\mathcal{H}}$. Let $j \leqslant \min(n,m)$, we have:



$= tr(\alpha_j \times \beta_j)$

where

$\beta_j =$



$\in \mathcal{P}_{2j}$.

We know that α_j and β_j are positive operators of $\mathcal{P}_{2j}$; thus, there exists two self-adjoint elements a_j,$b_j \in \mathcal{P}_{2j}$ such that $a_j \times a_j = \alpha_j$ and $b_j \times b_j = \beta_j$. If we look at the trace of the planar algebra $\mathcal{P}$:

$$\begin{aligned} tr(\alpha_j \times \beta_j) &= tr(a_j \times b_j \times b_j \times a_j) \\ &= \|a_j \times b_j\|_{\mathcal{H}}^2 \leqslant \|a_j\|_{\mathcal{P}_{2j}}^2 . \|b_j\|_{\mathcal{H}}^2 . \end{aligned}$$

Clearly, for any j we have that $\|b_j\|_{\mathcal{H}} = \|b\|_{\mathcal{H}}$. Hence,

$$\|a \star b\|_{\mathcal{H}} \leqslant (\sum_{0 \leqslant j \leqslant n} \|a_j\|_{\mathcal{P}_{2j}}) . \|b\|_{\mathcal{H}} .$$

Thus for any $m \geqslant 0$ and any $b \in \mathcal{P}_m$, $\|a \star b\|_{\mathcal{H}} \leqslant C \|b\|_{\mathcal{H}}$, where $C = \sum_{0 \leqslant j \leqslant n} \|a_j\|_{\mathcal{P}_{2j}}$. $\square$

We can define a representation of the $*$-algebra $Gr(\mathcal{P})$. Consider the left multiplication $\pi : Gr(\mathcal{P}) \longrightarrow \mathcal{B}(\mathcal{H})$, such that $\pi(a)(b) = a \star b$ for any $a \in Gr(\mathcal{P})$ and $b \in Gr(\mathcal{P})$. We write $M_{\mathcal{P}}$ the von Neumann algebra generated by $\pi(Gr(\mathcal{P}))$. We extend the representation to $M_{\mathcal{P}}$ and still denote it by π. The right multiplication is also bounded, this is giving us a representation of the opposite algebra: $\rho : M_{\mathcal{P}}^{\mathrm{op}} \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\rho(a)(b) = b \star a$.

Remark 1.1.5. The trace tr of the graded algebra $Gr(\mathcal{P})$ can be extend on the von Neumann algebra $M_{\mathcal{P}}$ with the formula $tr(a) = \langle a,1 \rangle$. It gives normal faithfull trace on $M_{\mathcal{P}}$ that we still denote by tr, hence $M_{\mathcal{P}}$ is a finite von Neumann algebra.

Consider the GNS representation of $M_{\mathcal{P}}$ on the Hilbert space $L^2(M_{\mathcal{P}})$ associated to the trace tr. This representation is conjugate with the representation $\pi : M_{\mathcal{P}} \longrightarrow \mathcal{B}(\mathcal{H})$. We identify those two representations. Furthermore, we identify the von Neumann algebra $M_{\mathcal{P}}$ and its dense image in the Hilbert space $L^2(M_{\mathcal{P}})$.

We remark that $\|a \bullet b\|_2 = \|a\|_2 \|b\|_2$ if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. By the triangle inequality, the bilinear function

$$\begin{aligned} Gr(\mathcal{P}) \times Gr(\mathcal{P}) &\longrightarrow Gr(\mathcal{P}) \\ (a,b) &\longmapsto a \bullet b \end{aligned}$$

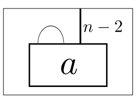
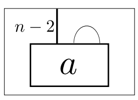
is continuous for the norm $\| \cdot \|_2$. We extend this operation on $L^2(M_{\mathcal{P}}) \times L^2(M_{\mathcal{P}})$ and still denote it by $\bullet$. In order to simplify the notations, we identify the *-algebra $Gr(\mathcal{P})$ with its image in the von Neumann algebra $M_{\mathcal{P}}$. We also identify $M_{\mathcal{P}}$ with its image in the Hilbert space $L^2(M_{\mathcal{P}})$.

1.2 The cup subalgebra

Let $A \subset M_{\mathcal{P}}$ be the abelian von Neumann algebra generated by the element cup $\cup \in \mathcal{P}_2$. We call it the cup subalgebra.

1.2.1 The bimodule structure of ${}_A L^2(M_{\mathcal{P}})_A$

Let $n \geqslant 2$ and V_n be the subspace of $\mathcal{P}_n$ of elements which vanish when a cap is placed at the top right and vanish when a cap is placed at the top left, i.e.

$V_n = \Big\{ a \in \mathcal{P}_n,$  $=$  $= 0 \Big\}.$

Let

$$V = \bigoplus_{n=2}^{\infty} V_n,$$

be the orthogonal direct sum in $L^2(M_{\mathcal{P}})$. We consider the bimodule generated by V that we denote by ${}_A\overline{V}_A$.

Let us write ${}_AL^2(M_{\mathcal{P}})_A$ as a direct sum of bimodules:

Proposition 1.2.1. *The bimodule ${}_AL^2(M_{\mathcal{P}})_A$ is isomorphic to the direct sum*

$${}_AL^2(A)_A \oplus {}_A\overline{\mathcal{P}_1}_A \oplus {}_A\overline{V}_A,$$

where ${}_A\overline{\mathcal{P}_1}_A$ is the bimodule generated by the 1*-box space* $\mathcal{P}_1$*.*

Proof. The subspace $L^2(A) \subset L^2(M_{\mathcal{P}})$ is a bimodule and, by definition, ${}_A\overline{\mathcal{P}_1}_A$ and ${}_A\overline{V}_A$ are subbimodules of $L^2(M_{\mathcal{P}})$. Hence to prove the proposition, it is sufficient to show that the Hilbert space $L^2(M_{\mathcal{P}})$ is equal to the orthogonal direct sum

$$L^2(A) \oplus {}_A\overline{\mathcal{P}_1}_A \oplus {}_A\overline{V}_A.$$

Consider the three closed vector subspaces E_1,E_2,E_3 of $L^2(M_{\mathcal{P}})$ where

- E_1 is spanned by the family of vectors $\{\cup^{\bullet k}, k \geqslant 0\}$,
- E_2 is spanned by the family of vectors $\{\cup^{\bullet r} \bullet b \bullet \cup^{\bullet k}, b \in \mathcal{P}_1, r,k \geqslant 0\}$ and
- E_3 is spanned by the family of vectors $\{\cup^{\bullet r} \bullet v \bullet \cup^{\bullet k}, r,k \geqslant 0, v \in V\}$.

Let us show that $E_1 = L^2(A)$. By definition of the space E_1, $\cup \in E_1$. It is easy to see that E_1 is a bimodule. Hence the bimodule generated by $\cup$, which is $L^2(A)$, is contained in E_1. For the converse inclusion, an easy induction on k shows that $\cup^{\bullet k} \in A$.

Let us show that $E_2 = {}_A\overline{\mathcal{P}_1}_A$. By definition of the space E_2, $\mathcal{P}_1 \subset E_1$. It is easy to see that E_2 is a bimodule. Hence the bimodule generated by $\mathcal{P}_1$, which is ${}_A\overline{\mathcal{P}_1}_A$, is contained in E_2. For the converse inclusion, an easy induction on r and k shows that $\cup^{\bullet r} \bullet b \bullet \cup^{\bullet k} \in {}_A\overline{\mathcal{P}_1}_A$ for any $b \in \mathcal{P}_1$.

Let us show that $E_3 = {}_A\overline{V}_A$. By definition, $V \subset E_3$ and clearly E_3 is stable by the actions of $\cup$, hence is a bimodule. Therefore, ${}_A\overline{V}_A \subset E_3$. For the converse inclusion, fix a $v \in V$. An easy induction show that for any $r,k \geqslant 0$, $\cup^{\bullet r} \bullet v \bullet \cup^{\bullet k} \in {}_A\overline{V}_A$.

Consider the Hilbert space equal to the sum $E = E_1 + E_2 + E_3$. Let us show that $E = L^2(M_{\mathcal{P}})$. To do this, we show that $\mathcal{P}_n \subset E$ for any $n \geqslant 0$. We proceed by induction on n.

By definition, $\mathcal{P}_0 \subset E_1$ and $\mathcal{P}_1 \subset E_2$. Consider $n \geqslant 2$ and suppose that $\mathcal{P}_{n-1}$ and $\mathcal{P}_{n-2}$ are include in E. Let us denote the orthogonal of V_n inside $\mathcal{P}_n$ by W_n. By [29][Lemma 4.5.], we have that W_n is the space spanned by element that can be written $y \bullet \cup$ and $\cup \bullet z$, where $y,z \in \mathcal{P}_{n-2}$. It is easy to see that for any $x \in E$, we have that $\cup \bullet x \in E$ and $x \bullet \cup \in E$. So $W_n \subset E$; thus, by definition of E_3, $V_n \subset E$. So $\mathcal{P}_n \subset E$, we have proved that E is a dense subspace of $L^2(M_{\mathcal{P}})$.

Let us show that the E_i are pairwise orthogonal. Let $n \geqslant 2$, $v \in V_n$, $b \in \mathcal{P}_1$, and $k,l,r,m \geqslant 0$. Consider the vectors $\cup^{\bullet r} \bullet v \bullet \cup^{\bullet k}$ and $\cup^{\bullet l} \bullet b \bullet \cup^{\bullet m}$. They are in the vector spaces $\mathcal{P}_{2(r+k)+n}$ and $\mathcal{P}_{2(l+m)+1}$. The spaces $\{\mathcal{P}_n, n \geqslant 0\}$ are pairwise orthogonal by definition of the inner product on the graded vector space $Gr(\mathcal{P})$. So $\cup^{\bullet r} \bullet v \bullet \cup^{\bullet k}$ and $\cup^{\bullet l} \bullet b \bullet \cup^{\bullet m}$ are orthogonal if $2(r + k) + n \neq 2(l + m) + 1$. Suppose $2(r + k) + n = 2(l + m) + 1$, by hypothesis $n \geqslant 2$, hence $l > r$ or $m > k$, in any case, v will get a cap at the top right or left in the inner product $\langle \cup^{\bullet r} \bullet v \bullet \cup^{\bullet k}, \cup^{\bullet l} \bullet b \bullet \cup^{\bullet m}\rangle$. Therefore, they are orthogonal and so are E_2 and E_3.

Let $r,k,l \geqslant 0$, $n \geqslant 2$ and $v \in V_n$. Consider the inner product $\langle \cup^{\bullet r} \bullet v \bullet \cup^{\bullet k}, \cup^{\bullet l}\rangle$. It is equal to 0 if $2r+2k+n \neq 2l$ by the orthogonality of the spaces P_i. Suppose $2r+2k+n = 2l$, we have that n is an even number and

$$\langle \cup^{\bullet r} \bullet v \bullet \cup^{\bullet k}, \cup^{\bullet l}\rangle = \delta^{r+k}\langle v, \cup^{\bullet \frac{n}{2}}\rangle.$$

This must be equal to 0 because v will get a cap at the top left in this inner product; thus, E_1 is orthogonal to E_3.

The space E_1 is contained in the orthogonal direct sum $\bigoplus_n \mathcal{P}_{2n}$, and E_2 is contained in the orthogonal direct sum $\bigoplus_n \mathcal{P}_{2n+1}$, therefore $E_1 \perp E_2$. Hence, the vector spaces E_i are in direct sum and their sum is equal to the Hilbert space $L^2(M_{\mathcal{P}})$. We have proved that $L^2(M_{\mathcal{P}})$ is equal to the orthogonal direct sum

$$L^2(A) \oplus {}_A\overline{\mathcal{P}_1}_A \oplus {}_A\overline{V}_A.$$

$\square$

Proposition 1.2.2. *The map*

$$\eta_V : {}_A\overline{V}_A \to \ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})$$
$$\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet v \bullet \cup^{\bullet r} \mapsto e_k \otimes v \otimes e_r$$

defined a unitary transformation such that

$$\eta_V \pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\eta_V^* = (s+s^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} \text{ and} \tag{1.1}$$

$$\eta_V \rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\eta_V^* = 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (s+s^*). \tag{1.2}$$

In other words, the map $\frac{\cup-1}{\delta^{\frac{1}{2}}} \longmapsto s+s^*$ *defines a normal faithful representation of the von Neumann algebra A on the Hilbert space $\ell^2(\mathbb{N})$ and the bimodule ${}_A\overline{V}_A$ is isomorphic to a direct sum of the coarse bimodule.*

Proof. This is the same proof that the one given in [29][theorem 4.9.] for a subfactor planar algebra. $\square$

This proposition is telling us that to understand the bimodule structure of ${}_AL^2(M_{\mathcal{P}})_A$ is the same that to understand the bimodule structure of ${}_A\overline{\mathcal{P}_1}_A$.

1.2.2 The bimodule generated by the 1-box space

We suppose, in this section, that $\mathcal{P}_1 \neq \{0\}$. Let us fix an element $b \in \mathcal{P}_1$ such that $\|b\|_2 = 1$. We denote the bimodule generated by b by ${}_A\overline{b}_A$.

Lemma 1.2.3. *The set*

$$E_b = \{\delta^{-\frac{r}{2}} b \bullet \cup^{\bullet r}, r \geqslant 0\} \cup \{\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}, k,r \geqslant 0\}$$

is an orthonormal basis of the Hilbert space ${}_A\overline{b}_A$, where

$$Z_b = \frac{\cup \bullet b - \delta^{-1} b \bullet \cup}{\sqrt{\delta - \delta^{-1}}}.$$

Proof. Let us prove that E_b is a family of unit vectors. Note that

$$\|a \bullet b\|_2 = \|a\|_2 \|b\|_2.$$

This implies that

$$\|\delta^{-\frac{r}{2}} b \bullet \cup^{\bullet r}\|_2 = \delta^{-\frac{r}{2}} \|b\|_2 \| \cup \|_2^r = \|b\|_2 = 1,$$

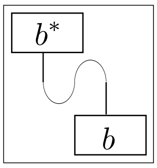
and

$$\|\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}\|_2 = \|Z_b\|_2.$$

Let us compute $\|Z_b\|_2$: We have that

$$\|Z_b\|_2^2 = \frac{1}{\delta - \delta^{-1}} (\| \cup \bullet b\|_2^2 + \delta^{-2} \|b \bullet \cup\|_2^2 - \delta^{-1} \langle \cup \bullet b, b \bullet \cup \rangle - \delta^{-1} \langle b \bullet \cup, \cup \bullet b \rangle).$$

The inner product

$$\langle \cup \bullet b, b \bullet \cup \rangle =$$  $$= \|b\|_2^2 = 1.$$

Hence, $\langle b \bullet \cup, \cup \bullet b \rangle = \overline{\langle \cup \bullet b, b \bullet \cup \rangle} = 1$; thus, $\|Z_b\|_2 = 1$. We have proved that E_b is a family of unit vectors.

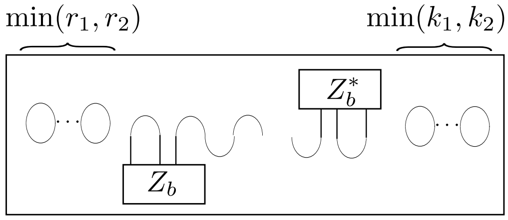
Let us show that E_b is an orthogonal family. Consider the family of vectors

$$\{\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}, k, r \geqslant 0\}.$$

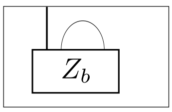
Let $k, r, k_1, r_1, k_2, r_2 \geqslant 0$ be some natural numbers. The spaces $\mathcal{P}_n$ are pairwise orthogonal and $\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r} \in \mathcal{P}_{2(k+r)+3}$, thus

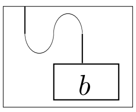
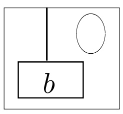
$$\langle \cup^{\bullet r_1} \bullet Z_b \bullet \cup^{\bullet k_1}, \cup^{\bullet r_2} \bullet Z_b \bullet \cup^{\bullet k_2} \rangle = 0 \text{ if } k_1 + r_1 \neq k_2 + r_2.$$

Suppose $k_1 + r_1 = k_2 + r_2$, and $r_1 \neq r_2$,

$$\langle \cup^{\bullet r_1} \bullet Z_b \bullet \cup^{\bullet k_1}, \cup^{\bullet r_2} \bullet Z_b \bullet \cup^{\bullet k_2} \rangle =$$ 

We have that



$$= \frac{1}{\sqrt{\delta - \delta^{-1}}} \Bigg($$  $$- \frac{1}{\delta}$$  $$\Bigg) = 0.$$

Thus,

$$\langle \cup^{\bullet r_1} \bullet Z_b \bullet \cup^{\bullet k_1}, \cup^{\bullet r_2} \bullet Z_b \bullet \cup^{\bullet k_2} \rangle = 0.$$

Hence, the inner product $\langle \cup^{\bullet r_1} \bullet Z_b \bullet \cup^{\bullet k_1}, \cup^{\bullet r_2} \bullet Z_b \bullet \cup^{\bullet k_2} \rangle$ is non null if and only if $r_1 = r_2$ and $k_1 = k_2$. This is telling us that $\{\cup^{\bullet r} \bullet Z_b \bullet \cup^{\bullet k}, r, k \geqslant 0\}$ is an orthogonal family and then an orthonormal family of vectors.

Let us show that $\{b \bullet \cup^{\bullet r}, r \geqslant 0\}$ is an orthogonal family. The elements $b \bullet \cup^{\bullet r_1}$ and $b \bullet \cup^{\bullet r_2}$ belong to $\mathcal{P}_{1+2r_1}$ and $\mathcal{P}_{1+2r_2}$, so they are orthogonal if $r_1 \neq r_2$.

Let us show that the two sets

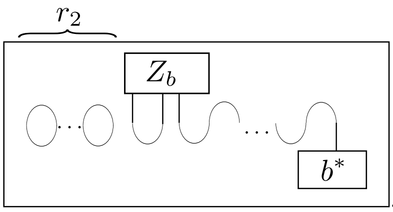
$$\{\delta^{-\frac{r}{2}} b \bullet \cup^{\bullet r}, r \geqslant 0\}$$

and

$$\{\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}, k,r \geqslant 0\}$$

are orthogonal. Consider $b \bullet \cup^{\bullet r_1}$ and $\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r_2}$. The vector $b \bullet \cup^{\bullet r_1}$ belongs to $\mathcal{P}_{2r_1+1}$ and $\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r_2} \in \mathcal{P}_{2(k+r_2)+3}$. These are orthogonal if $r_1 \neq k + r_2 + 1$. Supposing $r_1 = k + r_2 + 1$, we get that $r_1 \geqslant r_2 + 1$. Thus,

$$\langle \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r_2}, b \bullet \cup^{\bullet r_1} \rangle =$$



This is equal to zero because Z_b has a cap on the top right.

We have shown that E_b is an orthonormal family.

Let us show that the closed vector space spanned by E_b, that we denote by X_b, is equal to ${}_A\overline{b}_A$. We see clearly that X_b is stable by left and right multiplication by $\cup$, hence X_b is a bimodule. The space X_b contains b, therefore it contains the bimodule generated by b which is ${}_A\overline{b}_A$.
Let us show that X_b is contained in ${}_A\overline{b}_A$, which is equivalent to show that E_b is contained in ${}_A\overline{b}_A$. We remark that X_b is spanned by the family of vectors $\{\cup^{\bullet r} \bullet b \bullet \cup^{\bullet k}, r,k \geqslant 0\}$. An easy induction on r and k shows that $\cup^{\bullet r} \bullet b \bullet \cup^{\bullet k} \in {}_A\overline{b}_A$ for any r,k. $\square$

Proposition 1.2.4. *Consider the operator $\eta_b : {}_A\overline{b}_A \to \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ defined as follows:*

$$\eta_b(\delta^{-\frac{r}{2}} b \bullet \cup^{\bullet r}) = e_0 \otimes e_r \text{ and}$$
$$\eta_b(\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}) = e_{k+1} \otimes e_r.$$

This operator η_b is a unitary transformation. Furthermore,

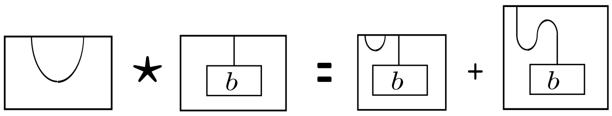
$$\eta_b \pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) \eta_b^* = \alpha + (s + s^*) \otimes 1 \text{ and} \tag{1.3}$$
$$\eta_b \rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) \eta_b^* = 1 \otimes (s + s^*), \tag{1.4}$$

where α is the operator of $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ defined as follows: for all $x \in \ell^2(\mathbb{N})$,

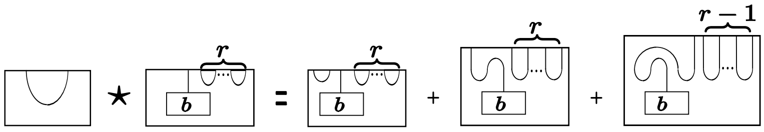
$$\alpha(e_0 \otimes x) = (\sqrt{1 - \delta^{-2}} - 1)(e_1 \otimes x) + \delta^{-1}(e_0 \otimes (s + s^*)(x))$$
$$\alpha(e_1 \otimes x) = (\sqrt{1 - \delta^{-2}} - 1)(e_0 \otimes x)$$
$$\alpha(e_k \otimes x) = 0 \text{ if } k \geqslant 2.$$

Proof. We have proved that E_b is an orthonormal basis of the Hilbert space ${}_A\overline{b}_A$; thus, the operator η_b sends an orthonormal basis to an other one. Hence η_b is a unitary transformation.

Let us show that η_b satisfies the equality 1.3: Let us look at the left action of cup. Let us compute $\cup \star (b \bullet \cup^{\bullet r})$:
If $r = 0$,



If $r \geqslant 1$,



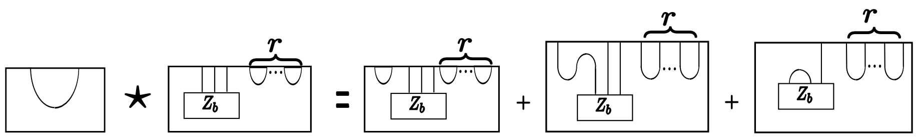
Thus for any $r \geqslant 0$,

$$\cup \star (b \bullet \cup^{\bullet r}) = \sqrt{\delta - \delta^{-1}}(Z_b \bullet \cup^{\bullet r}) + \delta^{-1}(b \bullet \cup^{\bullet r+1}) + (b \bullet \cup^{\bullet r}) + (b \bullet \cup^{\bullet r-1}).$$

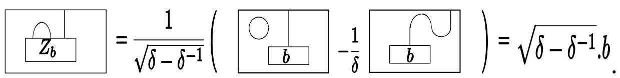
So

$$(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) \star (\delta^{-\frac{r}{2}} b \bullet \cup^{\bullet r}) = \sqrt{1 - \delta^{-2}}(\delta^{-\frac{r}{2}} Z_b \bullet \cup^{\bullet r}) + \delta^{-1}(\delta^{-\frac{r+1}{2}} b \bullet \cup^{\bullet r+1}) + \delta^{-1}(\delta^{-\frac{r-1}{2}} b \bullet \cup^{\bullet r-1}).$$

Let us compute $\cup \star (\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r})$:
If $k = 0$,



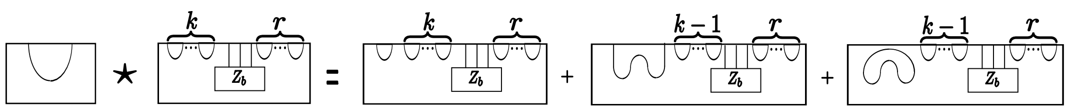
and



So,

$$(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) \star (\delta^{-\frac{r}{2}} Z_b \bullet \cup^{\bullet r}) = \delta^{-\frac{r+1}{2}}(\cup \bullet Z_b \bullet \cup^{\bullet r}) + \sqrt{1 - \delta^{-2}}(\delta^{-\frac{r}{2}} b \bullet \cup^{\bullet r}).$$

If $k \geqslant 1$,



Thus,

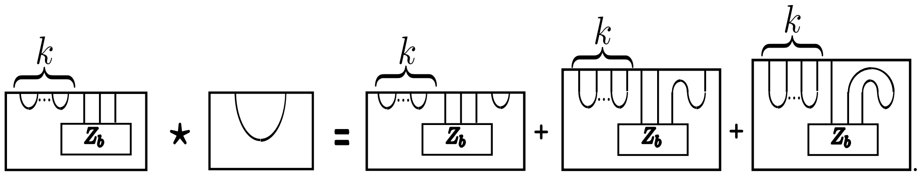
$$(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) \star (\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}) = (\delta^{-\frac{k+1+r}{2}} \cup^{\bullet k+1} \bullet Z_b \bullet \cup^{\bullet r}) + (\delta^{-\frac{k-1+r}{2}} \cup^{\bullet k-1} \bullet Z_b \bullet \cup^{\bullet r}),$$

where $k \geqslant 1$.

Let us look at the right action of cup:
Let us compute $(b \bullet \cup^{\bullet r}) \star \cup$:
If $r = 0$,

$$b \star \cup = (b \bullet \cup) + b.$$

If $r \geqslant 1$,
$$(b \bullet \cup^{\bullet r}) \star \cup = (b \bullet \cup^{\bullet r+1}) + (b \bullet \cup^{\bullet r}) + \delta(b \bullet \cup^{\bullet r-1}).$$
Let us compute $(\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}) \star \cup$:
If $r = 0$,



We have seen in the proof of lemma 1.2.3 that Z_b vanishes if it is capped off on the right. Thus,
$$(\cup^{\bullet k} \bullet Z_b) \star \cup = (\cup^{\bullet k} \bullet Z_b \bullet \cup) + (\cup^{\bullet k} \bullet Z_b).$$
If $r \geqslant 1$,
$$(\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}) \star \cup = (\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r+1}) + (\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}) + \delta(\cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r-1})$$
Thus for any $k \geqslant 0, r \geqslant 0$,
$$\begin{aligned}(\delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r}) \star (\frac{\cup - 1}{\delta^{\frac{1}{2}}}) &= \delta^{-\frac{k+r+1}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r+1} \\ &+ \delta^{-\frac{k+r-1}{2}} \cup^{\bullet k} \bullet Z_b \bullet \cup^{\bullet r-1}.\end{aligned}$$
$\square$

We want to show that the bimodules ${}_A\overline{b}_A$ is isomorphic to the coarse bimodule. By proposition 1.2.4, this is equivalent to saying that there exists a unitary u of the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ such that
$$u(\alpha + (s+s^*) \otimes 1)u^* = (s+s^*) \otimes 1 \quad \text{and}$$
$$u(1 \otimes (s+s^*))u^* = 1 \otimes (s+s^*).$$
So we are looking to a unitary u that conjugates $\alpha + (s+s^*) \otimes 1$ and $(s+s^*) \otimes 1$, and that commutes with $1 \otimes (s+s^*)$. Let us consider the abelian von Neumann algebra
$$D = \{1 \otimes (s+s^*)\}'' \subset \mathcal{B}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$$
generated by the operator $1 \otimes (s+s^*)$. It is equal to the von Neumann algebra $D = \mathbb{C}.1 \otimes D_0$, where $D_0 = \{s+s^*\}'' \subset \mathcal{B}(\ell^2(\mathbb{N}))$. Let us look at the distribution of the operator $s+s^*$.
Proposition 1.2.5. *The operator*
$$\frac{s+s^*}{2}$$
is semicircular in the sense of Voiculescu [39]. In particular, the distribution of $s+s^*$ is absolutely continuous with respect to the Lebesgue measure, is supported in $[-2;2]$ and equal to:
$$d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi} dt.$$
We have a unitary transformation $\eta_\nu : L^2([-2;2],\nu) \longrightarrow \ell^2(\mathbb{N})$ defined on the dense subspace of continuous functions $\mathcal{C}([-2;2])$ by $f \in \mathcal{C}([-2;2]) \longmapsto f(s+s^*)(e_0)$. This satisfies
$$\eta_\nu(s+s^*)\eta_\nu^*(f)(t) = tf(t) \tag{1.5}$$

for any $f \in \mathcal{C}([-2;2])$ *and* $t \in [-2;2]$. *Furthermore,* $\eta_\nu(P_n) = e_n$, *for all* $n \geqslant 0$ *where* $\{P_n\}_n$ *is the family of polynomials defined as follows:*

$$\begin{array}{l} P_0(X) = 1 \\ P_1(X) = X \\ P_n(X) = XP_{n-1}(X) - P_{n-2}(X), \textit{ for any } n \geqslant 2 \end{array},$$

where X *is an indeterminate.*

Proof. To show that the distribution of $s + s^*$ is $d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi}dt$, see [39][Example 3.4.2]. The equality 1.5 is obvious by definition of η_ν.

We prove that $\eta_\nu(P_n) = e_n$ by induction on n.

It is clear for $n = 0$.
For $n = 1$: $(s + s^*)(e_0) = e_1$, so it is true for $n = 1$.
Let $n \geqslant 2$ and suppose the result true for $n-1$ and $n-2$. We have:

$$\begin{aligned} P_n(s+s^*)(e_0) &= (s+s^*)P_{n-1}(s+s^*)(e_0) - P_{n-2}(s+s^*)(e_0) \\ &= (s+s^*)(e_{n-1}) - e_{n-2} = e_n + e_{n-2} - e_{n-2} \\ &= e_n. \end{aligned}$$

By definition, η_ν is an isometry and we just proved that the orthonormal basis $\{e_n, n \geqslant 0\}$ is in the image of η_ν. Hence, η_ν is surjective; thus, it is a unitary transformation. $\square$

The Hilbert space $\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$ is the constant field of the Hilbert space $\ell^2(\mathbb{N})$ over the probability space $([-2;2],\nu)$. We identify it with the Hilbert space of measurable functions $\xi : [-2;2] \longrightarrow \ell^2(\mathbb{N})$ that are square-integrable. We denote such a vector of $\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$ by the direct integral

$$\xi = \int_{[-2;2]}^{\oplus} \xi_t d\nu(t),$$

where $\xi_t \in \ell^2(\mathbb{N})$. A bounded measurable operator field $\{b_t, t \in [-2;2]\}$ defined a decomposable operator that we denote by

$$b = \int_{[-2;2]}^{\oplus} b_t d\nu(t).$$

It acts in the following way:

$$b(\xi) = \int_{[-2;2]}^{\oplus} b_t(\xi_t) d\nu(t).$$

We recall that the vector space of decomposable operators is a von Neumann algebra equal to the commutant of the diagonal algebra in $\mathcal{B}(\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu))$.

The unitary transformation

$$1 \otimes \eta_\nu : \ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu) \longrightarrow \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$$

conjugates the von Neumann algebra D and the diagonal algebra associates to this decomposition. The two operators $\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})$ and $\rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})$ commute, this implies that the operator

$$c := (1 \otimes \eta_\nu)(\alpha + (s+s^*) \otimes 1)(1 \otimes \eta_\nu)^*$$

commutes with the diagonal algebra, hence it is a decomposable operator. We give in the next proposition an explicit decomposition of the operator c.

Proposition 1.2.6. *The operator c is equal to the direct integral*

$$c = \int_{[-2;2]}^{\oplus} c_t d\nu(t),$$

where c_t acts on the Hilbert space $\ell^2(\mathbb{N})$ and is equal to:

$$c_t = \begin{pmatrix} \frac{t}{\delta} & \sqrt{1-\delta^{-2}} & 0 & 0 & 0 & 0 & ... \\ \sqrt{1-\delta^{-2}} & 0 & 1 & 0 & 0 & 0 & ... \\ 0 & 1 & 0 & 1 & 0 & 0 & ... \\ 0 & 0 & 1 & 0 & 1 & 0 & ... \\ 0 & 0 & 0 & 1 & 0 & 1 & ... \\ ... & ... & ... & ... & ... & ... & ... \end{pmatrix}$$

in the orthogonal standard basis of the Hilbert space $\ell^2(\mathbb{N})$.

Proof. Let us show that $\{c_t, t \in [-2;2]\}$ is a bounded measurable operator field. For this we have to show that $t \mapsto c_t$ is measurable for the Borel σ-algebra generated by the operator strong topology on $\mathcal{B}(\ell^2(\mathbb{N}))$ and that the set $\{\|c_t\|, t \in Y\}$ is bounded in $\mathbb{R}$.

Let $t \in [-2;2]$, the operator c_t is a finite rank perturbation of $s+s^*$, so it is a bounded operator. Consider the function from the compact interval $[-2;2]$ to the space of bounded linear operators $\mathcal{B}(\ell^2(\mathbb{N}))$:

$$\begin{aligned} [-2;2] &\longrightarrow \mathcal{B}(\ell^2(\mathbb{N})) \\ t &\longmapsto c_t. \end{aligned}$$

This function is clearly continuous for the norm topology on $\mathcal{B}(\ell^2(\mathbb{N}))$. This implies that its image is compact and then bounded in $\mathcal{B}(\ell^2(\mathbb{N}))$. This function is measurable for the Borel σ-algebra generated by the norm topology on $\mathcal{B}(\ell^2(\mathbb{N}))$ and then is measurable for the σ-algebra generated by the operator strong topology. This implies that the family $\{c_t, t \in [-2;2]\}$ is a bounded measurable field of operators.

Consider the decomposable operator

$$d = \int_{[-2;2]}^{\oplus} c_t d\nu(t)$$

acting on the Hilbert space $\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$. Let us show that the two operators c and d are equal, i.e.

$$c = \int_{[-2;2]}^{\oplus} c_t d\nu(t).$$

Let $\text{span}\{e_n, n \geqslant 0\}$ be the subspace of $\ell^2(\mathbb{N})$ spanned by the standard basis and let $\mathcal{C}([-2;2]) \subset L^2([-2;2],\nu)$ be the subspace of continuous functions. The algebraic tensor product of those vectors spaces,

$$\text{span}\{e_n, n \geqslant 0\} \otimes \mathcal{C}([-2;2]),$$

is a dense subspace of $\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$. Let us show that c and d coincide on this subspace.

Consider the operator

$$c - (s+s^*) \otimes 1 = (1 \otimes \eta_\nu) \circ \alpha \circ (1 \otimes \eta_\nu^*) \in \mathcal{B}(\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu))$$

and a vector $f \in \mathcal{C}([-2;2])$. Let us compute the vector $(c-(s+s^*)\otimes 1)(e_k \otimes f)$:
For $k=0$,

$$\begin{aligned}(c-(s+s^*)\otimes 1)(e_0\otimes f) &= (1\otimes \eta_\nu)(\alpha(e_0\otimes \eta_\nu^*(f)))\\ &= (1\otimes \eta_\nu)[(\sqrt{1-\delta^{-2}}-1)(e_1\otimes \eta_\nu^*(f))\\ &+\delta^{-1}e_0\otimes((s+s^*)(\eta_\nu^*(f)))]\\ &= (\sqrt{1-\delta^{-2}}-1)(e_1\otimes f)+\delta^{-1}(e_0\otimes(\eta_\nu(s+s^*)\eta_\nu^*)(f)).\end{aligned}$$

For $k=1$,

$$(c-(s+s^*)\otimes 1)(e_1\otimes f) = (\sqrt{1-\delta^{-2}}-1)(e_0\otimes f).$$

For any $k \geqslant 2$,

$$(c-(s+s^*)\otimes 1)(e_k\otimes f) = 0.$$

On the other hand let us compute

$$(d-(s+s^*)\otimes 1)(e_k\otimes f):$$

For $k=0$,

$$\begin{aligned}(d-(s+s^*)\otimes 1)(e_0\otimes f) &= \left(\int_{[-2;2]}^{\oplus}(c_t-s-s^*)d\nu(t)\right)\left(\int_{[-2;2]}^{\oplus}f(t)e_0 d\nu(t)\right)\\ &= \int_{[-2;2]}^{\oplus} f(t)(c_t-s-s^*)(e_0)d\nu(t)\\ &= \int_{[-2;2]}^{\oplus}(f(t)\frac{t}{\delta}e_0+f(t)(\sqrt{1-\delta^{-2}}-1)e_1)d\nu(t)\\ &= \delta^{-1}(e_0\otimes \eta_\nu(s+s^*)\eta_\nu^*(f))+(\sqrt{1-\delta^{-2}}-1)(e_1\otimes f)\\ &= (c-(s+s^*)\otimes 1)(e_0\otimes f).\end{aligned}$$

For $k=1$,

$$\begin{aligned}(d-(s+s^*)\otimes 1)(e_1\otimes f) &= \int_{[-2;2]}^{\oplus} f(t)(c_t-s-s^*)(e_1)d\nu(t)\\ &= \int_{[-2;2]}^{\oplus}(f(t)(\sqrt{1-\delta^{-2}}-1)e_0)d\nu(t)\\ &= (\sqrt{1-\delta^{-2}}-1)(e_0\otimes f)\\ &= (c-(s+s^*)\otimes 1)(e_1\otimes f).\end{aligned}$$

Let $k \geqslant 2$,

$$\begin{aligned}(d-(s+s^*)\otimes 1)(e_k\otimes f) &= \int_{[-2;2]}^{\oplus} f(t)(c_t-s-s^*)(e_k)d\nu(t)\\ &= \int_{[-2;2]}^{\oplus}(0)d\nu(t)=0\\ &= (c-(s+s^*)\otimes 1)(e_k\otimes f).\end{aligned}$$

The two operators

$$c \text{ and } \int_{[-2;2]}^{\oplus} c_t d\nu(t)$$

coincide on the dense subspace

$$\text{span}\{e_n, n \geqslant 0\}) \otimes \mathcal{C}([-2;2])$$

of $\ell^2(\mathbb{N}) \otimes L^2([-2;2],\nu)$; thus, they are equal. $\square$

Let us show that c_t is unitarily equivalent to $s+s^*$: Before proving it, let us recall some definitions and basic facts on spectral theory.

Definition 1.2.7. Let $\mathcal{H}$ be a Hilbert space with a inner product $\langle \cdot,\cdot \rangle$ and $a \in \mathcal{B}(\mathcal{H})$ a self-adjoint operator acting on $\mathcal{H}$. We denote the spectrum of a by $\sigma(a)$. The essential spectrum of a is the complement of the set of isolated eigenvalues of finite multiplicity in $\sigma(a)$. We denote it by $\sigma_{ess}(a)$.

Suppose that a is a self-adjoint operator. For any vector $\xi \in \mathcal{H}$, one can associate a Radon measure μ_ξ on the spectrum of a that is $\mu_\xi(f) = \langle f(a)\xi,\xi \rangle$ for any continuous functions on $\sigma(a)$, $f \in \mathcal{C}(\sigma(a))$. Let $\mathcal{H}_{ac}$ be the Hilbert space of vectors $\xi \in \mathcal{H}$ such that the measure μ_ξ is absolutely continuous with respect to the Lebesgue measure. Let $\mathcal{H}_{sc}$ be the Hilbert space of vectors $\xi \in \mathcal{H}$ such that the measure μ_ξ is singular with respect to the Lebesgue measure. Let $\mathcal{H}_{pp}$ be the Hilbert space of vectors $\xi \in \mathcal{H}$ such that the measure μ_ξ is a pure point measure. We have that $\mathcal{H} = \mathcal{H}_{ac} + \mathcal{H}_{sc} + \mathcal{H}_{pp}$ but the sum is not direct in general. We define the absolutely continuous spectrum of a by the spectrum of the operator a restricted to the Hilbert space $\mathcal{H}_{ac}$. We denote it by $\sigma_{ac}(a)$. We define the singular spectrum and the pure point spectrum in the similar way. We denote them by $\sigma_{sc}(a)$ and by $\sigma_{pp}(a)$.

The operator $a \in \mathcal{B}(\mathcal{H})$ is said to have uniform multiplicity equal to 1 if there exists a measure μ on the spectrum of a and a unitary transformation $w : \mathcal{H} \longrightarrow L^2(\sigma(a),\mu)$ such that for any function $f \in L^2(\sigma(a),\mu)$, $waw^*(f)(z) = zf(z)$, where $z \in \sigma(a)$. In that case, the spectrums $\sigma_{ac}(a)$, $\sigma_{sc}(a)$ and $\sigma_{pp}(a)$ form a partition of the spectrum of a. See [32] for more details.

Lemma 1.2.8. *Let $t \in [-2;2]$, then c_t is unitarily equivalent to $s+s^*$.*

Proof. Let us fix $t \in [-2;2]$, and consider the operator c_t. Let us show that the spectrum of the operator c_t, $\sigma(c_t)$, is equal to $[-2;2]$:

We remark that $k_t := c_t - (s+s^*)$ is a finite rank operator. The theorem of Weyl-von Neumann, see [32][p.523], shows that the essential spectrum of an operator acting on a Hilbert space is invariant under compact perturbation. So, the essential spectrum of c_t, $\sigma_{\text{ess}}(c_t)$, is equal to the essential spectrum of $s+s^*$, $\sigma_{\text{ess}}(s+s^*)$. The operator $s+s^*$ is semicircular, its spectrum is essential and equal to $[-2;2]$; thus, $\sigma_{\text{ess}}(c_t) = [-2;2]$. The complement of the essential spectrum inside the spectrum is equal to the set of isolated eigenvalues of finite multiplicity. It is called the discrete spectrum. Let us show that this complement is empty. The operator c_t is self-adjoint, so its spectrum is contained in the real line $\mathbb{R}$. If we show that c_t does not have any real eigenvalue with module strictly bigger than 2, we will have shown that the discrete spectrum is empty. Consider a real number $z > 2$ and $x = (x_n)_n \in \ell^2(\mathbb{N})$ such that $c_t(x) = zx$. For any $n \geqslant 1$, $x_n + x_{n+2} = zx_{n+1}$ the roots of the characteristic polynomials of this equation are

$$r = \frac{z - \sqrt{z^2-4}}{2}$$

and

$$l = \frac{z + \sqrt{z^2-4}}{2}.$$

Hence, there exists two complex numbers B,$C \in \mathbb{C}$, such that $x_n = Br^n + Cl^n$ for any $n \geqslant 1$. We notice that $|l| > 1$, and x is a square summable complex sequence; thus, $C = 0$. Hence, for any $n \geqslant 1$, $x_n = x_1 r^{n-1}$. The equality $c_t(x) = zx$ gives us the system of equations:

$$\begin{array}{c} \frac{t}{\delta}x_0 + \sqrt{1-\delta^{-2}}x_1 = zx_0 \\ \sqrt{1-\delta^{-2}}x_0 + x_2 = zx_1 \end{array}.$$

This implies that

$$[(1-\delta^{-2}) - (\frac{z-\sqrt{z^2-4}}{2} - z)(\frac{t}{\delta} - z)]x_1 = 0,$$

which means that $[(1-\delta^{-2}) - h(z)]x_1 = 0$ where

$$h(z) = (z - \frac{t}{\delta})\frac{z+\sqrt{z^2-4}}{2}.$$

Since the function h is strictly increasing on $\mathbb{R}_+$ and $h(2) \geqslant 2(1-\delta^{-1})$, we have

$$h(z) - (1-\delta^{-2}) > (\delta^{-1}-1)^2 > 0.$$

Hence $x_1 = 0$, that implies that for any $n \geqslant 1$, $x_n = 0$ and also that $x_0 = 0$. For the case where $z < -2$ it is the same proof where we replace z by $-z$. Hence, the spectrum of c_t, $\sigma(c_t) = \sigma_{\text{ess}}(c_t) = [-2; 2]$.

Let us show that c_t is of uniform multiplicity equal to 1. To do this, we show that the vector e_0 of the Hilbert space $\ell^2(\mathbb{N})$ is cyclic for the abelian von Neumann algebra generated by the self-adjoint operator c_t in $\mathcal{B}(\ell^2(\mathbb{N}))$. Consider the family of polynomials $\{S_{n,t}, n \geqslant 0\}$ defined as follows:

$$\begin{aligned} &S_{0,t}(X) = 1 \\ &S_{1,t}(X) = \frac{X - \frac{t}{\delta}}{\sqrt{1-\delta^{-2}}} \\ &S_{2,t}(X) = XS_{1,t}(X) - \sqrt{1-\delta^{-2}} \\ &S_{n,t}(X) = XS_{n-1,t}(X) - S_{n-2,t}(X), \text{ for any } n \geqslant 3, \end{aligned}$$

where X is an indeterminate.
Let us show that for any $n \geqslant 0$, $S_{n,t}(c_t)(e_0) = e_n$. We proceed by induction:
For $n = 0$ it is trivial.
For $n = 1$:

$$\begin{aligned} S_{1,t}(c_t)(e_0) &= \frac{1}{\sqrt{1-\delta^{-2}}}(c_t - \frac{t}{\delta}.1)(e_0) \\ &= \frac{1}{\sqrt{1-\delta^{-2}}}(\frac{t}{\delta}e_0 + \sqrt{1-\delta^{-2}}e_1 - \frac{t}{\delta}e_0) \\ &= e_1. \end{aligned}$$

For $n = 2$:

$$\begin{aligned} S_{2,t}(c_t)(e_0) &= c_t S_{1,t}(c_t)(e_0) - \sqrt{1-\delta^{-2}}e_0 = c_t(e_1) - \sqrt{1-\delta^{-2}}e_0 \\ &= \sqrt{1-\delta^{-2}}e_0 + e_2 - \sqrt{1-\delta^{-2}}e_0 \\ &= e_2. \end{aligned}$$

Let $n \geqslant 3$, we suppose the result is true for $n-1$ and $n-2$:

$$\begin{aligned} S_{n,t}(c_t)(e_0) &= c_t S_{n-1,t}(c_t)(e_0) - S_{n-2,t}(c_t)(e_0) \\ &= c_t(e_{n-1}) - e_{n-2}. \end{aligned}$$

We know that for any $k \geqslant 2$, $c_t(e_k) = e_{k-1} + e_{k+1}$, so $S_{n,t}(c_t)(e_0) = e_n$.
This show that the vector e_0 is a cyclic vector for the von Neumann algebra generated by c_t.

Consider the Radon measure ν^t defined as follows:

$$\int_{\mathbb{R}} f(z) d\nu^t(z) = \langle f(c_t)(e_0), e_0 \rangle,$$

for any continuous complex valued function $f : \mathbb{R} \to \mathbb{C}$. The measure ν^t is the distribution of the operator c_t and is supported on the spectrum of c_t that is $[-2;2]$. Consider the Hilbert space of square integrable functions $L^2([-2;2], \nu^t)$, we have an operator:

$$\begin{aligned} \eta_t : L^2([-2;2], \nu^t) &\longrightarrow \ell^2(\mathbb{N}) \\ f \in \mathcal{C}([-2;2]) &\longmapsto f(c_t)(e_0). \end{aligned}$$

This operator η_t is a unitary transformation by construction. It satisfies that

$$(\eta_t c_t \eta_t^*)(f)(z) = z f(z),$$

for any continuous function $f \in \mathcal{C}([-2;2])$ and for ν-almost everywhere $z \in [-2;2]$. Hence, c_t is of uniform multiplicity equal to 1.

This implies that the spectrum $\sigma(c_t)$ is equal to the disjoint union of its pure point spectrum $\sigma_{\text{pp}}(c_t)$, its singular spectrum $\sigma_{\text{sc}}(c_t)$, and its absolutely continuous spectrum $\sigma_{\text{ac}}(c_t)$.

Let us show that the spectrum of c_t is absolutely continuous. The theorem of Kato-Rosenblum [32][p. 540] says the following:
If a and b are self-adjoint operators acting on a Hilbert space $\mathcal{H}$, and a is trace class, then the absolutely continuous part of b and $a+b$ are unitarily equivalent.
The operator $k_t = c_t - (s + s^*)$ is of finite rank, so is trace class. Hence, the absolutely continuous part of c_t and $s + s^*$ are unitarily equivalent. In particular, the absolutely continuous spectrum of c_t and $s+s^*$ are equal. The operator $s+s^*$ is semicircular, so is equal to its absolutely continuous part. The spectrum of $s+s^*$ is equal to $[-2;2]$, so the absolutely continuous spectrum of c_t is equal to $[-2;2]$. This implies that $\sigma(c_t) = \sigma_{\text{ac}}(c_t) = [-2;2]$ and then $\sigma_{\text{pp}}(c_t) = \sigma_{\text{sc}}(c_t) = \emptyset$. Hence, c_t is equal to its absolutely continuous part.

Let us apply again the theorem of Kato-Rosenblum, we get that c_t and $s + s^*$ are unitarily equivalent.

□

Proposition 1.2.9. *There exists a unitary u acting on the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ that commutes with $1 \otimes (s+s^*)$ and conjugates $\alpha + (s+s^*) \otimes 1$ with $(s+s^*) \otimes 1$. Hence, the bimodule ${}_A\overline{b}_A$ is isomorphic to the coarse bimodule.*

Proof. By lemma 1.2.8, for any $t \in [-2;2]$, the operator c_t is unitarily equivalent to the operator $s + s^*$. By [11][Lemma 2,Chap.II,§2] there exists a measurable operator field of unitaries $\{u_t, t \in [-2;2]\}$ such that $u_t c_t u_t^* = s + s^*$. This defines a unitary operator

$$u = \int_{[-2;2]}^{\oplus} u_t d\nu(t),$$

such that u commutes with $1 \otimes (s + s^*)$ because it is decomposable. By construction $u(\alpha + (s+s^*) \otimes 1)u^* = (s+s^*) \otimes 1$.

Let us show that the bimodule ${}_A\overline{b}_A$ is isomorphic to the coarse bimodule. Consider the unitary transformation $\eta_b : {}_A\overline{b}_A \longrightarrow \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ given in proposition 1.2.4 and the unitary $u \in \mathcal{B}(\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}))$ that we just consider. Let $w = u \circ \eta_b$, it is a unitary transformation from the Hilbert space ${}_A\overline{b}_A$ into the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. Furthermore, by construction, it satisfies that

$$w\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})w^* = (s + s^*) \otimes 1$$

and

$$w\rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})w^* = 1 \otimes (s + s^*).$$

Hence, by definition, ${}_A\overline{b}_A$ is isomorphic to the coarse bimodule via the unitary transformation w. □

Consider ${}_A\overline{\mathcal{P}_1}_A$ the subbimodule of ${}_AL^2(M_{\mathcal{P}})_A$ generated by $\mathcal{P}_1$.

Corollary 1.2.10. *The bimodules ${}_A\overline{\mathcal{P}_1}_A$ is isomorphic to a direct sum of coarse bimodules.*

Proof. Let $\{b^i, i \in I\}$ be an orthonormal basis of $\mathcal{P}_1$, view as a subspace of the Hilbert space $L^2(M_{\mathcal{P}})$. The bimodule ${}_A\overline{\mathcal{P}_1}_A$ is isomorphic to the direct sum of bimodules:

$$\bigoplus_{i \in I} {}_A\overline{b^i}_A,$$

where ${}_A\overline{b^i}_A$ is the subbimodule of $L^2(M_{\mathcal{P}})$ generated by the vector b^i. By proposition 1.2.9 we have that the bimodules ${}_A\overline{b^i}_A$ is isomorphic to the coarse bimodule. Hence, ${}_A\overline{\mathcal{P}_1}_A$ is isomorphic to a direct sum of coarse bimodules. □

Theorem 1.2.11. *The bimodule ${}_AL^2(M_{\mathcal{P}})_A$ is isomorphic to the direct sum of the bimodule ${}_AL^2(A)_A$ and some copies of the coarse bimodule. In particular, $M_{\mathcal{P}}$ is a II_1 factor and $A \subset M_{\mathcal{P}}$ is a MASA.*

Proof. By proposition 1.2.2, the bimodule ${}_A\overline{V}_A$ is isomorphic to a direct sum of coarse bimodules and by proposition 1.2.9, the bimodule ${}_A\overline{\mathcal{P}_1}_A$ is isomorphic to a direct sum of coarse bimodules. By proposition 1.2.1, the bimodule ${}_AL^2(M_{\mathcal{P}})_A$ is isomorphic to the direct sum ${}_AL^2(A)_A \oplus {}_A\overline{\mathcal{P}_1}_A \oplus {}_A\overline{V}_A$. Therefore, the bimodule ${}_AL^2(M_{\mathcal{P}})_A$ is isomorphic to the direct sum of the bimodule ${}_AL^2(A)_A$ and a direct sum of coarse bimodules.

Let us show that $A \subset M_{\mathcal{P}}$ is a MASA. For this, it is sufficient to show that for any vector $\xi \in L^2(M_{\mathcal{P}})$ such that $\pi(\cup)\xi = \rho(\cup)\xi$ we have that $\xi \in L^2(A)$. Consider a vector in the orthogonal of $L^2(A)$, $\xi \in L^2(M_{\mathcal{P}}) \ominus L^2(A)$, and suppose that $\pi(\cup)\xi = \rho(\cup)\xi$. We have seen that the bimodule $L^2(M_{\mathcal{P}})$ is isomorphic to the direct sum of the bimodule ${}_AL^2(A)_A$ and some coarse bimodules. Hence, we can consider that ξ is a vector of the coarse bimodule $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ such that

$$((s + s^*) \otimes 1)(\xi) = (1 \otimes (s + s^*))(\xi). \tag{1.6}$$

Let $HS(\ell^2(\mathbb{N}))$ be the Hilbert space of Hilbert-Schmidt operators on the Hilbert space $\ell^2(\mathbb{N})$ and let $\Psi : \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N}) \longrightarrow HS(\ell^2(\mathbb{N}))$ be the anti-linear isomorphism defined such that

$$\Psi(x_1 \otimes x_2)(x) = \langle x, x_1 \rangle x_2,$$

where $x \in \ell^2(\mathbb{N})$ and $x_1 \otimes x_2 \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$. If we apply the transformation Ψ to the equation 1.6, we get that the two operators $\Psi(\xi)$ and $s + s^*$ commute. Suppose that ξ is a non null vector, then $\Psi(\xi)$ is a non null Hilbert-Schmidt operator. The Hilbert-Schmidt operators are compact operators. Hence $\Psi(\xi)$ acts by homothety on a non null finite dimensional vector space. Hence, the self-adjoint operator $s + s^*$ lets invariant a non null finite dimensional vector space. This implies that $s + s^*$ admits an eigenvalue, but the spectrum of $s + s^*$ is absolutely continuous, a contradiction. Therefore, $\xi = 0$ and so $A \subset M_{\mathcal{P}}$ is a MASA.

Let us show that the von Neumann algebra $M_{\mathcal{P}}$ is a factor. Let $a \in M_{\mathcal{P}}$ be an element in the center of $M_{\mathcal{P}}$, in particular, a commutes with A. Hence, $a \in A$ because $A \subset M_{\mathcal{P}}$ is a MASA. For any vector $\xi \in L^2(M_{\mathcal{P}})$, we have that $\pi(a)\xi = \rho(a)\xi$ because a commutes with $M_{\mathcal{P}}$. Consider the map $\frac{\cup - 1}{\delta^{\frac{1}{2}}} \longmapsto s + s^*$, it defines a faithful representation from the von Neumann algebra A on the Hilbert space $\ell^2(\mathbb{N})$, see proposition 1.2.2. Let $b \in \mathcal{B}(\ell^2(\mathbb{N}))$ be the image of a by this representation. The orthogonal of $L^2(A)$ is equal to a direct sum of coarse bimodules. Then, for any $x \otimes y \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$,

$$b(x) \otimes y = x \otimes b(y).$$

Therefore,

$$\|b(x) \otimes y\| = \|b(x)\|\|y\| = \|x \otimes b(y)\| = \|x\|\|b(y)\|.$$

Hence,

$$\|b(x) \otimes y\|^2 = \|b(x)\|\|b(y)\|\|x\|\|y\|.$$

On the other hand,

$$\begin{aligned}\|b(x) \otimes y\|^2 &= \langle b(x) \otimes y, x \otimes b(y)\rangle = \langle b(x), x\rangle\langle y, b(y)\rangle \\ &\leqslant \|b(x)\|\|b(y)\|\|x\|\|y\|,\end{aligned}$$

by the inequality of Cauchy-Schwartz. We notice that we are in the case of equality of Cauchy-Schwartz, hence the vectors $b(x)$ and x are homothetic. Therefore, b is an homothety, hence a is an homothety. □

1.3 Appendix

1.3.1 Subfactor planar algebra associated to an unshaded planar algebra

Consider an unshaded planar algebra $\mathcal{P} = \{\mathcal{P}_n, n \geqslant 0\}$. We have seen that we can associate to it a graded algebra $Gr(\mathcal{P}) = \bigoplus_{n \geqslant 0} \mathcal{P}_n$ and a II_1 factor $M_{\mathcal{P}}$ as in the subfactor planar algebra's case. Let us follow the article [29]. We define for any $k \geqslant 0$, a graded algebra $Gr_k(\mathcal{P}) = \bigoplus_{n \geqslant 0} \mathcal{P}_{n+2k}$. Then we can associate to $Gr_k(\mathcal{P})$ a von Neumann algebra M_k, it is a II_1 factor for the same reason that $M_{\mathcal{P}}$ is a II_1 factor. Hence, with those notations, $M_{\mathcal{P}} = M_0$ and $Gr(\mathcal{P}) = Gr_0(\mathcal{P})$. We obtain a tower of II_1 factors that is

$$M_0 \subset M_1 \subset \cdots M_k \subset \cdots$$

The question is, what is the subfactor planar algebra associated to the subfactor $M_0 \subset M_1$?

It is easy to see that the relative commutant $M_{k+l} \cap M_k'$ is equal to the direct sum of even box spaces

$$\bigoplus_{j=k+1}^{k+l} \mathcal{P}_{2j},$$

if $k \geqslant 0$ and $l \geqslant 1$. Therefore, the subfactor planar algebra associated to the subfactor $M_0 \subset M_1$ is equal to $\tilde{\mathcal{P}}$, where $\tilde{\mathcal{P}}$ is equal to the even box spaces of $\mathcal{P}$. It means that $\tilde{\mathcal{P}} = \{\mathcal{P}_{2n}, n \geqslant 0\}$.

Note that, if we apply the construction of the article [29] to the subfactor planar algebra $\tilde{\mathcal{P}}$, we get a tower of II_1 factors

$$\tilde{M}_0 \subset \tilde{M}_1 \subset \cdots \tilde{M}_k \subset \cdots$$

This provides a countable family of subfactors $\tilde{M}_k \subset M_k$ given by the inclusions $\bigoplus_n \mathcal{P}_{2k+2n} \subset \bigoplus_l \mathcal{P}_{2k+l}$.

1.3.2 Maximal abelian subalgebras

In this appendix, we review some definitions about maximal abelian subalgebras (MASAs). We present some invariants for those objects. We give a definition of coarse bimodule with respect to a finite von Neumann algebra. We prove a theorem that generalize the proof of the theorem 1.2.11 for diffuse abelian subalgebra of a finite von Neumann algebra. Then, we discuss a conjecture of MASAs in the free group factor that is relating to the cup subalgebra. Note that we stop using the symbol $\star$ for the multiplication in a von Neumann algebra. Let us review some definitions:

Definition 1.3.1. Let $A \subset M$ be a maximal abelian subalgebra (MASA). Two MASAs $A_1 \subset M_1$ and $A_2 \subset M_2$ are called isomorphic if there exists an isomorphism of von Neumann algebras $\phi : M_1 \longrightarrow M_2$ such that $\phi(A_1) = A_2$.

Group normalizer:

We define the group normalizer $N_M(A)$ that is the group of unitaries $u \in M$ such that $uAu^* = A$. If the von Neumann algebra generated by $N_M(A)$ is equal to A, the MASA is called singular. If the von Neumann algebra generated by $N_M(A)$ is equal to M, the MASA is called regular.

Suppose M is a II_1 factor with its unique faithful trace τ. Let $L^2(M)$ be the GNS Hilbert space associate to the trace τ.

Pukanszky invariant:

Consider the abelian von Neumann algebra $\mathcal{A}$ equal to the bicommutant

$$\{\pi(A),\rho(A)\}'' \subset \mathcal{B}(L^2(M)),$$

where π,ρ are the left and right action of M on the Hilbert space $L^2(M)$. Let P be the commutant of $\mathcal{A}$ acting on the orthogonal complement $L^2(M) \ominus L^2(A)$. The algebra P is a finite type I von Neumann algebra. Consider the subset of $n \in \mathbb{N} \cup \{\infty\}$ such that there exists a non null direct summand of type I_n in P. This set is the Pukanszky invariant of the MASA $A \subset M$, we denote it by $Puk(A \subset M)$.

Takesaki invariant:

Let $(Y,\mathcal{D},\nu)$ be a standard probability space such that A is isomorphic to the von Neumann algebra of bounded measurable complex valued functions $L^\infty(Y,\nu)$. Let π,ρ be the left and right action of M on the Hilbert space $L^2(M)$, i.e. $\pi(x)\rho(y)(z) = xzy$, where x,y,$z \in M$.

Consider a measurable field of Hilbert spaces $\{\mathcal{H}_t, t \in Y\}$ such that $L^2(M)$ is equal to the direct integral

$$\int_Y^{\oplus} \mathcal{H}_t d\nu(t),$$

such that $\rho(A)$ becomes the algebra of all diagonalizable operators. Let $B \subset M$ be a separable C^*-subalgebra that is dense for the ultraweak topology. Consider a measurable field of representations of B, $\{\pi_t, t \in Y\}$, such that

$$\pi|_B = \int_Y^{\oplus} \pi_t d\nu(t),$$

where $\pi|_B$ denotes the restriction to B of the standard representation. Let $\mathcal{R}$ be the equivalence relation on Y such that $(s,t) \in \mathcal{R}$ if and only if the representation π_s is unitarily equivalent to the representation π_t. Now let $\mathcal{R}$,$\mathcal{R}'$ be two equivalence relations on Y. We define an equivalence relation "$\equiv$" such that $\mathcal{R} \equiv \mathcal{R}'$ if and only if there exists a Borel null set $N \subset Y$ such that $\mathcal{R}\backslash N^2 = \mathcal{R}'\backslash N^2$. Let $\widehat{\mathcal{R}}$ be the equivalence class of $\mathcal{R}$ for "$\equiv$". It is the Takesaki invariant. We say that a MASA is Takesaki simple if $\widehat{\mathcal{R}}$ is equal to the equivalence class of the trivial equivalence relation.

Those definitions have been introduced by Dixmier [9], Pukanszky [53] and Takesaki [60].

Remark 1.3.2. The main result of chapter 3 proves that a MASA is singular if and only if it is simple. More generally, one can see the Takesaki invariant as the orbital equivalence relation of a countable discrete subgroup of the normalizer.

Definition 1.3.3. Let A be a finite von Neumann algebra with a faithful normal trace τ. Consider the GNS Hilbert space $L^2(A)$ associated to the trace τ. This Hilbert space has a natural structure of bimodule over A given by the multiplication on A, we denote this bimodule by ${}_AL^2(A)_A$. Let $\mathcal{K}$ be Hilbert space and consider the Hilbert space equal to the tensor product $L^2(A) \otimes \mathcal{K} \otimes L^2(A)$. We equip this Hilbert space with the following bimodule structure:

$$a_1.(b_1 \otimes \xi \otimes b_2).a_2 = (a_1a_2) \otimes \xi \otimes (b_2a_1),$$

where a_1,a_2,b_1,$b_2 \in A$ and $\xi \in \mathcal{K}$. We denote this bimodule by ${}_AL^2(A) \otimes \mathcal{K} \otimes L^2(A)_A$. If $\mathcal{K} = \mathbb{C}$ we denote the bimodule by ${}_AL^2(A) \otimes L^2(A)_A$ and call it the coarse bimodule associated to A.

Proposition 1.3.4. *Consider a finite von Neumann algebra M with a faithful trace τ and $A \subset M$ a diffuse abelian von Neumann subalgebra. Let $L^2(M)$ and $L^2(A)$ be the Hilbert spaces of the GNS construction associated to τ. Suppose there exists a Hilbert space $\mathcal{K}$ such that we have an isomorphism of bimodules:*

$${}_AL^2(M)_A \simeq {}_AL^2(A)_A \oplus ({}_AL^2(A) \otimes \mathcal{K} \otimes L^2(A)_A).$$

Then M is a II_1 factor, A is maximal abelian, singular, Takesaki simple, with Pukanszky invariant equal to the singleton $\{\dim \mathcal{K}\}$.

Proof. Consider $A \subset M$ and $\mathcal{K}$ as in the hypothesis of the proposition. Let us show that $A \subset M$ is maximal abelian. Let $(Y,\mathcal{D},\nu)$ be a standard probability space such that A is isomorphic to the von Neumann algebra $L^\infty(Y,\nu)$ of bounded, measurable, complex-valued functions. We have supposed that A is diffuse, hence ν is non-atomic. We denote by $\psi : A \longrightarrow L^\infty(Y,\nu)$ an isomorphisms of von Neumann algebras. It induces an isomorphisms of Hilbert spaces $\overline{\Psi} : L^2(A) \longrightarrow L^2(Y,\nu)$. Consider the Hilbert space $L^2(Y^2,\mathcal{K},\nu \otimes \nu)$ of

Borel measurable functions $f : Y^2 \longrightarrow \mathcal{K}$ that are square integrable for the product measure $\nu \otimes \nu$, i.e.

$$\int_{Y^2} \|f(s,t)\|_{\mathcal{K}}^2 d(\nu \otimes \nu)(s,t) < \infty.$$

We have a isomorphisms of Hilbert spaces defined as follows:

$$\begin{aligned} L^2(Y,\nu) \otimes \mathcal{K} \otimes L^2(Y,\nu) &\longrightarrow L^2(Y^2,\mathcal{K},\nu \otimes \nu) \\ g_1 \otimes \xi \otimes g_2 &\longmapsto ((s,t) \in Y^2 \mapsto g_1(s)g_2(t)\xi). \end{aligned}$$

The Hilbert space $L^2(Y^2,\mathcal{K},\nu\otimes\nu)$ as a structure of bimodule over the von Neumann algebra $L^\infty(Y,\nu)$ that is:

$$(f_1.h.f_2)(s,t) = f_1(s)f_2(t)h(s,t),$$

for any $f_1,f_2 \in L^\infty(Y,\nu)$ and any $h \in L^2(Y^2,\mathcal{K},\nu \otimes \nu)$. Consider the Hilbert space $L^2(Y,\nu)$, it has a structure of bimodule that is:

$$(f_1.h.f_2)(t) = f_1(t)f_2(t)h(t),$$

for any $f_1,f_2 \in L^\infty(Y,\nu)$ and any $h \in L^2(Y,\nu)$. We suppose those Hilbert spaces equipped with the bimodule structures that we just described. We have an isomorphism of bimodules:

$$\varphi : {}_A L^2(M)_A \longrightarrow L^2(Y,\nu) \oplus L^2(Y^2,\mathcal{K},\nu \otimes \nu), \tag{1.7}$$

where A is identified with $L^\infty(Y,\nu)$ and $L^2(A)$ with $L^2(Y,\nu)$.
Let π be the standard representation of M on $L^2(M)$ and ρ the standard representation of the opposite algebra M^{op} on $L^2(M)$. If we identify the two bimodules ${}_A L^2(M)_A$ and

$$L^2(Y,\nu) \oplus L^2(Y^2,\mathcal{K},\nu \otimes \nu)$$

we have that

$$\pi(f)(h \oplus k)(s,t) = (f(t)h(t)) \oplus (f(s)h(s,t))$$

and

$$\rho(f)(h \oplus k)(s,t) = (f(t)h(t)) \oplus (f(t)h(s,t)),$$

where $f \in L^\infty(Y,\nu)$, $h \in L^2(Y,\nu)$ and $k \in L^2(Y^2,\mathcal{K},\nu \otimes \nu)$.

Let us show that $A \subset M$ is maximal abelian:

Consider an element $x \in M$ that commutes with A such that its conditional expectation onto A is equal to zero, i.e. $E_A(x) = 0$. Let us identify the von Neumann algebra M and its image in the Hilbert space $L^2(M)$. The vector $\varphi(x)$ is orthogonal to $L^2(Y,\nu)$ because $E_A(x) = 0$, so there exists a vector $g \in L^2(Y^2,\mathcal{K},\nu\otimes\nu)$ such that $\varphi(x) = g$. Let $f \in L^2(Y,\nu)$ be an injective function, we have that $\pi(f)(g) = \rho(f)(g)$ because x commutes with A. Hence, $f(s)g(s,t) = f(t)g(s,t)$ almost everywhere for the product measure $\nu \otimes \nu$.

This implies that g is supported on the diagonal $\Delta Y = \{(t,t), t \in Y\}$. The measure ν is non-atomic; thus, $(\nu \otimes \nu)(\Delta Y) = 0$. Therefore, $g = 0$, hence $x = 0$. It means that $A' \cap M$ is contained in A, so $A \subset M$ is maximal abelian.

Let us show that M is a factor. Consider a central element $x \in M \cap M'$, we have that $x \in M \cap A' = A$. The element x commutes with M; thus, for any vector $\eta \in L^2(M)$, $\pi(x)(\eta) = \rho(x)(\eta)$. We denote the identity of the algebra M by 1. Let $1 \otimes \xi \otimes 1 \in L^2(A) \otimes \mathcal{K} \otimes L^2(A)$, we have that $\pi(x)(1 \otimes \xi \otimes 1) = x \otimes \xi \otimes 1$ and $\rho(x)(1 \otimes \xi \otimes 1) = 1 \otimes \xi \otimes x$. By identification we get that $x \in \mathbb{C}1$, so M is a factor.

The equality 1.7 shows that the bimodule $L^2(M)\ominus L^2(A)$ is isomorphic to $L^2(Y^2,\mathcal{K},\nu\otimes\nu)$. Thus, it is the direct integral of measurable fields of Hilbert spaces $\{\mathcal{K}_{s,t},(s,t)\in Y^2\}$ over the probability space $(Y^2,\nu\otimes\nu)$, where for any $(s,t)\in Y^2$, $\mathcal{K}_{s,t}=\mathcal{K}$. The Pukanszky invariant is, by definition, the essential value of the dimension function $d(s,t)=\dim\mathcal{K}_{s,t}$. In our case, it is clearly equal to the singleton $\{\dim\mathcal{K}\}$.

Let us prove that $A\subset M$ is Takesaki simple [60].

Let B be a separable, ultraweakly dense, C^*-subalgebra of M. Consider the abelian C^*-algebra of continuous complex valued function $\mathcal{C}(Y)$, view as a subalgebra of A. We suppose that $\mathcal{C}(Y)\subset B$. We begin by diagonalizing the abelian von Neumann algebra $\rho(A)$. Let $\mathcal{H}_0$ be the Hilbert space equal to the orthogonal direct sum $\mathbb{C}\oplus L^2(Y,\mathcal{K},\nu)$, where $L^2(Y,\mathcal{K},\nu)$ is the Hilbert space of measurable functions $g:Y\longrightarrow\mathcal{K}$ such that

$$\int_Y \|g(s)\|^2_{\mathcal{K}}d\nu(s)<\infty.$$

Consider the tensor product of Hilbert spaces $\mathcal{H}:=\mathcal{H}_0\otimes L^2(Y,\nu)$. We have an isomorphism

$$\phi:\mathcal{H}_0\otimes L^2(Y,\nu)\longrightarrow L^2(Y,\nu)\oplus L^2(Y^2,\mathcal{K},\nu\otimes\nu)$$

such that

$$\phi((z\oplus f)\otimes g)(s,t)=(g(t)z)\oplus(f(s)g(t)),$$

where $z\in\mathbb{C}$, $f\in L^2(Y,\mathcal{K},\nu)$, $g\in L^2(Y,\nu)$. The isomorphism ϕ conjugates the right action of A, $\rho(A)$, and the diagonal algebra. If $\xi\otimes g\in\mathcal{H}_0\otimes L^2(Y,\nu)$ and $f\in L^\infty(Y,\nu)$, then

$$(\phi^*\rho(f)\phi)(\xi\otimes g)(t)=\xi\otimes f(t)g(t).$$

Let $\pi|_B$ be the restriction to B of the standard representation. We have that $\pi(B)$ commutes with the diagonal algebra $\rho(A)$, hence there exists a measurable field $\{\pi_t,t\in Y\}$ of representations unique almost everywhere such that

$$\pi|_B=\int_Y^\oplus \pi_t d\nu(t).$$

We want to prove that $A\subset M$ is Takesaki simple. We need to show that there exists a Borel null set $N\subset Y$ such that for any $s,t\in Y\backslash N$, π_s is unitarily equivalent to π_t if and only if $s=t$. Here, we denote the set $\{t\in Y,t\notin N\}$ by $Y\backslash N$.

Consider an injective continuous function $f\in\mathcal{C}(Y)$. Fix a $t_0\in Y$, and consider the operator f_{t_0} acting on the Hilbert space $\mathcal{H}_0$ as follows:

$$f_{t_0}(z\oplus g)(s)=(f(t_0)z)\oplus(f(s)g(s)),$$

for any $z\in\mathbb{C}$, $g\in L^2(Y,\mathcal{K},\nu)$ and $s\in Y$. The collection of operators $\{f_t,t\in Y\}$ is a bounded measurable operator field. Consider the decomposable operator

$$D_f:=\int_Y^\oplus f_t d\nu(t)$$

acting on $\mathcal{H}_0\otimes L^2(Y,\nu)$. We clearly have that $D_f=\phi\pi(f)\phi^*$.

Let us show that f_{t_1} is unitarily conjugate to f_{t_2} if and only if $t_1=t_2$. To do this, we prove that the operator f_t has a unique eigenvalue equal to $f(t)$. Consider the vector

$$z\oplus 0\in\mathbb{C}\oplus L^2(Y,\mathcal{K},\nu),$$

where z is a complex number different from zero. We have that $f_t(z \oplus 0) = f(t)z \oplus 0$, so $f(t)$ is an eigenvalue of the operator f_t. Let us show this is the only eigenvalue of f_t, to do this it is sufficient to show that the restriction of f_t to the Hilbert space $L^2(Y,\mathcal{K},\nu)$ does not have any eigenvalue. Let $z \in \mathbb{C}$ and $h \in L^2(Y,\mathcal{K},\nu)$ such that $f_t(h) = zh$. We have that $f(s)h(s) = zh(s)$ almost everywhere. The function f is injective, so $f^{-1}(\{z\})$ is empty or is a singleton. The measure ν is non-atomic; thus, $\nu(f^{-1}(\{z\})) = 0$. Therefore, $h(s) = 0$ almost everywhere, hence $h = 0$. We have proved that the set of eigenvalues of the operator f_t is equal to the singleton $\{f(t)\}$. The set of eigenvalues of an operator is invariant under unitary conjugacy. Hence by injectivity of the function f, we get that f_{t_1} and f_{t_2} are unitarily equivalent if and only if $t_1 = t_2$.

By uniqueness of the decomposition of an operator, there exists a Borel null set $N \subset Y$ such that $\pi_t(f) = f_t$ for any $t \in Y \backslash N$. Let $s,t \in Y \backslash N$, and suppose that the two representation π_s and π_t are unitarily equivalent. In particular, $\pi_s(f)$ and $\pi_t(f)$ are unitarily equivalent; thus, f_t and f_s are unitarily equivalent. This implies that $s = t$. We have proved that $A \subset M$ is Takesaki simple.

A simple MASA is singular by [60][Theorem 4.1] (and it is equivalent by the main theorem of chapter 3). □

Remark 1.3.5. By a result of Popa [45], if $A \subset M$ is a MASA such that 1 is not in the Pukanszki invariant we have that $A \subset M$ is singular. Hence, in the case where $\dim \mathcal{K} > 1$ we have a simpler proof of this proposition.

Remark 1.3.6. The study of the cup subalgebra has been motivated by the following conjecture:
Consider the free group $\mathbb{F}_2$ generated by a,b. Let $L(\mathbb{F}_2)$ be the group von Neumann algebra associated, called the free group factor with two generators. Let $L(\mathbb{Z}) \subset L(\mathbb{F}_2)$ be the abelian von Neumann algebra generated by a, it is a MASA called the generator MASA. Consider the abelian subalgebra $A \subset L(\mathbb{F}_2)$ generated by the element $h = a+a^*+b+b^*$, it is a MASA called the radial MASA. The conjecture asks if the generator and the radial MASA are isomorphic or not. It is know that those two MASAs are singular, with Pukanzsky invariant equal to the singleton $\{\infty\}$. It is also known that generator and the radial MASAs are maximal hyperfinite, see [43] and [3]. Using proposition 1.3.4, one can show that the cup subalgebra have the same invariants. Furthermore, we show in the chapter 2 that the cup subalgebra is maximal hyperfinite in the case of $\mathcal{P}$ is a subfactor planar algebra.

If we take the subfactor planar algebra of non commutative polynomials of two variables with monomials of even degree, the cup subalgebra is isomorphic to the generator MASA. Furthermore, let us take the unshaded planar algebra $\mathcal{P}$ of all non commutative polynomials of two variables. The von Neumann algebra $M_{\mathcal{P}}$ is isomorphic to the free group factor with two generators $L(\mathbb{F}_2)$. The cup subalgebra has a very similar form of the radial MASA. One question is that if this cup subalgebra is isomorphic to the radial MASA?

More generally, consider the free group factor factor $L(\mathbb{F}_2)$ with generators a,b. Let f be a measurable bounded function on $\mathbb{C}$ and let $h_f := f(a) + f(b) \in L(\mathbb{F}_2)$ obtained by functional calculus. In which case h_f generates a MASA of $A_f \subset L(\mathbb{F}_2)$ and which condition on f could assure that the MASA $A_f \subset L(\mathbb{F}_2)$ is isomorphic to the radial MASA. Note that the cup subalgebra is of this form when $\mathcal{P}$ is the unshaded planar algebra of non commutative polynomials.

Chapitre 2

Standard invariant give abelian maximal hyperfinite subalgebras

Introduction

Dixmier [9] was the first to study maximal abelian subalgebra (MASA) $A \subset M$ in a von Neumann algebra. He introduced an invariant coming from the group normalizer.

Definition 2.0.7. Two MASAs $A_1 \subset M_1$ and $A_2 \subset M_2$ are called isomorphic if there exists an isomorphism of von Neumann algebras $\phi : M_1 \longrightarrow M_2$ such that $\phi(A_1) = A_2$.

Group normalizer:

We define the group normalizer $N_M(A)$ that is the group of unitaries $u \in M$ such that $uAu^* = A$. If the von Neumann algebra generated by $N_M(A)$ is equal to A, the MASA is called singular. If the von Neumann algebra generated by $N_M(A)$ is equal to M, the MASA is called regular.

Kadison asked if given a II_1 factor M and a unitary operator $u \in M$, does there exists a copy of the hyperfinite II_1 factor R such that $u \in R \subset M$? Popa answered negatively to this question in [43] by giving a maximal hyperfinite MASA in a II_1 factor. Here is the counterexample:

Consider the free group with $n \geqslant 2$ generators $a_1, \cdots, a_n$ that we denote by $\mathbb{F}_n$. Consider the free group factor $L(\mathbb{F}_n)$ equal to the group von Neumann algebra associated to $\mathbb{F}_n$. Let $L(\mathbb{Z}) \subset L(\mathbb{F}_n)$ be the abelian subalgebra generated by the element a_1. This subalgebra is maximal hyperfinite, hence cannot be contained in a copy of the hyperfinite II_1 factor inside $L(\mathbb{F}_2)$.

To prove the maximal hyperfiniteness, Popa introduces a notion called the asymptotic orthogonality property, or shortly AOP.

Definition 2.0.8. Let $A \subset M$ be a subalgebra in a finite von Neumann algebra, and τ a fixed faithful trace on M. Let ω be a free ultrafilter on the natural numbers $\mathbb{N}$, M^ω the ultrapower of M and $L^2(M^\omega)$ the GNS Hilbert space obtained from the extension of τ on M^ω. We denote this trace by τ_ω and identify M with its image in M^ω given by the diagonal embedding. Say that $A \subset M$ has the approximation orthogonality property if for any $x \in A' \cap M^\omega$, in the relative commuant such that $E_{A^\omega}(x) = 0$, and for any $y \in M$ such that $E_A(y) = 0$, we have that xy is orthogonal to yx for the inner product given by the faithful trace τ_ω.

Notation 2.0.9. We denote the conditional expectation on A with respect to the trace τ by $E_A : M \longrightarrow A$ and the conditional expectation on the ultrapower of A with respect to the trace τ_ω by $E_{A^\omega} : M^\omega \longrightarrow A^\omega$.

Let (M,τ) be a finite von Neumann algebra together with a trace, if $x,y \in M$ are two vectors in M, we say that they are orthogonal (with respect to the trace τ) if $\tau(xy^*) = 0$. In that case we write $x \perp y$. It is equivalent to say that their image in the GNS Hilbert space $L^2(M)$ are orthogonal.

Popa shows the following result:

Theorem 2.0.10. *Let $A \subset M$ be a singular MASA in a II_1 factor with the AOP, then $A \subset M$ is maximal hyperfinite.*

By mimicking the strategy introduced by Popa, new examples of abelian maximal hyperfinite subalgebras have been provided by different authors. Consider the self-adjoint operator

$$h = \sum_{i=0}^{n} (a_i + a_i^*)$$

in the free group factor $L(\mathbb{F}_n)$. It generates a MASA $A \subset L(\mathbb{F}_n)$ called the radial MASA or Laplacian MASA. In [3], Cameron et al. show that this MASA is maximal hyperfinite. Some other examples can be found in Jolissaint [23] and Shen [57].

Note that maximal hyperfinite von Neumann algebras are sometime called maximal injective or maximal amenable. Those alternative denominations are justified by the fundamental theorem of Connes [5], that says that a von Neumann algebra is hyperfinite if and only if it is injective if and only if it is amenable.

To deal with the asymptotic orthogonality property, we work in the algebra of central sequences of a von Neumann algebra. The study of this space comes from Murray and von Neumann in [38] with the introduction of property Γ. It has been formalized, using ultrafilter, by McDuff in [35].

In this paper we will construct MASAs using planar algebras. The modern theory of subfactors has been initiated by Jones [24]. He introduced the standard invariant that has been formalized as a λ-lattice by Popa [47] and as a subfactor planar algebra by Jones [28], (see also Peters [42] for an introduction to planar algebras). Popa [46, 47, 49] proved that any standard invariant comes from a subfactor, Popa and Shlyakhtenko proved [52] that the subfactor can be realized in the infinite free group factor $L(\mathbb{F}_\infty)$. Guionnet et al. [20, 21] gave a planar proof of this result and show, in the finite depth case, that the subfactor can be realized in an interpolated free group factor $L(\mathbb{F}_t)$, see the definition of an interpolated free group factor by Dykema and Radulescu [13, 55], where t is a function of the Jones index and the global index. Note that those results have been partially proved independently by Sunder and Kodiyalam in [33, 34]. In [29], Jones et al. gave a simpler construction of this process. This is the construction that we will refer to.

In this paper, we consider a subfactor planar algebra $\mathcal{P}$. Following [29], we can associate a tower $\{M_i, i \geqslant 0\}$ of II_1 factors. We will be interested by the first von Neumann algebra M_0 that appears in this construction and denote it by $M_{\mathcal{P}}$.

We consider an element in $M_{\mathcal{P}}$ that we call cup and denote it by $\cup$. This element generates a MASA $A \subset M_{\mathcal{P}}$ that we call the cup subalgebra. The main result of this paper is:

Theorem 2.0.11. *The cup subalgebra is maximal hyperfinite.*

Here is the plan of the paper:

In section 2.1, we recall the proof of theorem 2.0.10 of Popa for convenience of the readers. We consider a singular MASA in a II_1 factor, $B \subset M$ and suppose that it has the AOP. We prove in theorem 2.1.4, that this MASA is maximal hyperfinite. To do this, we consider an intermediate hyperfinite von Neumann algebra $B \subset L \subset M$. Using the fact that any MASA in a finite type I von Neumann algebra is regular, we show that the finite type I direct summand of L is included in B. Then, using the intertwining lemma of Popa, [50], we show that L cannot have a non null type II_1 direct summand. Hence, $B = L$; thus, $B \subset M$ is maximale hyperfinite.

In section 2.2, we consider a subfactor planar algebra $\mathcal{P}$ and fix some notations. We recall the construction of [29] that associate a II_1 factor to $\mathcal{P}$. We denote this II_1 factor by $M_{\mathcal{P}}$ and by $L^2(M_{\mathcal{P}})$ the GNS Hilbert space obtained from the unique trace of the II_1 factor $M_{\mathcal{P}}$. We denote the multiplication of $M_{\mathcal{P}}$ by $\star$ in reference to the construction. The standard representation of the II_1 factor $M_{\mathcal{P}}$ on the Hilbert space $L^2(M_{\mathcal{P}})$ is denote by π. We denote the right action of the the II_1 factor $M_{\mathcal{P}}$ on $L^2(M_{\mathcal{P}})$ by ρ and identify $M_{\mathcal{P}}$ and its dense image in the Hilbert space $L^2(M_{\mathcal{P}})$. Hence, $\pi(x)\rho(z)(y) = x\star y\star z$ for $x,y,z \in M_{\mathcal{P}}$.

In section 2.2.1, we define the cup subalgebra $A \subset M_{\mathcal{P}}$. It is an abelian von Neumann algebra generated by the self-adjoint element cup denoted by the symbol $\cup$.

We study the $A - A$-bimodule structure of ${}_AL^2(M_{\mathcal{P}})_A$. The Hilbert space $L^2(M_{\mathcal{P}})$ has a natural bimodule structure over A by restricting the left and right action, π,ρ, to A. Consider the set of natural numbers $\mathbb{N} = \{0,1,2,\cdots\}$, the Hilbert space of square summable complex sequences $\ell^2(\mathbb{N})$ and the unilateral shift operator $s(e_n) = e_{n+1}$, where $\{e_n, n \geqslant 0\}$ is the orthonormal standard orthonormal basis of $\ell^2(\mathbb{N})$. In proposition 2.2.1, we review a theorem of [29] that describes the bimodule structure of ${}_AL^2(M_{\mathcal{P}})_A$ and gives an explicit unitary transformation

$$\psi : L^2(M_{\mathcal{P}}) \longrightarrow \ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})),$$

where V is a Hilbert space. This transformation satisfies that

$$(\psi\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\psi^*)(x \otimes v \otimes y) = (s + s^*)(x) \otimes v \otimes y \tag{2.1}$$

$$(\psi\rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}})\psi^*)(x \otimes v \otimes y) = x \otimes v \otimes (s + s^*)(y), \tag{2.2}$$

for any $x,y \in \ell^2(\mathbb{N})$ and $v \in V$. Note that $\delta > 1$ is the modulus of the subfactor planar algebra $\mathcal{P}$. Also we have that $\psi(L^2(A)) = \ell^2(\mathbb{N})$. This bimodule structure implies that the cup subalgebra is a singular MASA, see corollary 2.2.2. To prove the singularity, we use the Pukanszky invariant.

Definition 2.0.12. Let $A \subset M$ be a MASA in a separable II_1 factor, $L^2(M)$ the GNS Hilbert space induced by the unique trace of M. Consider the abelian von Neumann algebra $\mathcal{A}$ generated by the left and right action of A on the Hilbert space $L^2(M)$. Let e_A be the orthogonal projection from $L^2(M)$ onto $L^2(A)$, where $L^2(A)$ is the completion of A in $L^2(M)$. Consider the commutant of the von Neumann algebra $\mathcal{A}(1 - e_A)$. It is a type I von Neumann algebra. The Pukanszky invariant of $A \subset M$ is the set of $n \in \mathbb{N} \cup \{\infty\}$ such that the commutant of the von Neumann algebra $\mathcal{A}(1 - e_A)$ has a non-zero type I_n direct summand.

In section 2.3.1, we look at the self-adjoint operator $s + s^*$. Let us introduce some notations. We fix a free ultrafilter ω on the natural numbers and consider convergence with respect to this ultrafilter.

Definition 2.0.13. Let X be a topological space, $(x_n)_n$ a sequence of elements of X, and $x \in X$. The sequence $(x_n)_n$ converges to x with respect to the ultrafilter ω if for any neighborhood V of x there exists a set $D \in \omega$ such that for any $n \in D$, $x_n \in V$. We write

$$\lim_{n \to \omega} x_n = x.$$

If the sequence $(x_n)_n$ converges to x for the classical convergence, we write

$$\lim_{n\to\infty} x_n = x.$$

If $\mathcal{K}$ is a Hilbert space, we denote the identity function on it by $1_{\mathcal{K}}$. If the context is clear, we simply denote this operator by 1. Let $i \geqslant 0$, and $e_i \in \ell^2(\mathbb{N})$ be an element of the standard orthonormal basis. We denote the rank one projection onto the complex line $\mathbb{C}e_i$ by q_{e_i}.

The aim of this section is to prove the following result of proposition 2.3.5: If $(\eta^{(n)})_n$ is a sequence of vectors in the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ such that

$$\lim_{n\to\omega}((s+s^*)\otimes 1 - 1\otimes(s+s^*))\eta^{(n)} = 0,$$

then for any $i \geqslant 0$,

$$\lim_{n\to\omega}(q_{e_i}\otimes 1)\eta^{(n)} = \lim_{n\to\omega}(1\otimes q_{e_i})\eta^{(n)} = 0.$$

In section 2.3.2, we conclude and prove that the cup subalgebra is maximal hyperfinite. In order to do that, we first prove that the cup subalgebra has the AOP. We consider an element in the relative commutant $x \in M^\omega \cap A'$ such that $E_{A^\omega}(x) = 0$. Let $(x^{(n)})_n$ be a bounded sequence of M such that for any $n \geqslant 0$, $E_A(x^{(n)}) = 0$ and $(x^{(n)})_n$ is a representant of x in the ultrapower M^ω. Let $J \geqslant 0$ and $Z_J \subset L^2(M_{\mathcal{P}})$ be the closed vector space spanned by the set of vectors

$$\{\psi^*(e_k \otimes v \otimes e_r),\ \min(k,r) \leqslant J, v \in V\},$$

where ψ is the unitary transformation from $L^2(M_{\mathcal{P}})$ into $\ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N}))$. We denote the orthogonal projection onto this subspace by Q_J. Proposition 2.3.7 implies that

$$\lim_{n\to\omega} Q_J(x^{(n)}) = 0. \tag{2.3}$$

To prove that the cup subalgebra has the AOP, we need to consider product of elements. Let $J \geqslant 0$, $\tilde{y} \in M \cap \bigotimes_{j\leqslant J}\mathcal{P}_j$ such that $E_A(\tilde{y}) = 0$ and $\tilde{x} \in M$ such that $Q_J(\tilde{x}) = E_A(\tilde{x}) = 0$. Then, lemma 2.3.8 implies that

$$\tilde{x} \star \tilde{y} \perp \tilde{y} \star \tilde{x}. \tag{2.4}$$

We are able to prove that $A \subset M_{\mathcal{P}}$ has the AOP. Let $y \in M$ such that $E_A(y) = 0$, by approximation in the Hilbert space $L^2(M_{\mathcal{P}})$, one can suppose that there exists $J \geqslant 0$ such that $y \in \bigotimes_{j\leqslant J}\mathcal{P}_j$. Let $x = (x^{(n)})_n$ be the sequence in the relative commutant $A' \cap M^\omega$ such that $E_{A^\omega}(x) = 0$. If we combine equation 2.3 and equation 2.4, we get that

$$x \star y \perp y \star x.$$

Hence, the cup subalgebra has the AOP. By corollary 2.2.2, the cup subalgebra is a singular MASA, hence by theorem 2.1.4, it is maximal hyperfinite.

2.1 AOP and maximal hyperfiniteness

In this section, we recall the proof of Popa that says that a singular MASA with the AOP in a II_1 factor is maximal hyperfinite. Be aware that in this section, we denote the product of two elements $x,y \in M$ by xy.

Lemma 2.1.1. *A MASA in a finite type I von Neumann algebra is regular.*

Proof. Let $B \subset M$ be a MASA in a finite type I von Neumann algebra. By the structure of finite type I von Neumann algebras, there exists a family of abelian von Neumann algebras A_n such that M is isomorphic to the direct sum:

$$\bigoplus_{n \geqslant 0} A_n \otimes \mathcal{M}_n(\mathbb{C}),$$

where $\mathcal{M}_n(\mathbb{C})$ is the complex algebra of n by n matrices. By the structure of MASAs in finite type I von Neumann algebras, see [58][Chap. 2.4], $B \subset M$ is unitarily conjugate to the inclusion

$$(\bigoplus_{n \geqslant 0} A_n \otimes \mathcal{D}_n(\mathbb{C})) \subset (\bigoplus_{n \geqslant 0} A_n \otimes \mathcal{M}_n(\mathbb{C})),$$

where $\mathcal{D}_n(\mathbb{C})$ is the abelian complex algebra of n by n diagonal matrices. It is clear that the MASA is regular because the MASA $\mathcal{D}_n(\mathbb{C}) \subset \mathcal{M}_n(\mathbb{C})$ is regular. □

Lemma 2.1.2. *Let L be a finite von Neumann algebra such that there exists an increasing sequence of von Neumann subalgebras $L_k \subset L$ such that there union is ultraweakly dense, i.e.*

$$\{\bigcup_{k \geqslant 0} L_k\}'' = L,$$

and for any $k \geqslant 0$ the relative commutant $L_k' \cap L$ is a type II_1 von Neumann algebra. Then, there is no MASA in L that has the AOP.

Proof. Consider a finite von Neumann algebra L and such a sequence $\{L_k, k \geqslant 0\}$ as in the hypothesis of the lemma. Let $B \subset L$ be an abelian von Neumann subalgebra. Let us show that for any integer $k \geqslant 0$ there exists a unitary in the relative commutant $u_k \in L_k' \cap L$ such that

$$\|E_B(u_k)\|_2 \leqslant \frac{1}{k+1}.$$

Suppose there exists an integer $k \geqslant 0$ and a positive constant $c > 0$ such that for any unitary $u \in L_k' \cap L$ we have

$$\|E_B(u)\|_2 > c.$$

The intertwining lemma of Popa [50] tells us that we can embed a non null corner of the von Neumann algebra $L_k' \cap L$ into a corner of B. This would implies an embedding of a II_1 von Neumann algebra in a finite type I von Neumann algebra, a contradiction. Therefore, there exists some unitaries $u_k \in L_k' \cap L$ such that $\|E_B(u_k)\|_2 \leqslant \frac{1}{k+1}$. Consider the unitary $u = (u_k)_k$ in the ultraprower L^ω. By definition of the u_k, we have that u belongs to the relative commutant $L^\omega \cap L'$, in particular, $u \in L^\omega \cap A'$. We have that the conditional expectation of u onto B^ω is equal to the null vector. Let $x \in L$ such that $E_B(x) = 0$, and x different from the null vector. Such a x exists because the existence of the sequence L_k implies that L is not abelian and then different from B. If $B \subset L$ has the AOP, we would get that ux is orthogonal to xu. But, $xu = ux$ because $u \in L^\omega \cap L'$ and xu is different from the null vector because u is a unitary. We have a contradiction. □

Lemma 2.1.3. *Let $B \subset M$ be a singular MASA in a II_1 factor that has the AOP. Consider a projection $p \in B$ and an intermediate von Neumann algebra $B \subset L \subset M$. Then $pB \subset pLp$ is a singular MASA with the AOP.*

Proof. This is obvious. □

We are able to proof the theorem of this section:

Theorem 2.1.4. *Let $B \subset M$ be a singular MASA in a II_1 factor with the AOP. Then $B \subset M$ is maximal hyperfinite.*

Proof. Consider an intermediate hyperfinite von Neumann algebra $B \subset L \subset M$. We want to show that $B = L$. Let us divide the work in two parts, and look at the type I and the type II summand of the von Neumann algebra L. Let p be the maximal central projection of L such that pL is a type II_1 von Neumann algebra. The subalgebra $B \subset M$ is maximal abelian, hence $p \in B$. We have that pL is a type II_1 von Neumann algebra and $(1-p)L$ a finite type I von Neumann algebra.

Let us show that $(1-p)B = (1-p)L$. By lemma 2.1.3, $(1-p)B \subset (1-p)L$ is a singular MASA in a finite type I von Neumann algebra. Lemma 2.1.1 is telling us that this MASA is regular, hence $(1-p)B = (1-p)L$.

Let us show that the projection p is null. Suppose that p is a non null projection. The von Neumann algebra pL is an hyperfinite type II_1 von Neumann algebra. Hence, by uniqueness of the hyperfinite II_1 factor, see [37], there exists an abelian von Neumann algebra C such that pB is isomorphic to the tensor product $D := C \otimes R$ where R is the hyperfinite II_1 factor. Let us write R as the inductive limit of the tensor product of 2 by 2 matrices

$$R = \bigotimes_{i=0}^{\infty} \mathcal{M}_2(\mathbb{C}).$$

Let R_k be the finite dimensional subalgebra of R equal to the tensor product

$$R_k = \bigotimes_{i=0}^{k} \mathcal{M}_2(\mathbb{C}).$$

We denote by D_k the von Neumann algebra $C \otimes R_k$. We remark that the relative commutant $D_k' \cap D$ is isomorphic to D, hence is a type II_1 von Neumann algebra. Therefore, the von Neumann algebra pL satisfies the hypothesis of the lemma 2.1.2. Hence it does not contain a MASA with the AOP. But by lemma 2.1.3, $pB \subset pL$ is a MASA with the AOP, a contradiction. Hence, $B = L$; thus, $B \subset M$ is maximal hyperfinite. $\square$

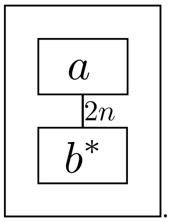
Remark 2.1.5. Let M be a McDuff factor, meaning that M is a II_1 factor and is isomorphic to the tensor product of von Neumann algebras $M \otimes R$ where R is the hyperfinite II_1 factor. We have that M satisfies the hypothesis of lemma 2.1.2, hence does not contain a MASA with the AOP. Furthermore, there exists some McDuff factors that contain maximal hyperfinite MASAs, see the example of Shen [57].

2.2 Construction of a II_1 factor associated to a subfactor planar algebra

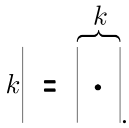
We assume that the reader is familiar with planar algebras. We will hence use the standard notation of planar algebras for the rest of the paper. For further details, see the paper of Jones [28] or the introduction of Peters [42]. We follow the setup of [29]. Let $\mathcal{P} = (\mathcal{P}_n)_{n \geqslant 0}$ be a subfactor planar algebra of modulus $\delta > 1$. Let $Gr(\mathcal{P})$ be the graded vector space equal to the algebraic direct sum $\bigoplus_{n \geqslant 0} \mathcal{P}_n$. We consider the inner product

$\langle \cdot,\cdot \rangle$ on each $\mathcal{P}_n$ that is:

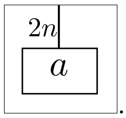
$$\langle a, b \rangle =$$



We extend this inner product on $Gr(\mathcal{P})$ in such a way that the spaces $\mathcal{P}_n$ are pairwise orthogonal. We still write $\mathcal{P}_n$ when it is considered as the n-graded part of $Gr(\mathcal{P})$. Let $\mathcal{H}$ be the Hilbert space equal to the completion of $Gr(\mathcal{P})$ for its prehilbert structure. Note that $\mathcal{H}$ is the Hilbert space equal to the orthogonal direct sum of the spaces $\mathcal{P}_n$. To simplify the pictures, as in the article of Kodiyalam and Sunder [33] we decorate strands in a planar tangle with non-negative integers to represent cabling of that strand. For example:

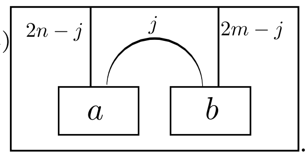


An element $a \in \mathcal{P}_n$ will be represent as a box:

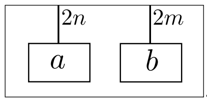


We assume that the distinguished first interval is at the top left of the box. We define a multiplication on $Gr(\mathcal{P})$ by requiring that if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$, then $a \star b$ is given by

$$a \star b = \sum_{j=0}^{min(2n,2m)}$$



Let us fix an element $a \in Gr(\mathcal{P})$, the map $b \in Gr(\mathcal{P}) \longmapsto a \star b \in Gr(\mathcal{P})$ is bounded for the inner product $\langle \cdot,\cdot \rangle$. Hence, can be extend as an operator on the Hilbert space $\mathcal{H}$. This gives us a representation of the $*$-algebra $Gr(\mathcal{P})$ on $\mathcal{H}$. We denote by $M_{\mathcal{P}}$ the von Neumann algebra equal to the bicommutant of this representation. It is a II_1 factor by [29]. We identify the graded algebra $Gr(\mathcal{P})$ and its image in the von Neumann algebra $M_{\mathcal{P}}$. The unique faithful normal trace tr of $M_{\mathcal{P}}$ is the one coming from the planar algebra structure of $\mathcal{P}$. It is equal to the formula $tr(a) = \langle a,1 \rangle$ where 1 is the unity of $Gr(\mathcal{P})$. We remark that the trace of an element of $Gr(\mathcal{P})$ is its zero-graded piece. Let $L^2(M_{\mathcal{P}})$ be the Hilbert space coming from the GNS construction over the faithful trace tr. Note that the standard representation of the von Neumann algebra $M_{\mathcal{P}}$ on the Hilbert space $L^2(M_{\mathcal{P}})$ is conjugate to the action of $M_{\mathcal{P}}$ on the Hilbert space $\mathcal{H}$. We will identify those two representations. To simplify the notation, we identify the von Neumann algebra $M_{\mathcal{P}}$ and its image in the Hilbert space $L^2(M_{\mathcal{P}})$. We denote the multiplication of $M_{\mathcal{P}}$ by $\star$, in reference to this construction. The left and right action of $M_{\mathcal{P}}$ in the GNS construction $L^2(M_{\mathcal{P}})$ are denote by π and ρ, i.e. $\pi(x)\rho(y)(z) = x \star z \star y$, for $x,y,z \in M_{\mathcal{P}}$. If a confusion is possible, we denote by $\| \cdot \|_2$ the norm of $L^2(M_{\mathcal{P}})$ and by $\| \cdot \|$ the norm of M. We define a multiplication on $Gr(\mathcal{P})$ by requiring that if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$, then $a \bullet b \in \mathcal{P}_{n+m}$ is given by

$a \bullet b =$ .

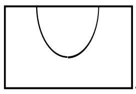
We remark that $\|a \bullet b\|_2 = \|a\|_2 \|b\|_2$ if $a \in \mathcal{P}_n$ and $b \in \mathcal{P}_m$. By the triangle inequality, the bilinear function

$$\begin{aligned} Gr(\mathcal{P}) \times Gr(\mathcal{P}) &\longrightarrow Gr(\mathcal{P}) \\ (a,b) &\longmapsto a \bullet b \end{aligned}$$

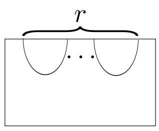
is continuous for the norm $\|\cdot\|_2$. We extend this operation on $L^2(M_{\mathcal{P}}) \times L^2(M_{\mathcal{P}})$ and still denote it by $\bullet$.

2.2.1 The cup subalgebra

The cup subalgebra $A \subset M_{\mathcal{P}}$ is the abelian von Neumann algebra generated by the self-adjoint element cup:

.

We denote cup by the symbol $\cup$ and the element



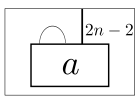
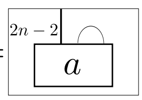
by the symbol $\cup^{\bullet r}$. We use the convention that $0 = \cup^{\bullet k}$ for $k < 0$ and $1 = \cup^{\bullet 0}$. In particular, $x \bullet \cup^{\bullet r} = 0$ if $r \leqslant -1$, for any x. The conditional expectation on A is denote by E_A.

Note that the closure of A in $L^2(M_{\mathcal{P}})$, that we denote by $L^2(A)$, admits the following orthonormal basis:

$$\{\delta^{-\frac{r}{2}} \cup^{\bullet r}, r \geqslant 0\}.$$

Let us look at the bimodule ${}_A L^2(M_{\mathcal{P}})_A$:

We proceed as in [29]. Let $n \geqslant 1$ and V_n be the subspace of $\mathcal{P}_n$ of elements which vanish when a cap is placed at the top right and vanish when a cap is placed at the top left, i.e.

$V_n = \Big\{ a \in \mathcal{P}_n,$  $=$  $= 0 \Big\}.$

We denote by V the orthogonal direct sum of the V_n, i.e.

$$V = \bigoplus_{n=1}^{\infty} V_n.$$

We give a decomposition of the bimodule ${}_A L^2(M_{\mathcal{P}})_A$.

Proposition 2.2.1. *The map*

$$\begin{array}{rccccc} \psi: & L^2(M_{\mathcal{P}}) & \longrightarrow & \ell^2(\mathbb{N}) & \oplus & (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})) \\ & \delta^{-\frac{r}{2}} \cup^{\bullet r} & \longmapsto & e_r & \oplus & 0 \\ & \delta^{-\frac{k+r}{2}} \cup^{\bullet k} \bullet v \bullet \cup^{\bullet r} & \longmapsto & 0 & \oplus & e_k \otimes v \otimes e_r \end{array}$$

defines a unitary transformation, where $r,k \geqslant 0$ *and* $v \in V$. *We have that*

$$\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) = \begin{pmatrix} s + s^* - q_{e_0} & 0 \\ 0 & (s + s^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} \end{pmatrix}$$

and

$$\rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) = \begin{pmatrix} s + s^* - q_{e_0} & 0 \\ 0 & 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (s + s^*) \end{pmatrix},$$

where q_{e_0} *is the rank one projection on* $\mathbb{C}e_0$.

Proof. See [29][theorem 4.9.]. □

Corollary 2.2.2. *The cup subalgebra is a singular MASA.*

Proof. The fact that the cup subalgebra is maximal abelian has been already proved in [29]. Let us show that the cup subalgebra is singular. The decomposition of the last proposition tells us that the Pukanszky invariant of the MASA $A \subset M_{\mathcal{P}}$ is equal to the singleton $\{\infty\}$. Popa has proved in [45] that a MASA is singular if its Pukanszky invariant does not contained the natural number 1. Hence, the cup subalgebra is singular. □

2.3 The cup subalgebra is maximal hyperfinite

2.3.1 Property of the operator $s + s^*$

In this section, we study the operator $s + s^*$. We begin by looking at its spectrum in proposition 2.3.2, then we will look at sequences in the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ that are asymptotically in the kernel of the derivation of $s + s^*$. It means that we consider sequences $\eta^{(n)} \in \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ such that

$$\lim_{n \to \omega} ((s + s^*) \otimes 1 - 1 \otimes (s + s^*))\eta^{(n)} = 0.$$

Notation 2.3.1. If a is a self-adjoint operator and f is a measurable function defined on the spectrum of a, we denote the operator obtained by functional calculus by $f(a)$.

Let us give a spectral decomposition of the operator $s + s^*$.

Proposition 2.3.2. *The operator*

$$\frac{s + s^*}{2}$$

is semicircular in the sense of Voiculescu [39]. In particular, the distribution of $s + s^*$ *is absolutely continuous with respect to the Lebesgue measure, is supported in* $[-2; 2]$ *and equal to:*

$$d\nu(t) = \frac{\sqrt{4 - t^2}}{2\pi} dt.$$

We have a unitary transformation $\phi : L^2([-2; 2],\nu) \longrightarrow \ell^2(\mathbb{N})$ *defined on the dense subspace of continuous function* $\mathcal{C}([-2; 2])$ *by* $f \in \mathcal{C}([-2; 2]) \longmapsto f(s + s^*)(e_0)$. *This satisfies*

$$(\phi(s + s^*)\phi^*)(f)(t) = tf(t) \tag{2.5}$$

for any $f \in \mathcal{C}([-2;2])$ and almost everywhere in $t \in [-2;2]$. Furthermore, $\phi(P_n) = e_n$, for all $n \geqslant 0$ where

$$\begin{array}{l} P_0(X) = 1 \\ P_1(X) = X \\ P_n(X) = XP_{n-1}(X) - P_{n-2}(X), \text{ for any } n \geqslant 2 \end{array},$$

and X is an indeterminate.

Proof. To show that the distribution of $s + s^*$ is $d\nu(t) = \frac{\sqrt{4-t^2}}{2\pi}dt$, see [39][Example 3.4.2]. The equality 2.5 is obvious by definition of ϕ. We prove that $\phi(P_n) = e_n$ by induction on n.

It is clear for $n = 0$.
For $n = 1$: $(s + s^*)(e_0) = e_1$, so it is true for $n = 1$.
Let $n \geqslant 2$ and suppose the result true for $n - 1$ and $n - 2$. We have:

$$\begin{aligned} P_n(s + s^*)(e_0) &= (s + s^*)P_{n-1}(s + s^*)(e_0) - P_{n-2}(s + s^*)(e_0) \\ &= (s + s^*)(e_{n-1}) - e_{n-2} = e_n + e_{n-2} - e_{n-2} \\ &= e_n. \end{aligned}$$

By construction, ϕ is an isometry. We just proved that the standard orthonormal basis $\{e_n, n \geqslant 0\}$ of $\ell^2(\mathbb{N})$ is in the image of ϕ, hence ϕ is surjective. Therefore, ϕ is a unitary transformation. $\square$

Let $(\eta^{(n)})_{n\geqslant 0}$ be a sequence of vectors of the Hilbert space $\ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$ such that for any $n \geqslant 0$, $\|\eta^{(n)}\| < 1$. We suppose that

$$\lim_{n\to\omega}((s + s^*) \otimes 1 - 1 \otimes (s + s^*))(\eta^{(n)}) = 0.$$

The objective of this section is to show that for any $i \geqslant 0$,

$$\lim_{n\to\omega}(q_{e_i} \otimes 1)\eta^{(n)} = \lim_{n\to\omega}(1 \otimes q_{e_i})\eta^{(n)} = 0.$$

Lemma 2.3.3. *For any $i \geqslant 0$,*

$$\lim_{n\to\omega} \|(q_{e_0} \otimes P_i(s + s^*))\eta^{(n)} - (q_{e_i} \otimes 1)\eta^{(n)}\| = 0,$$

where $\{P_i\}_i$ is the family of polynomials defined in proposition 2.3.2.

Proof. By definition of the family of polynomials $\{P_i\}_i$ we have that for any $i \geqslant 0$,

$$P_i(s + s^*)(e_0) = e_i.$$

The polynomials P_i have real coefficient and $s + s^*$ is self-adjoint, hence the $P_i(s + s^*)$ are self-adjoint operators. Let $k \geqslant 0$,

$$\langle P_i(s + s^*)(e_k), e_0\rangle = \langle e_k, P_i(s + s^*)(e_0)\rangle = \langle e_k, e_i\rangle.$$

Therefore $q_{e_0} \circ P_i(s + s^*)(e_i) = e_i$ and $q_{e_0} \circ P_i(s + s^*)(e_k) = 0$ if $k \neq i$. It means that $q_{e_0} \circ P_i(s + s^*) = q_{e_i}$, where the composition of operators is denoted by $\circ$.

Let us show that for any polynomials P,

$$\lim_{n\to\omega}(P(s + s^*) \otimes 1 - 1 \otimes P(s + s^*))(\eta^{(n)}) = 0.$$

To do this, it is sufficient to show that for any $k \geqslant 0$,

$$\lim_{n\to\omega}((s+s^*)^k \otimes 1 - 1 \otimes (s+s^*)^k)(\eta^{(n)}) = 0.$$

We proceed by induction on k:
It is evident for $k = 0$, and by hypothesis it is true for $k = 1$. Suppose it is true for k. Remark that the two operators

$$((s+s^*)^{k+1} \otimes 1 - 1 \otimes (s+s^*)^{k+1})$$

and

$$((s+s^*) \otimes 1 + 1 \otimes (s+s^*)) \circ ((s+s^*)^k \otimes 1 - 1 \otimes (s+s^*)^k)$$

are equal. Let C be the norm of the operator $((s+s^*) \otimes 1 + 1 \otimes (s+s^*))$. We get that

$$\|((s+s^*)^{k+1} \otimes 1 - 1 \otimes (s+s^*)^{k+1})(\eta^{(n)})\| \leqslant C\|((s+s^*)^k \otimes 1 - 1 \otimes (s+s^*)^k)(\eta^{(n)})\|.$$

Hence,

$$\lim_{n\to\omega}((s+s^*)^{k+1} \otimes 1 - 1 \otimes (s+s^*)^{k+1})(\eta^{(n)}) = 0.$$

Thus, for any $i \geqslant 0$,

$$\lim_{n\to\omega}(P_i(s+s^*) \otimes 1 - 1 \otimes P_i(s+s^*))(\eta^{(n)}) = 0.$$

Therefore,

$$\lim_{n\to\omega}(q_{e_0} \otimes 1) \circ (P_i(s+s^*) \otimes 1 - 1 \otimes P_i(s+s^*))(\eta^{(n)}) = 0.$$

It means that

$$\lim_{n\to\omega}(q_{e_i} \otimes 1 - q_{e_0} \otimes P_i(s+s^*))(\eta^{(n)}) = 0.$$

□

Lemma 2.3.4. *Let $j \geqslant 0$, and consider the real valued function $S_j : [-2;2] \longrightarrow \mathbb{R}$ such that $S_j(t) = \sum_{i=0}^{j} P_i(t)^2$. Then the sequence $(S_j)_{j\geqslant 0}$ converges uniformly to $+\infty$ (for the classical convergence on the natural number $\mathbb{N}$).*

Proof. Let us prove the simple convergence to $+\infty$.

Suppose there exists $t_0 \in [-2;2]$ such that the sequence $(S_j(t_0))_k$ does not converge to $+\infty$. The polynomial P_i have real coefficient, hence for any $t \in [-2;2]$, $P_i(t)$ is real; thus, $(S_j(t_0))_k$ is an increasing sequence in $\mathbb{R}$. If this sequence does not diverge, then it is bounded. Then, the sequence $(P_i(t_0))_i$ is square summable. In particular,

$$\lim_{i\to\infty} P_i(t_0) = 0.$$

Let $\varepsilon_i = P_i(t_0)$. We have that $\varepsilon_{i+1} = t_0\varepsilon_i - \varepsilon_{i-1}$ and

$$\lim_{i\to\infty} \varepsilon_i = 0.$$

There is only one sequence that satisfies those axioms and it is the sequence equal to zero. Since $0 \neq 1 = P_0(t_0) = \varepsilon_0$, we arrive to a contradiction and thus, $\lim_{j\to\infty} S_j(t) = +\infty$ for any $t \in [-2;2]$.

To conclude we use the following well known result due to Dini: Let $(f_j)_j$ be a sequence of continuous functions from a compact topological space K to $\mathbb{C}$ such that $f_j \leqslant f_{j+1}$. If for any $t \in K$, $\lim_{j\to\infty} f_j(t) = +\infty$, then the sequence $(f_j)_j$ converges uniformly to $+\infty$. □

Proposition 2.3.5. *For any $i \geqslant 0$, we have:*

$$\lim_{n\to\omega}(q_{e_i}\otimes 1)(\eta^{(n)}) = 0$$

and

$$\lim_{n\to\omega}(1\otimes q_{e_i})(\eta^{(n)}) = 0.$$

Proof. Let us show that

$$\lim_{n\to\omega}(q_{e_0}\otimes 1)(\eta^{(n)}) = 0.$$

Let us fix $\varepsilon > 0$, we have to find an element of the ultrafilter $E \in \omega$ such that for any $n \in E$, $\|(q_{e_0}\otimes 1)(\eta^{(n)})\| < \varepsilon$. Let $i \geqslant 0$, by the triangle inequality, we have

$$\|(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)})\| \leqslant \|(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)}) - (q_{e_i}\otimes 1)(\eta^{(n)})\| + \|(q_{e_i}\otimes 1)(\eta^{(n)})\|.$$

The theorem of Pythagoras tells us that

$$\|\eta^{(n)}\|^2 = \sum_{i\geqslant 0}\|(q_{e_i}\otimes 1)(\eta^{(n)})\|^2$$

and we know by hypothesis that $1 > \|\eta^{(n)}\|$. Hence $\|(q_{e_i}\otimes 1)(\eta^{(n)})\| < 1$; thus,

$$\begin{aligned}\|(q_{e_i}\otimes 1)(\eta^{(n)})\|^2 \geqslant{} & \|(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)})\|^2 \\ & - \|(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)}) - (q_{e_i}\otimes 1)(\eta^{(n)})\|^2 \\ & - 2\|(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)}) - (q_{e_i}\otimes 1)(\eta^{(n)})\|.\end{aligned} \tag{2.6}$$

By lemma 2.3.4, there exists an integer $J \in \mathbb{N}$ such that

$$\inf_{t\in[-2;2]} S_J(t) > 2/\varepsilon.$$

Consider the unitary transformation ϕ defined in proposition 2.3.2. We identify the Hilbert space $\ell^2(\mathbb{N})\otimes L^2([-2;2],\nu)$ with the Hilbert space of measurable square-integrable functions from $[-2;2]$ to $\ell^2(\mathbb{N})$. Let

$$z_n = (q_{e_0}\otimes\phi^*)(\eta^{(n)}),$$

view as a a function $z_n : [-2;2] \longrightarrow \ell^2(\mathbb{N})$. We have that

$$\begin{aligned}\sum_{i=0}^{J}\|(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)})\|^2 &= \sum_{i=0}^{J}\|(1\otimes P_i(s+s^*))\circ(q_{e_0}\otimes 1)(\eta^{(n)})\|^2 \\ &= \sum_{i=0}^{J}\int_{[-2;2]}\|P_i(t)z_n(t)\|^2 d\nu(t) \\ &= \int_{[-2;2]}(\sum_{i=0}^{J}|P_i(t)|^2)\|z_n(t)\|^2 d\nu(t) \\ &\geqslant 2/\varepsilon\|z_n\|^2 = 2/\varepsilon\|(q_{e_0}\otimes 1)(\eta^{(n)})\|^2.\end{aligned} \tag{2.7}$$

By lemma 2.3.3, there exists an element of the ultrafilter $E \in \omega$ such that for any $n \in E$, for any $i \in \{0,\cdots,J\}$, we have that

$$\|(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)}) - (q_{e_i}\otimes 1)(\eta^{(n)})\| < \frac{1}{4(J+1)}. \tag{2.8}$$

If we combine the inequalities 2.7, and 2.8, we get that

$$\|(q_{e_i}\otimes 1)(\eta^{(n)})\|^2 \geqslant \|(q_{e_0}\otimes P_i(s+s^*)(\eta^{(n)})\|^2 - ((\frac{1}{4(J+1)})^2 - 2\frac{1}{4(J+1)}). \tag{2.9}$$

Let us reuse the theorem of Pythagoras and the inequality **??** and 2.9.

$$\begin{aligned}
\|\eta^{(n)}\|^2 &= \sum_{i\geqslant 0}\|(q_{e_i}\otimes 1)(\eta^{(n)})\|^2 \geqslant \sum_{i=0}^{J}\|(q_{e_i}\otimes 1)(\eta^{(n)})\|^2\\
&\geqslant \sum_{i=0}^{J}\|(q_{e_0}\otimes P_i(s+s^*)(\eta^{(n)})\|^2 - (J+1)((\frac{1}{4(J+1)})^2 - 2\frac{1}{4(J+1)})\\
&\geqslant \frac{2}{\varepsilon}\|(q_{e_0}\otimes 1)(\eta^{(n)})\| - (J+1)((\frac{1}{4(J+1)})^2 - 2\frac{1}{4(J+1)}).
\end{aligned}$$

By hypothesis $\|\eta^{(n)}\|\leqslant 1$ and

$$(J+1)((\frac{1}{4(J+1)})^2 - 2\frac{1}{4(J+1)}) < 1.$$

Therefore,

$$\|(q_{e_0}\otimes 1)(\eta^{(n)})\| \leqslant \varepsilon,$$

for any $n\in E$. We have proved that

$$\lim_{n\to\omega}(q_{e_0}\otimes 1)(\eta^{(n)}) = 0.$$

Hence,

$$\lim_{n\to\omega}(1\otimes P_i(s+s^*))(q_{e_0}\otimes 1)(\eta^{(n)}) = \lim_{n\to\omega}(q_{e_0}\otimes P_i(s+s^*))(\eta^{(n)}) = 0.$$

By lemma 2.3.3, this implies that

$$\lim_{n\to\omega}(q_{e_i}\otimes 1)(\eta^{(n)}) = 0,$$

for any $i\geqslant 0$.

Consider the flip $\theta(x_1\otimes x_2) = x_2\otimes x_1$ defined on the Hilbert space $\ell^2(\mathbb{N})\otimes\ell^2(\mathbb{N})$. We can apply the same proof to the sequence of vectors $(\theta(\eta^{(n)}))_n$, and this is telling us that

$$\lim_{n\to\omega}(1\otimes q_{e_i})(\eta^{(n)}) = 0,$$

for any $i\geqslant 0$. □

2.3.2 End of the proof

Let us prove a general lemma on convergence. It will be useful to apply the result of proposition 2.3.5 to sequences in the Hilbert space $\ell^2(\mathbb{N})\otimes V\otimes\ell^2(\mathbb{N})$.

Lemma 2.3.6. *Let $\mathcal{H}$,$\mathcal{K}$ be two Hilbert spaces. Let a,$b\in\mathcal{B}(\mathcal{H})$ be two linear bounded operators on $\mathcal{H}$. We suppose that for any sequence of vectors $x_n\in\mathcal{H}$ such that $\lim_{n\to\omega} a(x_n) = 0$ we have that $\lim_{n\to\omega} b(x_n) = 0$. Then for any sequence of vectors $z_n\in\mathcal{H}\otimes\mathcal{K}$ such that $\lim_{n\to\omega}(a\otimes 1_{\mathcal{K}})(z_n) = 0$ we have that $\lim_{n\to\omega}(b\otimes 1_{\mathcal{K}})(z_n) = 0$.*

Proof. Consider a,b,$\mathcal{H}$,$\mathcal{K}$ as in the hypothesis of the lemma. Let us show that $a(x) = 0$ implies that $b(x) = 0$. Let $x \in \mathcal{H}$ such that $a(x) = 0$, consider the constant sequence $(x_n)_n$ such that $x_n = x$ for any n. We have that $\lim_{n\to\omega} a(x_n) = 0$, so $\lim_{n\to\omega} b(x_n) = 0$, meaning that $b(x) = 0$.

Let us show that there exists a positive constant $C > 0$ such that for any $x \in \mathcal{H}$,

$$\|b(x)\| \leqslant C\|a(x)\|. \tag{2.10}$$

Suppose it is not true, for any $n \geqslant 1$, there exists a vector $x_n \in \mathcal{H}$ such that $\|b(x_n)\| > n\|a(x_n)\|$. The strict inequality implies that $b(x_n) \neq 0$; thus, $a(x_n) \neq 0$. Consider

$$z_n = \frac{x_n}{\|a(x_n)\|\sqrt{n}},$$

we have that $\|a(z_n)\| = \frac{1}{\sqrt{n}}$, so $\lim_{n\to\omega} a(z_n) = 0$. Hence, $\lim_{n\to\omega} b(z_n) = 0$. On the other hand $\|b(z_n)\| \geqslant \sqrt{n}$, a contradiction.

Let us prove the lemma. Let $C > 0$ as in the equation 2.10. Let $\{\varepsilon_i, i \in I\}$ be an orthonormal basis of the Hilbert space $\mathcal{K}$. Let $(z_n)_n$ be a sequence of vectors in $\mathcal{H}\otimes\mathcal{K}$ such that

$$\lim_{n\to\omega}(a \otimes 1_{\mathcal{K}})(z_n) = 0.$$

For any n, there exists some vectors $z_{n,i} \in \mathcal{H}$ such that z_n is decomposed as

$$z_n = \sum_{i\in I} z_{n,i} \otimes \varepsilon_i.$$

We have that

$$\|(a \otimes 1_{\mathcal{K}})(z_n)\|^2 = \|\sum_{i\in I} a(z_{n,i}) \otimes \varepsilon_i\|^2 = \sum_{i\in I} \|a(z_{n,i})\|^2,$$

by the theorem of Pythagoras. On the other hand,

$$\begin{aligned}\|(b \otimes 1_{\mathcal{K}})(z_n)\|^2 = \sum_{i\in I} \|b(z_{n,i})\|^2 &\leqslant \sum_{i\in I} C^2\|a(z_{n,i})\|^2 \\ &\leqslant C^2\|(a \otimes 1_{\mathcal{K}})(z_n)\|^2.\end{aligned}$$

Thus, $\lim_{n\to\omega}(b \otimes 1_{\mathcal{K}})(z_n) = 0$. □

Let x be an element of the relative commutant $M^\omega \cap A'$ such that $E_{A^\omega}(x) = 0$. Consider a representant of x which is a bounded sequence $(x^{(n)})_n$ in M. We choose this sequence such that for any $n \geqslant 0$, $E_A(x^{(n)}) = 0$.

Proposition 2.3.7. *Let $J \geqslant 0$ be an integer and Z_J the closed subspace of $L^2(M_{\mathcal{P}})$ spanned by the set of vectors*

$$\{\cup^{\bullet k} \bullet v \bullet \cup^{\bullet r}, \min(k,r) \leqslant J, v \in V\}.$$

Consider the projection Q_J on this subspace Z_J. Then

$$\lim_{n\to\omega} Q_J(x^{(n)}) = 0.$$

Proof. Consider the unitary transformation

$$\psi : L^2(M_{\mathcal{P}}) \longrightarrow \ell^2(\mathbb{N}) \oplus (\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N}))$$

defined in proposition 2.2.1. By hypothesis, $x^{(n)}$ is in the orthogonal complement of $L^2(A)$, hence its image $\psi(x^{(n)})$is in the vector space $\ell^2(\mathbb{N}) \otimes V \otimes \ell^2(\mathbb{N})$. Let us denote the sequence $\psi(x^{(n)})$ by $\eta^{(n)}$. The sequence $(x^{(n)})_n$ commutes asymptotically with the action of cup, in particular

$$\lim_{n\to\omega}(\pi(\frac{\cup - 1}{\delta^{\frac{1}{2}}}) - \rho(\frac{\cup - 1}{\delta^{\frac{1}{2}}}))(x^{(n)}) = 0.$$

If we conjugate this expression by ψ, we get that

$$\lim_{n\to\omega}((s + s^*) \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} - 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes (s + s^*))(\eta^{(n)}) = 0,$$

by proposition 2.2.1. We are almost in the hypothesis of proposition 2.3.7, but we have an extra tensor by the Hilbert space V. Thus, we apply lemma 2.3.6 with $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \ell^2(\mathbb{N})$, $\mathcal{K} = V$, $a = s + s^*$ and $b = q_{e_i} \otimes 1_{\ell^2(\mathbb{N})}$ or $b = 1_{\ell^2(\mathbb{N})} \otimes q_{e_i}$. We get that for any $i \geqslant 0$,

$$\lim_{n\to\omega}(q_{e_i} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})})(\eta^{(n)}) = 0$$

and

$$\lim_{n\to\omega}(1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes q_{e_i})(\eta^{(n)}) = 0.$$

The projection Q_J is equal to the finite sum of projections

$$\psi^*(\sum_{i=0}^{J}(q_{e_i} \otimes 1_V \otimes 1_{\ell^2(\mathbb{N})} + 1_{\ell^2(\mathbb{N})} \otimes 1_V \otimes q_{e_i}))\psi.$$

Therefore, $\lim_{n\to\omega} Q_J(x^{(n)}) = 0$. ☐

So far, we have only used the bimodule structure of ${}_AL^2(M_{\mathcal{P}})_A$ given in proposition 2.2.1. We show, in the last proposition, that a large part of an element that commutes asymptotically with the cup subalgebra is vanishing. In order to prove the AOP for the cup subalgebra we need to look at the product of elements. In the next lemma, we use the graduation given by the element cup and the product $\bullet$.

Lemma 2.3.8. *Consider an integer $J \geqslant 1$, and the subspace $X_J \subset L^2(M_{\mathcal{P}})$, spanned by the set of vectors:*

$$\{\cup^{\bullet k} \bullet v \bullet \cup^{\bullet r}, k,r \geqslant J, v \in V\}.$$

We have that X_J is the orthogonal complement of Z_{J-1} in $L^2(M_{\mathcal{P}}) \ominus L^2(A)$, i.e.

$$L^2(M_{\mathcal{P}}) = L^2(A) \oplus X_J \oplus Z_{J-1}.$$

On the other hand, consider the Hilbert space Y_J spanned by the set of vectors:

$$\{\cup^{\bullet k} \bullet v \bullet \cup^{\bullet r}, k,r \leqslant J, v \in V\}.$$

Let $\tilde{x} \in X_J \cap M_{\mathcal{P}}$ and $\tilde{y} \in Y_{J-1} \cap M_{\mathcal{P}}$. Then, $\tilde{x} \star \tilde{y}$ is orthogonal to $\tilde{y} \star \tilde{x}$.

Proof. Using the decomposition of the Hilbert space $L^2(M_\mathcal{P})$ given in proposition 2.2.1, it is easy to see that

$$L^2(M_\mathcal{P}) = L^2(A) \oplus X_J \oplus Z_{J-1}.$$

Consider some integers $k,r \geqslant J$, $i,j \leqslant J-1$, and $v,\tilde{v} \in V \cap M_\mathcal{P}$. We have that

$$(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j}) \star (\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r}) = (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k} \bullet \tilde{v} \bullet \cup^{\bullet r}) + (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j+k-1} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \cdots$$
$$+ \delta^j (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet k-j} \bullet \tilde{v} \bullet \cup^{\bullet r}) + \delta^j (\cup^{\bullet i} \bullet v \bullet \cup^{\bullet k-j-1} \bullet \tilde{v} \bullet \cup^{\bullet r}).$$

It is easy to see that $v \bullet \cup^{\bullet n} \bullet \tilde{v}$ is an element of V, whatever the value of $n \geqslant 0$. Hence, the product $(\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j}) \star (\cup^{\bullet k} \bullet \tilde{v} \bullet \cup^{\bullet r})$ is in the vector space

$$\operatorname{span}\{\cup^{\bullet l} \bullet w \bullet \cup^{\bullet m}, l \leqslant J-1, m \geqslant J, w \in V\}.$$

Let $\tilde{x} \in X_J \cap M_\mathcal{P}$ and $\tilde{y} \in Y_{J-1} \cap M_\mathcal{P}$. We have that $\tilde{y} \star \tilde{x}$ is in the closed vector space

$$\overline{\operatorname{span}\{\cup^{\bullet k} \bullet v \bullet \cup^{\bullet r}, k \leqslant J-1, r \geqslant J, v \in V\}}.$$

A similar computation shows that

$$\tilde{x} \star \tilde{y} \in \overline{\operatorname{span}\{\cup^{\bullet l} \bullet v \bullet \cup^{\bullet m}, l \geqslant J, m \leqslant J-1, v \in V\}}.$$

Therefore, $\tilde{x} \star \tilde{y}$ and $\tilde{y} \star \tilde{x}$ are orthogonal. $\square$

Theorem 2.3.9. *The cup subalgebra has the AOP.*

Proof. Let $(x^{(n)})_n$ as defined in the beginning of the section. The multiplication by x in $M_\mathcal{P}^\omega$ and the involution $*$ are weakly continuous, hence $y \mapsto x \star y \star x^* \star y^*$ is weakly continuous. The trace of $M_\mathcal{P}^\omega$ is weakly continuous, hence the function

$$y \in M_\mathcal{P} \longmapsto \lim_{n \to \omega} \langle y \star x^{(n)}, x^{(n)} \star y \rangle$$

is weakly continuous. The algebraic sum of the $\mathcal{P}_j$ is dense in $M_\mathcal{P}$ for the weak topology. Thus, to show that $A \subset M_\mathcal{P}$ has the AOP, it is sufficient to show that for any y in the algebraic sum of the $\mathcal{P}_j$, such that $E_A(y) = 0$, we have that: $\lim_{n \to \omega} \langle y \star x^{(n)}, x^{(n)} \star y \rangle = 0$.

Let $J \geqslant 0$ and consider

$$y \in \bigoplus_{j=0}^{J} \mathcal{P}_j,$$

such that $E_A(y) = 0$. If $i,j \geqslant 0$ and $v \in V_k$, we have that the vector $\cup^{\bullet i} \bullet v \bullet \cup^{\bullet j}$ is in $\mathcal{P}_{i+j+k}$, where $k \geqslant 1$. Hence, y is necessarily in the space Y_{J-1}. The vector $x^{(n)} \in L^2(M_\mathcal{P}) \ominus L^2(A)$, and by lemma 2.3.8, $L^2(M_\mathcal{P}) \ominus L^2(A) = Z_J \oplus X_{J+1}$. Hence, $x^{(n)} = Q_J(x^{(n)}) \oplus Q_J^\perp(x^{(n)})$, where $Q_J^\perp$ is the orthogonal projection on the space X_{J+1}.

By lemma 2.3.8, for any $n \geqslant 0$, $Q_J^\perp(x^{(n)}) \star y$ is orthogonal to $y \star Q_J^\perp(x^{(n)})$. We have:

$$\begin{aligned} |\langle y \star x^{(n)}, x^{(n)} \star y \rangle| &= |\langle y \star (Q_J(x^{(n)}) + Q_J^\perp(x^{(n)})), (Q_J(x^{(n)}) + Q_J^\perp(x^{(n)})) \star y \rangle| \\ &\leqslant |\langle y \star Q_J(x^{(n)}), x^{(n)} \star y \rangle| + |\langle y \star Q_J^\perp(x^{(n)}), Q_J(x^{(n)}) \star y \rangle| \\ &\leqslant 2\|y\|_{M_\mathcal{P}}^2 \|Q_J(x^{(n)})\| \|x^{(n)}\|. \end{aligned}$$

By proposition 2.3.7, $\lim_{n \to \omega} Q_J(x^{(n)}) = 0$. Hence, we have that

$$\lim_{n \to \omega} \langle y \star x^{(n)}, x^{(n)} \star y \rangle = 0.$$

Therefore, the cup subalgebra $A \subset M_\mathcal{P}$ has the AOP. $\square$

Theorem 2.3.10. *The cup subalgebra is maximal hyperfinite.*

Proof. The cup subalgebra is a singular MASA by corollary 2.2.2. Furthermore, it has the AOP by theorem 2.3.9. Hence, by theorem 2.1.4, the cup subalgebra is maximal hyperfinite. $\square$

Chapitre 3

The Takesaki invariant is the equivalence relation induced by the normalizer

Introduction

Let $A \subset M$ be a maximal abelian subalgebra (MASA) in a II_1 factor with separable predual. The study of MASAs began with the work of Dixmier in 1954 [9], where he looked at the von Neumann algebra generated by the group normalizer.

Definition 3.0.11. Two MASAs $A_1 \subset M_1$ and $A_2 \subset M_2$ are called isomorphic if there exists an isomorphism of von Neumann algebras $\phi : M_1 \longrightarrow M_2$ such that $\phi(A_1) = A_2$.

Group normalizer:
We define the group normalizer $N_M(A)$ that is the group of unitaries $u \in M$ such that $uAu^* = A$. If the von Neumann algebra generated by $N_M(A)$ is equal to A, the MASA is called singular. If the von Neumann algebra generated by $N_M(A)$ is equal to M, the MASA is called regular, which is equivalent to be a Cartan subalgebra in the II_1 case.

There are surprising results in the study of regular MASAs. Voiculescu proved that a free group factor does not contain any Cartan subalgebras [63]. This result has been improved with the introduction of the notion of solidity and strong solidity [40, 41, 22]. On the other hand, the theorem of Connes-Feldman-Weiss [7] shows that the hyperfinite II_1 factor contains a unique Cartan subalgebra (up to automorphism). One driving question was if there exists some II_1 factors with two non isomorphic Cartan subalgebras, which was confirmed by Connes and Jones in [8]. On the other hand Popa showed [44] that many factors contain at least one singular MASA, the technics to study them are very different than those for Cartan subalgebras. Some analytic characterizations, like the WHAP or the strong singularity [44, 48], turns out to be equivalent to singular [59, 58]. It seems that the notion of singularity is stronger that it appears.

A measure-theoretic invariant for MASAs with separable predual was introduced by Takesaki in 1963 [60]. An explicit presentation of this invariant is given below. Note that we adapt the definition of Takesaki to the specific case of MASAs in II_1 factors.

Let us first define the Takesaki equivalence relation. Let $(Y,\mathcal{D},\nu)$ be a standard probability space such that A is isomorphic to the von Neumann algebra of bounded measurable complex valued functions $L^\infty(Y,\nu)$. Let $L^2(M)$ be the GNS Hilbert space associate to the unique trace of M and $x \mapsto x\Omega$ the embedding of M in $L^2(M)$. Let π,ρ be the left and right action of M on the Hilbert space $L^2(M)$, i.e. $\pi(x)\rho(y)(z\Omega) = xzy\Omega$. Consider a measurable field of Hilbert spaces $\{\mathcal{K}_t, t \in Y\}$ such that $L^2(M)$ is equal to the direct integral

$$\int_Y^{\oplus} \mathcal{K}_t d\nu(t),$$

such that $\rho(A)$ becomes the algebra of all diagonalizable operators. Let $B \subset M$ be a separable C^*-subalgebra that is dense for the ultraweak topology. Consider a measurable field of representations of B, $\{\pi_t, t \in Y\}$, such that

$$\pi|_B = \int_Y^{\oplus} \pi_t d\nu(t),$$

where $\pi|_B$ denotes the restriction to B of the standard representation.

Definition 3.0.12. Let $\mathcal{R}$ be the equivalence relation on Y such that $(s,t) \in \mathcal{R}$ if and only if the representation π_s is unitarily equivalent to π_t. It is the Takesaki equivalence relation.

Note that in this definition we use the following fact:

Theorem 3.0.13. *Let B be a separable C^*-algebra, let $(Y,\mathcal{D},\nu)$ be Borel space with a σ-finite Borel measure. Consider*

$$\mathcal{K} = \int_Y^{\oplus} \mathcal{K}_t d\nu(t)$$

a direct integral of Hilbert spaces. Let $\pi : B \longrightarrow \mathcal{B}(\mathcal{K})$ be a representation of the C^*-algebra B that commutes with the diagonal algebra. Then, there exists a measurable field of representations of the C^*-algebra B over the measured space (Y,ν), $\{\pi_t, t \in Y\}$, such that

$$\pi = \int_Y^{\oplus} \pi_t d\nu(t).$$

See Dixmier [11] or Takesaki [61] for a proof. We write $\pi_s \simeq \pi_t$ to say that the two representations are unitarily equivalent. We identify an equivalence relation and the subset of Y^2 associates. Note that this equivalence relation is an analytic subset of Y^2, see [60][Theorem 1.3]. Before giving the Takesaki invariant, let us define a partial order on the subsets of Y^2.

Definition 3.0.14. Let $E,F \subset Y^2$ be some subsets, we say that E is almost include in F if there exists a null set $N \subset Y$ such that $E \backslash N^2 \subset F$, where $E \backslash N^2 = \{x \in E, x \notin N^2\}$. We denote this by $E \prec F$, its defines a partial order. We say that E is equivalent to F if $E \prec F$ and $F \prec E$ and denote it by $E \equiv F$. This defines an equivalence relation on the subsets of Y^2.

Let us give the definition of the Takesaki invariant.

Definition 3.0.15. Let $\widehat{\mathcal{R}}$ be the equivalence class of $\mathcal{R}$ for $\equiv$. It is an invariant for the MASA $A \subset M$ that we call the Takesaki invariant. In particular it does not depend of the choice of the C^*-algebra B. We say that a MASA is Takesaki simple if $\mathcal{R} \equiv \Delta Y$, where $\Delta Y = \{(t,t), t \in Y\}$ is the diagonal of Y.

Consider the normalizer $N_M(A)$, it is acting by automorphism on A as follows:

$$ad : N_M(A) \longrightarrow Aut(A)$$

with $ad(u)(f) = ufu^*$ for any $u \in N_M(A)$ and $f \in A$. The group of automorphisms of the von Neumann algebra A is denoted by $Aut(A)$. Let $\mathfrak{I}(Y,\nu)$ the group of measure preserving Borel automorphism of $(Y,\mathcal{D},\nu)$. The group action ad induced a group action on the probability space $(Y,\mathcal{D},\nu)$. We get that there exists a group homomorphism

$$\begin{aligned}\Theta : N_M(A) &\longrightarrow \mathfrak{I}(Y,\nu)\\ u &\longmapsto \Theta_u,\end{aligned}$$

such that $(ufu^*)(t) = f(\Theta_u(t))$ for any $u \in N_M(A)$, $f \in A$ and any $t \in Y$.

Hence, for any countable subgroup $G < N_M(A)$ we can associate the orbital equivalence relation

$$\mathcal{N}_G = \{(\Theta_u(t),t), t \in Y, u \in G\}.$$

Note that we call a set countable if it can be embedded in the set of natural numbers, hence a finite set is countable. Takesaki shows that for any countable subgroup $G < N_M(A)$, we have that $\mathcal{N}_G \prec \mathcal{R}$. The main result of this paper is to show the converse. It means that there exists a countable subgroup $G < N_M(A)$ such that $\mathcal{N}_G \equiv \mathcal{R}$.

To do this we will introduce two other subsets of Y^2 that we call the weak Takesaki equivalence relation $\mathcal{WR}$ and the set of atoms $\mathcal{Y}$ that are coming from the bimodule structure of $L^2(M)$. We show that there exists a countable subgroup $G_{\max} < N_M(A)$ such that

$$\{G_{\max} \cup A\}'' = N_M(A)''$$

as von Neumann subalgebras of M. This implies that for any countable subgroup $H < N_M(A)$ we have that the orbital equivalence relation $\mathcal{N}_H \prec \mathcal{N}_{G_{\max}}$. We consider such a subgroup $G_{\max}$ and show that

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}_{G_{\max}}.$$

We show also that the Takesaki equivalence relation is the same equivalence relation considered by Feldman and Moore in there work on Cartan subalgebras [15, 16]. Then we look at tensor product of MASAs, $A \subset M$ where $A = \bigotimes_l A_l$ and $M = \bigotimes M_l$. We compute the Takesaki invariant and give an alternative proof of a result of Chifan [4], that says that the von Neumann subalgebra of M generated by $N_M(A)$ is equal to the tensor product of the von Neumann subalgebras of the M_l generated by the normalizers $N_{M_l}(A_l)$, i.e.

$$N_M(A)'' = \bigotimes_l N_{M_l}(A_l)''.$$

Notation 3.0.16. If $\mathcal{K}$ is a Hilbert space, the von Neumann algebra of bounded linear operators is denoted by $\mathcal{B}(\mathcal{K})$. If M is a subset of $\mathcal{B}(\mathcal{K})$, we denote the commutant of M by M' and the bicommutant of M by M''. If we consider a subset S of M, like a subgroup of $N_M(A)$, we denote by S'' the von Neumann subalgebra of M generated by S.

Finally, we consider inclusions of discrete countable groups $H < G$ such that the corresponding inclusion of there group von Neumann algebras $L(H) \subset L(G)$ is a MASA. Using elementary technics, we prove that the Takesaki equivalence relation is equivalent under the relation "$\equiv$" to the equivalence relation induced by the action of the group normalizer

$$N_G(H) = \{g \in G, gHg^{-1} = H\}.$$

We give a new proof of a result of Fang et al. [14] that says that $N_G(H)$ and $N_{L(G)}(L(H))$ generate the same von Neumann algebra.

Here is a detail plan of the paper.

In all the paper we consider a fix Hausdorff compact space Y with a Borel probability measure ν such that A is isomorphic to the space of measurable bounded complex valued functions $L^\infty(Y,\nu)$. We identify the two von Neumann algebras A and $L^\infty(Y,\nu)$. We fix a separable and ultraweakly dense C^*-subalgebra $B \subset M$ and suppose that it contains the abelian C^*-algebra of continuous complex valued functions $\mathcal{C}(Y)$. Thus, we have the following square of inclusions:

$$\begin{array}{ccc} L^\infty(Y,\nu) & \subset & M \\ \cup & & \cup \\ \mathcal{C}(Y) & \subset & B \end{array}.$$

In section 3.1, we introduce and study a subset $\mathcal{Y}$ of Y^2 that we call the set of atoms. Let $\mathcal{A} = \{\pi(A),\rho(A)\}'' \subset \mathcal{B}(L^2(M))$ be the abelian von Neumann subalgebra generated by the left and right action of A on $L^2(M)$.

Following Feldman and Moore [16][proof of theorem 1, p336-337], we have that the inclusion $A \subset \mathcal{A}$ is isomorphic to $\imath : L^\infty(Y,\nu) \hookrightarrow L^\infty(Y^2,\mu)$, where μ is a Borel probability measure on Y^2. Consider the automorphism ϕ of $\mathcal{A}$ defined such that $\phi(\pi(f)) = \rho(f)$ and $\phi(\rho(f)) = \pi(f)$ for any $f \in A$. This automorphism induces the flip $\theta(s,t) = (t,s)$ on Y^2 and lets invariant the class of the measure μ. Thus $\theta_*\mu$ and μ are equivalent measures, where $\theta_*\mu(E) = \mu(\theta^{-1}(E))$ is the push-forward measure.

Definition 3.0.17. Let ν_1,ν_2 be two measures on a measurable space X, we say that ν_1 is equivalent to ν_2 if for any measurable subset $N \subset X$, $\nu_1(N) = 0$ if and only if $\nu_2(N) = 0$. In that case we write $\nu_1 \approx \nu_2$.

The inclusions $f \in A \longmapsto \pi(f) \in \mathcal{A}$ and $f \in A \longmapsto \rho(f) \in \mathcal{A}$ induce the coordinate projections from Y^2 into Y. That tells us that the push-forward measures $p_{k*}\mu$ are equivalent to ν, where $p_1(s,t) = s$ and $p_2(s,t) = t$. We want to disintegrate μ with respect to the function p_2 in the following sense:

Definition 3.0.18. Let $(Z,\mathcal{E},\mu)$ be a probability space, $(Y,\mathcal{D},\nu)$ a measure space with a $\sigma-$finite measure and $p : Z \longrightarrow Y$ a measurable map. We call a disintegration of μ with respect to (p,ν) a family $\{\mu_t, t \in Y\}$ of probability measure on Z, such that μ_t is supported on the measurable set $p^{-1}(\{t\})$ for any $t \in Y$. For any positive measurable function $f : Z \longrightarrow \mathbb{R}_+$, the map $t \mapsto \mu_t(f)$ is measurable, and

$$\mu(f) = \int_Y \mu_t(f) d\nu(t).$$

Such a family of measures $\{\mu_t, t \in Y\}$ is also called the family of conditioning measures.

Such a disintegration exists for μ and is unique almost everywhere, see Bourbaki [2][Chap. 6, §3]. Thus there exists a family $\{\mu_t, t \in Y\}$ of probability measures on Y^2 such that their supports are included in $Y \times \{t\}$. We view them as measures on Y. Consider the set of atoms

$$\mathcal{Y} = \{(s,t) \in Y^2, \mu_t(\{s\}) > 0, \mu_s(\{t\}) > 0\}.$$

Definition 3.0.19. We call $\mathcal{Y}$ the set of atoms of the MASA $A \subset M$.

The consideration of this set is the main idea of this paper. It satisfies some nice properties and we will easily embeds the Takesaki equivalence relation in it.

In proposition 3.1.1, we show that $\mathcal{Y}$ is measurable such that for any $t \in Y$, the set $\mathcal{Y}(t) = \{s, (t,s) \in \mathcal{Y}\}$ and $\theta(\mathcal{Y})(t) = \{s, (s,t) \in \mathcal{Y}\}$ are countable.

Notation 3.0.20. Let X be a subset of Y^2, we write $X(t) = \{s \in Y, (t,s) \in X\}$.

In proposition 3.1.2, we prove the following. Suppose that $h : Z \subset Y \longrightarrow Y$ is a measurable function such that its graph Γ_h is contained in $\mathcal{Y}$. Then for any measurable subset $N \subset Y$, N is a null set for the measure ν if and only if its image $h(N)$ is a null set for the measure ν.

Notation 3.0.21. If $h : X \longrightarrow Z$ is a function, we denote its graph $\{(h(t),t), t \in X\}$ by Γ_h.

We conclude this section by using a theorem of Feldman and Moore [15] to show that there exists a countable family of elements in $\mathfrak{I}(Y,\nu)$ such that $\mathcal{Y}$ is equivalent, for the relation "$\equiv$", with the union of their graphs.

In section 3.2, we study the left action of the C^*-algebra of continuous functions $\mathcal{C}(Y)$ on the Hilbert space $L^2(M)$ with respect to the right action of A. Hence, in section 3.2.1, we find a well adapted decomposition of $L^2(M)$ with respect to the right action of A. To do this, we consider an abstract decomposition of $L^2(M)$ with respect to to the abelian von Neumann algebra $\mathcal{A}$. It means that we look at a measurable field of Hilbert spaces $\{\mathcal{H}_{s,t}, (s,t) \in Y^2\}$ over the measure space (Y^2,μ) and a unitary transformation

$$\psi : L^2(M) \longrightarrow \int_{Y^2}^{\oplus} \mathcal{H}_{s,t} d\mu(s,t)$$

such that the von Neumann algebra $\mathcal{A}$ becomes the diagonal algebra. Then, we consider the decomposition of the measure μ that is

$$\mu = \int_Y \mu_t d\nu(t).$$

For any $t \in Y$, we have that $\{\mathcal{H}_{s,t}, s \in Y\}$ is a measurable field of Hilbert spaces over the measure space (Y,μ_t). Hence, we can define the direct integral of

$$\mathcal{K}_t = \int_Y^{\oplus} \mathcal{H}_{s,t} d\mu_t(s).$$

It turns out that the family $\{\mathcal{K}_t, t \in Y\}$ is a measurable field of Hilbert spaces over the measure space (Y,ν), we denote the direct integral by

$$\mathcal{K} = \int_Y^{\oplus} \mathcal{K}_t d\nu(t).$$

In this decomposition, the von Neumann algebra $\rho(A)$ is conjugate to the diagonal algebra. Let us denote the unitary transformation from the Hilbert space $L^2(M)$ onto $\mathcal{K}$ by ϕ. All those technicalities have been study by Guichardet [19][Proposition 1]. Therefore, we get a unitary transformation $\phi = \varphi \circ \psi$ from $L^2(M)$ into $\mathcal{K}$ such that for any $f \in A$ and any $\zeta \in \mathcal{K}$,

$$\phi\rho(f)\phi^*\zeta = \int_Y^{\oplus} f(t)\zeta_t d\nu(t).$$

This is the decomposition that we were looking for. It is well adapted to look at the left action of A.

Consider the restriction to the C^*-algebra B of the left action on $L^2(M)$ that we denote by $\pi|_B$. Using the decomposition of the Hilbert space $L^2(M)$ that we just give, we get a measurable field of representations $\{\pi_t, t \in Y\}$ over the measure space (Y,ν), where $\pi_t : B \longrightarrow \mathcal{B}(\mathcal{K}_t)$ and for any $b \in B$ and $\zeta \in \mathcal{K}$,

$$\phi\pi(b)\phi^*\zeta = \int_Y^{\oplus} \pi_t(b)\zeta_t d\nu(t).$$

We are able to define the Takesaki equivalence relation which is

$$\mathcal{R} = \{(s,t) \in Y^2, \pi_s \simeq \pi_t\}$$

the set of couple (s,t) such that those specific representations π_s and π_t are unitarily equivalent. Let us restrict the representations π_t to the C^*-subalgebra of continuous functions $\mathcal{C}(Y) \subset B$, we denote those restrictions by $\pi_t|_{\mathcal{C}(Y)}$.

Definition 3.0.22. Consider the equivalence relation $\mathcal{WR}$ which is defined such that $(s,t) \in \mathcal{WR}$ if the representation $\pi_t|_{\mathcal{C}(Y)}$ is unitarily equivalent to $\pi_s|_{\mathcal{C}(Y)}$. We call it the weak Takesaki equivalence relation.

In section 3.2.2, we give an explicit formula of the representations $\pi|_{\mathcal{C}(Y)}$. Then we prove in proposition 3.2.1 that for any continuous function $f \in \mathcal{C}(Y)$, the scalar $f(t)$ is an eigenvalue of the operator $\pi_t(f)$ ν-almost everywhere.

In section 3.3, we consider the action induced by the normalizer $N_M(A)$ on the space $(Y,\mathcal{D},\nu)$. Via the group homomorphism $\Theta : N_M(A) \longrightarrow \mathfrak{I}(Y,\nu)$, we can associate a measure-preserving Borel automorphism $\Theta_u \in \mathfrak{I}(Y,\nu)$ to any unitary $u \in N_M(A)$.

We want to consider the orbital equivalence relation associated , but the group $N_M(A)$ is not countable. Hence we consider countable subgroups $G < N_M(A)$ and the given orbital equivalence relation

$$\mathcal{N}_G = \{(\Theta_u(t),t), t \in Y, u \in G\}.$$

In lemma 3.4.3, we show that for any $h \in \mathfrak{I}(Y,\nu)$ such that its graph $\Gamma_h \subset \mathcal{Y}$, there exists $u \in N_M(A)$ such that $\Theta_u = h$ ν-almost everywhere. This allows us to prove in proposition 3.3.2 that two countable subgroups $G,H < N_M(A)$ satisfies the inclusion of von Neumann algebras

$$\{H \cup A\}'' \subset \{G \cup A\}''$$

if and only if we have $\mathcal{N}_H \prec \mathcal{N}_G$. Then we prove in proposition 3.3.3 that there exists a countable subgroup $G_{\max} < N_M(A)$ such that

$$\{G_{\max} \cup A\}'' = N_M(A)''.$$

This implies that for any countable subgroup $H < N_M(A)$, we have that $\mathcal{N}_H \prec \mathcal{N}_{G_{\max}}$.

Notation 3.0.23. We denote by $\mathcal{N}$ the orbital equivalence relation associated to a fix countable subgroup $G_{\max} < N_M(A)$ such that $\{G_{\max} \cup A\}'' = N_M(A)''$.

In section 3.4, we prove the main result of this paper.

Theorem 3.4.1 claims that all the sets considered are equivalent, meaning that

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}.$$

To prove the theorem we will prove the following chain of inclusions:

$$\mathcal{R} \prec \mathcal{WR} \prec \mathcal{Y}.$$

Then we construct a countable subgroup $G < N_M(A)$ such that $\mathcal{Y} \prec \mathcal{N}_G$. By maximality of $G_{\max}$, we have that $\mathcal{Y} \prec \mathcal{N}$. Then we show, by using a result of Takesaki, that for any countable subgroup $H < N_M(A)$ we have that $\mathcal{N}_H \prec \mathcal{R}$. Therefore, $\mathcal{N} \prec \mathcal{R}$. This implies that

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}.$$

Here is a more detailed explanation of the proof of this result: The first inclusion $\mathcal{R} \prec \mathcal{WR}$ is evident by definition.

To prove the second one, $\mathcal{WR} \prec \mathcal{Y}$, we use the fact that the bimodule ${}_AL^2(M)_A$ contains the bimodule ${}_AL^2(A)_A$. We consider an injective continuous function $f \in \mathcal{C}(Y)$. By proposition 3.2.1, we have that $f(t)$ is an eigenvalue of $\pi_t(f)$ for any t outside of a null set $N \subset Y$. We suppose that $(s,t) \in \mathcal{WR}$ and $s,t \notin N$, hence $\pi_s(f)$ and $\pi_t(f)$ are unitarily conjugate. Thus, $f(s)$ is an eigenvalue of the operator $\pi_t(f)$. We use the fact that $\mathcal{K}_t$ is equal to the direct integral

$$\int_{Y^2}^{\oplus} \mathcal{H}_{s,t} d\mu_t(s)$$

as defined in section 3.2.1. We show in lemma 3.4.2 that if $f(s)$ is an eigenvalue of $\pi_t(f)$, then s is an atom of the measure μ_t. By symmetry of the problem, t is an atom of the measure μ_s, therefore $\mathcal{WR} \prec \mathcal{Y}$.
The third inclusion, $\mathcal{Y} \prec \mathcal{N}$ is technical, it has been partially done in [15], [51] and [36]. We consider an automorphism $h \in \mathfrak{I}(Y,\nu)$ such that its graph $\Gamma_h \subset \mathcal{Y}$. We prove in lemma 3.4.3 that there exists a unitary in the normalizer $u \in N_M(A)$ such that $\Theta_u = h$ ν-almost everywhere. Hence, the graph $\Gamma_h \prec \mathcal{N}_{\langle u \rangle}$, where $\langle u \rangle$ is the subgroup of $N_M(A)$ generated by u. We use theorem 3.1.3, we have that there exists a countable family $\{h_n, n \geqslant 0\}$ such that $h_n \in \mathfrak{I}(Y,\nu)$, $\Gamma_{h_n} \subset \mathcal{Y}$ and

$$\mathcal{Y} \equiv \bigcup_{n \geqslant 0} \Gamma_{h_n}.$$

For any n there exists a unitary $u_n \in N_M(A)$ such that $\Theta_{u_n} = h_n$ ν-almost everywhere. Therefore, we have that $\mathcal{Y} \prec \mathcal{N}_G$, where $G < N_M(A)$ is the countable subgroup generated by the u_n. Then by maximality of $G_{\max}$, we have that $\mathcal{N}_G \prec \mathcal{N}$.
To show that $\mathcal{N} \prec \mathcal{R}$ we use the theorem of Takesaki [60]. Hence, we have proved the equivalence of the four sets.

As a direct corollary we have that a MASA in a II_1 factor is singular if and only if it is simple.

We can interpret the main theorem in terms of the full group of the Takesaki equivalence relation. Let us give some definitions.

Definition 3.0.24. Let $\mathcal{R}_0 \subset Y \times Y$ be an equivalence relation. We consider the set of measure-preserving Borel isomorphisms of the space (Y,ν) that is $\mathfrak{I}(Y,\nu)$. We define an equivalence relation $\doteq$ on the elements of $\mathfrak{I}(Y,\nu)$ such that $h_1 \doteq h_2$ if and only if $h_1(t) = h_2(t)$ ν-almost everywhere. Let $[\mathcal{R}_0]$ be the set of equivalence classes for $\doteq$ of elements $h \in \mathfrak{I}(Y,\nu)$ such that the graph $\Gamma_h \prec \mathcal{R}_0$. It is a group that we call the full group of the equivalence relation $\mathcal{R}_0$.

Definition 3.0.25. Consider the normalizer $N_M(A)$ and the group of unitaries of A that we denote by $U(A)$. The quotient group of $N_M(A)$ by $U(A)$ is called the Weyl group, we denote it by $W_M(A)$.

As a corollary of the fact that $\mathcal{R} \equiv \mathcal{N}$, we have that the Weyl group of $A \subset M$ is isomorphic to the full group of the Takesaki equivalence relation via the map Θ.

In section 3.4.1 we look at the equivalence relation considered by Feldman and Moore in [15, 16]. Consider an equivalence relation $\mathcal{R}_{FM} \subset Y \times Y$ that is a Borel subset, is quasi-invariant and has countable orbits. Let σ be a 2-cocycle on $\mathcal{R}_{FM}$ with value in the circle group. Then we can construct a Cartan subalgebra $A \subset M$ with this equivalence relation together with the 2-cocycle, we denote this Cartan subalgebra by $L^\infty(Y,\nu) \subset M(\mathcal{R}_{FM},\sigma)$. We show that this equivalence relation $\mathcal{R}_{FM}$ is equivalent for "$\equiv$" to the Takesaki equivalence relation in the case where $M(\mathcal{R}_{FM},\sigma)$ is a II_1 factor.

In section 3.5, we consider a family of MASAs in II_1 factors $\{A_l \subset M_l, l \in \Lambda\}$, where Λ is a countable set. Let $A = \bigotimes_l A_l$ and $M = \bigotimes_l M_l$ be the von Neumann algebras equal to the tensor product of those von Neumann algebras. We have that the inclusion $A \subset M$ is a MASA in a II_1 factor. Using the main theorem 3.4.1, we compute easily the Takesaki equivalence relation of this tensor product in function of the Takesaki equivalence relation of the $A_l \subset M_l$ in theorem 3.5.1. As a corollary of this theorem and the theorem 3.4.1 we show that the von Neumann algebra generated by the normalizer $N_M(A)$ is equal to the tensor product of the von Neumann algebras generated by the normalizers $N_{M_l}(A_l)$, i.e.

$$N_M(A)'' = \bigotimes_{l \in \Lambda} N_{M_l}(A_l)''.$$

It is a new proof of a result due to Chifan [4].

In section 3.6, we look at an inclusion of countable discrete groups $H < G$ such that the inclusion of the group von Neumann algebras $L(H) \subset L(G)$ is a MASA. We are looking to the Takesaki equivalence relation for this particular case. Our standard probability space $(Y,\mathcal{D},\nu)$ is replaced by $(\widehat{H},\mathcal{D},\nu)$, where $\widehat{H}$ is the dual group of H, $\mathcal{D}$ its Borel σ-algebra and ν its left invariant Haar measure. The separable weakly dense C^*-algebra $B \subset M$ is replaced by the reduced group C^*-algebra $C^*_r(G) \subset L(G)$ and the C^*-algebra of continuous functions $\mathcal{C}(Y)$ is replaced by the group C^*-algebra $C^*(H)$. Note that H is abelian, hence amenable so its reduced and universal group C^*-algebras coincide. We consider the action of the group normalizer

$$N_G(H) = \{k \in G, kHk^{-1} = H\}$$

on the dual group $\widehat{H}$. It is the action $ad : N_G(H) \longrightarrow Aut(\widehat{H},\nu)$ equal to

$$ad_k(t)(h) = t(khk^{-1}),$$

where $k \in N_G(H)$, $t \in \widehat{H}$ and $h \in H$. We show that the Takesaki equivalence relation is equivalent under the relation "$\equiv$" to the orbital equivalence relation of $N_G(H)$ on $\widehat{H}$ that we denote by $\mathcal{N}$. In particular, this implies that the bicommutant of the normalizer $N_{L(G)}(L(H))$ is equal to the bicommutant of the group normalizer $N_G(H)$. It is a new proof of a result of Fang et al. [14].

3.1 The set of atoms

We keep the notations given in the introduction. We assume that the measures are complete. It means that they are defined on the completion of the σ-algebras. Note that this does not change the L^∞ and L^2 spaces.

Proposition 3.1.1. *The set $\mathcal{Y}$ is measurable. Furthermore, for any $t \in Y$, the sets $\mathcal{Y}(t)$ and $\theta(\mathcal{Y})(t)$ are countable.*

Proof. See Mukherjee [36][Proposition 3.3] for the measurability of $\mathcal{Y}$. By hypothesis, the μ_t are probability measures, so they are finite. Then they have countably many atoms, telling us that $\mathcal{Y}(t)$ and $\theta(\mathcal{Y})(t)$ are countable for any t. □

We begin by showing that any null set of the set of atoms $\mathcal{Y}$ is contained in the square of a null set of Y, where a null set is a set of measure zero. In particular, any measurable function that has its graph in $\mathcal{Y}$ conserves the class of the measure ν.

Proposition 3.1.2. *Let $X \subset \mathcal{Y}$ be a measurable subset. Then, the following assertions are equivalent:*

1. *X is a null set for μ;*

2. $p_1(X)$ is a null set for ν;

3. $p_2(X)$ is a null set for ν.

In particular, if h is a measurable function and $N \subset Y$ a measurable subset such that the restricted graph $\Gamma_N = \{(h(t),t), t \in N\} \subset \mathcal{Y}$, we have that N is a null set for ν if and only if $h(N)$ is a null set for ν. In particular, if $\tilde{N} \subset \mathcal{Y}$ is a null set for the measure μ, then there exists a null set $N \subset Y$ such that $\tilde{N} \subset N^2$.

Proof. We have that

$$\mu(X) = \int_Y \mu_t(X) d\nu(t) = \int_{p_2(X)} \mu_t(X) d\nu(t).$$

The set X is contained in $\mathcal{Y}$, hence for any $t \in p_2(X)$, $\mu_t(X) > 0$. Therefore $\mu(X) = 0$ if and only if $p_2(X)$ is a null set. We know that the class of the measure μ is invariant under the flip; thus, $\mu(X) = 0$ if and only if $\mu(\theta(X)) = 0$ if and only if $p_1(X)$ is a null set. Let h and N be defined as in the proposition. We have that $p_1(\Gamma_N) = h(N)$ and $p_2(\Gamma_N) = N$. Thus by the first affirmation of the proposition we have that $\nu(N) = 0$ if and only if $h(N)$ is a null set.

Let $\tilde{N} \subset \mathcal{Y}$ such that $\mu(\tilde{N}) = 0$, we have that $p_1(\tilde{N})$ and $p_2(\tilde{N})$ are null sets for the measure ν. Hence,

$$N = p_1(\tilde{N}) \cup p_2(\tilde{N})$$

is a null set and $\tilde{N} \subset N^2$. □

Theorem 3.1.3. *There exists a countable family $\{h_n, n \in \mathbb{N}\}$ of measure-preserving Borel automorphisms $h_n : (Y,\nu) \longrightarrow (Y,\nu)$ such that*

$$\mathcal{Y} \equiv \bigcup_{n \in \mathbb{N}} \Gamma_{h_n}.$$

Proof. The set of atoms $\mathcal{Y}$ is measurable by proposition 3.1.1; thus, there exists a null set $\tilde{N} \subset Y^2$ such that $\mathcal{Y} \backslash \tilde{N}$ is Borel. The Borel set $X := \mathcal{Y} \backslash \tilde{N}$ is a standard Borel space such that the fibers, $X(t)$ and $\theta(X)(t)$ are countable for any t. Following Feldman and Moore [15][Proof of theorem 1], there exists a countable family $\{h_n, n \in \mathbb{N}\}$ of Borel isomorphisms such that

$$\mathcal{Y} \backslash \tilde{N} = \bigcup_{n \in \mathbb{N}} \Gamma_{h_n}.$$

By proposition 3.1.2, the maps h_n conserve the class of the measure ν; thus, are some measure-preserving Borel isomorphisms. Furthermore, by the last proposition, there exists a null set $N \subset Y$ such that $\tilde{N} \cap \mathcal{Y} \subset N^2$. Hence, we have that

$$\mathcal{Y} \equiv \bigcup_{n \in \mathbb{N}} \Gamma_{h_n}.$$

□

3.2 Left action of $\mathcal{C}(Y)$ in the module $L^2(M)_A$

3.2.1 The module structure of $L^2(M)_A$

The result of this section has been proved by Guichardet [19][Proposition 1].

The abelian von Neumann algebra $\mathcal{A}$ equal to the bicommutant of $\{\pi(A),\rho(A)\}$ in $\mathcal{B}(L^2(M))$ is acting on the Hilbert space $L^2(M)$, and we can write it as a direct integral such that $\mathcal{A}$ is the diagonal algebra. It means that there exists a measurable field of Hilbert spaces $\{\mathcal{H}_{s,t}, (s,t) \in Y^2\}$ and a unitary transformation

$$\psi : L^2(M) \longrightarrow \int_{Y^2}^{\oplus} \mathcal{H}_{s,t} d\mu(s,t),$$

such that for any

$$\xi = \int_{Y^2}^{\oplus} \xi_{s,t} d\mu(s,t) \in \int_{Y^2}^{\oplus} \mathcal{H}_{s,t} d\mu(s,t),$$

and any $f,g \in A$,

$$\psi\pi(f)\rho(g)\psi^*\xi = \int_{Y^2}^{\oplus} f(s)\xi_{s,t}g(t)d\mu(s,t).$$

We denote this direct integral of Hilbert spaces by $\mathcal{H}$.

Consider a disintegration of the measure μ with respect to the second coordinate projection p_2 as in definition 3.0.18:

$$\mu = \int_Y \mu_t d\nu(t).$$

Let us fix a $t \in Y$ and consider the family $\{\mathcal{H}_{s,t}, s \in Y\}$. It is a measurable field of Hilbert spaces over the measure space (Y,μ_t). Hence, we can define the Hilbert space of the direct integral of this family

$$\mathcal{K}_t = \int_Y^{\oplus} \mathcal{H}_{s,t} d\mu_t(s).$$

The family $\{\mathcal{K}_t, t \in Y\}$ is a measurable field of Hilbert spaces over the measure space (Y,ν). Consider its direct integral,

$$\mathcal{K} = \int_Y^{\oplus} \mathcal{K}_t d\nu(t).$$

If

$$\xi = \int_{Y^2}^{\oplus} \xi_{s,t} d\mu(s,t) \in \mathcal{H},$$

one can define for any $t \in Y$ the vector

$$\zeta_t = \int_Y^{\oplus} \xi_{s,t} d\mu_t(s) \in \mathcal{K}_t.$$

We have that the family $\{\zeta_t, t \in Y\}$ is a measurable vector field over the measure space (Y,ν) and defines a vector of $\mathcal{K}$

$$\zeta = \int_Y^{\oplus} \zeta_t d\nu(t).$$

The function $\xi \longmapsto \zeta$ that we just explain defined a unitary transformation from $\mathcal{H}$ into $\mathcal{K}$ that we denote by φ. Furthermore, if $\phi = \varphi \circ \psi$, then for any $f \in A$ and $\zeta \in \mathcal{K}$,

$$\phi\rho(f)\phi^*\zeta = \int_Y^{\oplus} f(t)\zeta_t d\nu(t).$$

3.2.2 Disintegration of the left action of $\mathcal{C}(Y)$ in $L^2(M)_A$

Consider the left action of B on the Hilbert space $L^2(M)$ that we denote by $\pi|_B$. The decomposition

$$\phi : L^2(M) \longrightarrow \int_Y^{\oplus} \mathcal{K}_t d\nu(t)$$

satisfies that

$$\phi\rho(f)\phi^*\zeta = \int_Y^{\oplus} f(t)\zeta_t d\nu(t).$$

The C^*-algebra B is separable, hence by theorem 3.0.13, the representation $\pi|_B$ admits a decomposition with respect to the unitary transformation ϕ. It means that there exists a measurable representation field $\{\pi_t, t \in Y\}$ over the measure space (Y,ν) such that $\pi_t : B \longrightarrow \mathcal{B}(\mathcal{K}_t)$ is a representation of the C^*-algebra B on the Hilbert space $\mathcal{K}_t$ such that for any $b \in B$,

$$\phi\pi(b)\phi^* = \int_Y^{\oplus} \pi_t(b)d\nu(t).$$

We consider a fixed decomposition $\{\pi_t, t \in Y\}$ in the rest of the paper. This allow us to consider the Takesaki equivalence relation $\mathcal{R}$ and the weak Takesaki equivalence relation $\mathcal{WR}$ associated to this decomposition, see definitions 3.0.15 and 3.0.22.

Let $f \in \mathcal{C}(Y)$ be a continuous function. Consider the operator

$$\begin{aligned} f_t : \mathcal{K}_t &\longrightarrow \mathcal{K}_t \\ \int_Y^{\oplus} \eta_s d\mu_t(s) &\longmapsto \int_Y^{\oplus} f(s)\eta_s d\mu_t(s). \end{aligned}$$

We remark that $\{f_t, t \in Y\}$ is a bounded measurable operator field, and that

$$\phi\pi(f)\phi^* = \int_Y^{\oplus} f_t d\nu(t).$$

Furthermore,

$$\begin{aligned} r_t : \mathcal{C}(Y) &\longrightarrow \mathcal{B}(\mathcal{K}_t) \\ f &\longmapsto f_t, \end{aligned}$$

is a representation of C^*-algebra. Furthermore, the family $\{r_t, t \in Y\}$ is a measurable field of representations of $\mathcal{C}(Y)$ over the measure space (Y,ν) that satisfies that for any $f \in \mathcal{C}(Y)$,

$$\phi\pi(f)\phi^* = \int_Y^{\oplus} r_t(f)d\nu(t).$$

Hence, by uniqueness of disintegration of representation, we have that the restriction of π_t to the C^*-algebra $\mathcal{C}(Y)$ is equal to r_t ν-almost everywhere.

Proposition 3.2.1. *Let $f \in \mathcal{C}(Y)$, the scalar $f(t)$ is an eigenvalue of the operator $\pi_t(f)$ ν-almost everywhere.*

Proof. The Hilbert space $L^2(M)$ is equal to the direct sum $L^2(A) \oplus P$ where P denotes the orthogonal complement of $L^2(A)$ in $L^2(M)$. Consider a disintegration with respect to $\rho(A)$:

$$P = \int_Y^{\oplus} P_t d\nu(t),$$

of the Hilbert space P. The Hilbert space $L^2(A)$ is isomorphic to $L^2(Y,\nu)$, hence the decomposition of this Hilbert space with respect to the algebra $\rho(A)$ is the constant field of one dimensional Hilbert space. Then,

$$L^2(M) = \int_Y^{\oplus} \mathbb{C}_t d\nu(t) \oplus \int_Y^{\oplus} P_t d\nu(t).$$

We denote the one dimensional complex Hilbert space in the fiber associates to $t \in Y$ by $\mathbb{C}_t$. The Hilbert spaces $L^2(A)$ and P are bimodules over A; thus, $\pi(f)$ admits a disintegration with respect to $\rho(A)$ as follows:

$$\pi(f) = \int_Y^{\oplus} (b_t \oplus c_t) d\nu(t),$$

such that b_t acts on $\mathbb{C}_t$ and c_t acts on P_t. The operator b_t is $f(t)$ times the identity function on $\mathbb{C}_t$. Therefore, $f(t)$ is an eigenvalue of b_t. Thus by uniqueness almost everywhere of the disintegration of $\pi(f)$, t is an eigenvalue of $\pi_t(f)$ ν-almost everywhere. $\square$

Remark 3.2.2. Let $N \subset Y$ be a null set such that for any $t \in Y \backslash N$, $f(t)$ is an eigenvalue of $\pi_t(f)$. If $(s,t) \in \mathcal{R} \backslash N^2$, then $f(s)$ is an eigenvalue of $\pi_t(f)$. The operator $\pi_t(f)$ admits countably many eigenvalues. This implies that the Takesaki equivalence relation $\mathcal{R}$ has countable orbits almost everywhere.

3.3 Action of the normalizer, equivalence relation

Consider the action of the normalizer $\Theta : N_M(A) \longrightarrow \mathfrak{I}(Y,\nu)$ on the space $(Y,\mathcal{D},\nu)$. We consider in this section some countable subgroup $G < N_M(A)$ and the orbital equivalence relation induced.

Lemma 3.3.1. *Let $u \in N_M(A)$ and $E \subset Y$ a Borel subset such that if $t \in E$, $\Theta_u(t) \neq t$ almost everywhere. Then, $tr(u\chi_E) = 0$, where χ_E is the characteristic function of the set E, view as an element of the von Neumann algebra A.*

Proof. Let us show that $E_A(u\chi_E) = 0$. Let $f \in L^\infty(Y,\nu)$ be an injective function. We have that

$$\begin{aligned} fE_A(u\chi_E) &= E_A(fu\chi_E) = E_A(u(u^*fu)\chi_E) = E_A(u(f \circ \Theta_u)\chi_E) \\ &= E_A(u\chi_E)(f \circ \Theta_u) = (f \circ \Theta_u)E_A(u\chi_E). \end{aligned}$$

We identify $E_A(u\chi_E)$ with a function of the algebra $L^\infty(Y,\nu)$. We have that $(f - f \circ \Theta_u)(t)E_A(u\chi_E)(t) = 0$ almost everywhere. The function f is injective and $\Theta_u(t) \neq t$ almost everywhere in E. Therefore, $E_A(u\chi_E)(t) = 0$ almost everywhere, hence $E_A(u\chi_E) = 0$. This implies that $tr(u\chi_E) = tr \circ E_A(u\chi_E) = 0$. $\square$

The next proposition shows us that the partial order "$\prec$" is compatible with the inclusion of von Neumann algebras generated by the subgroups of the normalizer $N_M(A)$.

Proposition 3.3.2. *Let $G,H < N_M(A)$ be two countable subgroups. We have the following inclusion of von Neumann algebras*

$$\{H \cup A\}'' \subset \{G \cup A\}'',$$

if and only if the equivalence relations associated satisfy

$$\mathcal{N}_H \prec \mathcal{N}_G.$$

Proof. Suppose that $\mathcal{N}_H \prec \mathcal{N}_G$, let $u \in \{H \cup A\}''$ be a unitary. Let us show that $u \in \{G \cup A\}''$. The group G is countable, we can enumerate it $G = \{v_k, k \geqslant 1\}$. Let

$$E_k = \{t \in Y, \Theta_{v_k}(t) = \Theta_u(t)\}$$

and

$$F_k = E_k \backslash \bigcup_{j<k} E_j.$$

The sets F_k are measurable. Let $p_k = \chi_{F_k}$, it is a projection of A and consider the sum

$$\sum_{k=1}^{\infty} v_k p_k.$$

By lemma 3.3.1, if $k \neq l$ then $v_k p_k \perp v_l p_l$. This sum converges in the von Neumann algebra $\{G \cup A\}''$ to an element v. The graph of Θ_u, $\Gamma_{\Theta_u} \prec \mathcal{N}_G$ by hypothesis. Hence, the disjoint union $\bigcup_k F_k = Y$ up to a null set. Thus, $v = \sum_k v_k p_k$ is a unitary in $\{G \cup A\}''$ and by construction $\Theta_u = \Theta_v$ ν-almost everywhere. Hence, the unitary vu^* is in the kernel of the group homomorphism Θ. Therefore, vu^* is a unitary of A, thus $u \in \{G \cup A\}''$.

Suppose that $\{H \cup A\}'' \subset \{G \cup A\}''$. Consider a unitary $u \in H$ and the set

$$E = \{t \in Y, (\Theta_u(t),t) \notin \mathcal{N}_G\}.$$

The set E is measurable, let $v = up$ where $p = \chi_E$. Consider a unitary $w \in G$ or in the unitary group of A. Let us show that v is orthogonal to w, i.e. $tr(w^*v) = 0$. We have that $\Theta_{w^*u} = \Theta_w^{-1} \circ \Theta_u$. By assumption, for any $t \in E$, $\Theta_u(t) \neq \Theta_w(t)$, hence $\Theta_{w^*u}(t) \neq t$. We can apply lemma 3.3.1; thus, $tr(w^*v) = 0$. Therefore, the partial isometry v is orthogonal to the von Neumann algebra $\{G \cup A\}''$. This implies that $p = 0$ and so the graph of Θ_u is almost include in the equivalence relation $\mathcal{N}_G$. So, $\mathcal{N}_H \prec \mathcal{N}_G$. $\square$

The next proposition allows us to define an equivalence relation associated to the action of the normalizer $N_M(A)$ on the space (Y,ν).

Proposition 3.3.3. *There exists a countable subgroup $G_{max} < N_M(A)$ such that*

$$\{G_{max} \cup A\}'' = N_M(A).$$

In particular, for any countable subgroup $H < N_M(A)$ we have that

$$\mathcal{N}_H \prec \mathcal{N}_{G_{max}}.$$

Proof. Suppose there is no countable subgroup $G < N_M(A)$ such that $\{G \cup A\}'' = N_M(A)''$. Therefore, there exists an uncountable set $(I, <)$ with a partial order and a family of countable subgroup of $N_M(A)$ that is $\{G_i, i \in I\}$ such that the von Neumann algebra $\{G_i \cup A\}''$ is strictly contained in the von Neumann algebra $\{G_j \cup A\}''$ if $i < j$ and $i \neq j$. This implies that the Hilbert space $L^2(M)$ is not separable, a contradiction.

The second assertion of the proposition is trivial by using proposition 3.3.2. $\square$

We fix a countable subgroup $G_{\max} < N_M(A)$ that satisfies the hypothesis of the last proposition and denote by $\mathcal{N}$ its orbital equivalence relation, i.e. $\mathcal{N}_{G_{\max}} = \mathcal{N}$.

Remark 3.3.4. Note that the equivalence class of $\mathcal{N}$ for the relation "$\equiv$" is an invariant for the MASA $A \subset M$.

3.4 The main result

Theorem 3.4.1. *Consider the equivalence relations $\mathcal{R}$, $\mathcal{WR}$, $\mathcal{N}$ with the set of atoms $\mathcal{Y}$. Then,*

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}.$$

Proof. We proceed as we explained in the introduction. By definition, $\mathcal{R} \subset \mathcal{WR}$.

Let us show that $\underline{\mathcal{WR} \prec \mathcal{Y}}$.

Let $f \in \mathcal{C}(Y)$ be an injective continuous function. Let us link the eigenvalues of $\pi_t(f)$ and the set of atoms $\mathcal{Y}$.

Lemma 3.4.2. *There exists a null set $N \subset Y$ such that for any $t \in Y \backslash N$, if $f(s_0)$ is an eigenvalue of $\pi_t(f)$ then s_0 is an atom of the measure μ_t.*

Proof of lemma 3.4.2. Consider the operator

$$f_t : \mathcal{K}_t \longrightarrow \mathcal{K}_t$$
$$\int_Y^{\oplus} \xi_s d\mu_t(s) \longrightarrow \int_Y^{\oplus} f(s)\xi_s d\mu_t(s),$$

by section 3.2 we know that

$$\phi\pi(f)\phi^* = \int_Y^{\oplus} f_t d\nu(t).$$

By uniqueness of the disintegration there exists a null set $N \subset Y$ such that $\pi_t(f) = f_t$ for any $t \in Y \backslash N$. Suppose $t \in Y \backslash N$ and $f(s_0)$ is an eigenvalue of $\pi_t(f)$. Then there exists a non null vector

$$\eta = \int_Y^{\oplus} \eta_s d\mu_t(s) \in \mathcal{K}_t$$

such that $f_t(\eta) = f(s_0)\eta$, meaning that $(f(s) - f(s_0))\eta_s = 0$ μ_t-almost everywhere. This implies that s_0 is an atom of μ_t because f is injective. □

Let $N_0 \subset Y$ be a null set such that $\pi_t(f) = f_t$ for any $t \in Y \backslash N_0$. Let $N_1 \subset Y$ be a null set such that $f(t)$ is an eigenvalue of $\pi_t(f)$ for any $t \in Y \backslash N_1$. Let $N = N_0 \cup N_1$ and $(s,t) \in \mathcal{WR} \backslash N^2$. We have that $\pi_s(f)$ is unitarily equivalent to $\pi_t(f)$ thus $f(s)$ is an eigenvalue of $\pi_t(f)$ and then by proposition 3.2.1, $f(s)$ is an atom of μ_t. If we exchange the roles of s and t we get that $f(t)$ is an eigenvalue of the operator $\pi_s(f)$. Hence,

$$\mathcal{WR} \prec \mathcal{Y}.$$

Let us show that $\underline{\mathcal{Y} \prec \mathcal{N}}$.

To do this, we construct a big enough countable subgroup $G < N_M(A)$ such that $\mathcal{Y} \prec \mathcal{N}_G$.

Lemma 3.4.3. *Consider a measure-preserving Borel automorphism $h \in \mathfrak{I}(Y,\nu)$ such that its graph $\Gamma_h \subset \mathcal{Y}$. Then, there exists a unitary in the normalizer of A, $u \in N_M(A)$ such that $\Theta_u = h$ ν-almost everywhere.*

Proof. Let us write $L^2(M)$ as a direct integral of Hilbert spaces over the measure space (Y^2,μ) as in section 3.2:

$$\psi : L^2(M) \longrightarrow \int_{Y^2}^{\oplus} \mathcal{H}_{s,t} d\mu(s,t).$$

Consider a vector

$$\xi^0 = \int_{Y^2}^{\oplus} \xi^0_{s,t} d\mu(s,t)$$

such that $\|\xi^0_{s,t}\|_{\mathcal{H}_{s,t}} = 1$ μ-almost everywhere. Then the bimodule generated by the vector ξ^0, that is the closure of $A\xi^0 A$ in the Hilbert space $L^2(M)$, is giving us an embedding of bimodules $L^2(Y^2,\mu) \subset L^2(M)$, where the bimodule structure of $L^2(Y^2,\mu)$ is

$$(f_1.(g).f_2)(s,t) = f_1(s)g(s,t)f_2(t)$$

for any $f_1,f_2 \in A$ and $g \in L^2(Y^2,\mu)$.

Let

$$\xi = \chi_{\Gamma_h} \in L^\infty(Y^2,\mu) \subset L^2(M)$$

be the characteristic function of the graph of h. We have that $\|\xi\|_2^2 = \mu(\Gamma_h)$, it is different from zero by lemma 3.1.2, and satisfies that for any $f \in A$, $\pi(f)(\xi) = \rho(f \circ h)(\xi)$. By [58][Appendix B] we can do a spectral decomposition of $\xi = u|\xi|$, where $u \in M$ is a unitary and $|\xi| \in \overline{M^+}$ is in the closure of the cone of positive operators. By [36][Proposition 4.7] we have that $|\xi| \in L^2(A)$ because it satisfies that $A|\xi| = |\xi|A$. Let $f \in A$, we get that

$$\pi(u^* f u - f \circ h)(|\xi|) = 0. \tag{3.1}$$

Let us show that this implies $u^* f u = f \circ h$. Consider $g \in A$ a non null vector, there exists $\varepsilon > 0$ and a measurable subset $E \subset Y$ such that $\nu(E) > 0$ and for any $t \in E$, $|g(t)| > \varepsilon$. Thus, the square of the norm

$$\begin{aligned}\|\rho(g)(\xi)\|_2^2 &= \int_{Y^2} \|g(t)\xi_{s,t}\|^2 d\mu(s,t) \\ &\geqslant \varepsilon^2 \int_{Y\times E} \|\xi_{s,t}\|^2 d\mu(s,t) = \varepsilon^2 \mu(\Gamma_h \cap (Y \times E)).\end{aligned}$$

We have that $\nu(p_2(\Gamma_h \cap (Y \times E))) = \nu(E) > 0$. By proposition 3.1.2, $\mu(\Gamma_h \cap (Y \times E)) > 0$ and then $\rho(g)(\xi) \neq 0$. Hence, $\rho(g)(u|\xi|) \neq 0$ and so $\rho(g)(|\xi|) \neq 0$. Using the adjoint, we get that $g \neq 0$ implies that $\pi(g)(|\xi|) \neq 0$. Hence, equation 3.1 implies that $u^* f u = f \circ h$ ν-almost everywhere, for any $f \in A$. Therefore, $\Theta_u = h$ ν-almost everywhere. $\square$

By theorem 3.1.3 there exists a countable family $\{h_n, n \in \mathbb{N}\}$ in $\mathfrak{I}(Y,\nu)$ such that

$$\mathcal{Y} \equiv \bigcup_n \Gamma_{h_n}.$$

By lemma 3.4.3, there exists some unitaries $\{u_n, n \in \mathbb{N}\}$ such that for any $n \in \mathbb{N}$, $\Theta_{u_n} = h_n$ ν-almost everywhere. Let G be the subgroup of $N_M(A)$ generated by the u_n, we have that $\mathcal{Y} \prec \mathcal{N}_G$. By maximality of the equivalence relation $\mathcal{N}$, we get that $\mathcal{Y} \prec \mathcal{N}$.

Let us show that $\underline{\mathcal{N} \prec \mathcal{R}}$.

To prove this, we consider a countable subgroup $H < N_M(A)$ and show that $\mathcal{N}_H \prec \mathcal{R}$. Let $u \in H$, by [60][theorem 1.2] there exists a null set $N_u \subset Y$ such that

$$\{(\Theta_u(t),t), t \in Y \backslash N_u\} \subset \mathcal{R}.$$

Let

$$N = \bigcup_{u \in H} N_u,$$

it is a null set and $\mathcal{N}_H \backslash N^2 \subset \mathcal{R}$. Hence, $\mathcal{N}_H \prec \mathcal{R}$. In particular, $\mathcal{N} = \mathcal{N}_{G_{\max}} \prec \mathcal{R}$. Hence we have proved that

$$\mathcal{R} \prec \mathcal{WR} \prec \mathcal{Y} \prec \mathcal{N} \prec \mathcal{R}.$$

Therefore they are all equivalent, i.e.

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{Y} \equiv \mathcal{N}.$$

$\square$

Remark 3.4.4. Theorem 3.4.1 and proposition 3.3.2 implies that for any countable subgroup $G < N_M(A)$,

$$\{G \cup A\}'' = N_M(A)''$$

if and only if $\mathcal{N}_G \equiv \mathcal{R}$.

Here is a corollary that answers the question of Takesaki:

Corollary 3.4.5. *Let $A \subset M$ be a MASA in a II_1 factor, then it is singular if and only if it is simple.*

Proof. Let $G < N_M(A)$ be a countable subgroup such that $\{G \cup A\}'' = N_M(A)''$. By theorem 3.4.1, we have that $\mathcal{N}_G \equiv \mathcal{R}$. If $A \subset M$ is singular, we can take the trivial group $G = \{1\}$, hence $\mathcal{R} \equiv \mathcal{N}_G \equiv \Delta Y$. Therefore, $A \subset M$ is simple. If $A \subset M$ is simple, then $\mathcal{N}_G \equiv \mathcal{R} \equiv \Delta Y$. This implies that for any $u \in N_M(A)$, we have that $\Theta_u(t) = t$ ν-almost everywhere. Hence, $u \in A$, therefore $A \subset M$ is singular. $\square$

Let us give a corollary on the full group of the Takesaki equivalence relation.

Corollary 3.4.6. *The map $\Theta : N_M(A) \longrightarrow \mathfrak{I}(Y,\nu)$ induces a group isomorphism between the Weyl group of $A \subset M$ and the full group of the Takesaki equivalence relation.*

Proof. If $u \in N_M(A)$, we have that $\Theta_u \in \mathfrak{I}(Y,\nu)$ and by the fact that $\mathcal{N} \prec \mathcal{R}$, we have that graph of Θ_u is almost included in $\mathcal{R}$. Hence, by composing with the quotient map, the map Θ defines a group morphism from the normalizer $N_M(A)$ to the full group $[\mathcal{R}]$. We denote this morphism by $\tilde{\Theta}$. The fact that $\mathcal{R} \prec \mathcal{N}$ tells us that $\tilde{\Theta}$ is surjective. Furthermore, the kernel of the group morphism $\tilde{\Theta}$ is clearly equal to the group of unitaries of A. Therefore, $\tilde{\Theta}$ induces an isomorphism from the Weyl group $W_M(A)$ onto the full group $[\mathcal{R}]$. $\square$

Remark 3.4.7. We have proved that the Takesaki equivalence relation is induced by the action of the normalizer and this equivalence relation has got countable orbits almost everywhere.

We have consider the weak Takesaki equivalence relation $\mathcal{WR}$. We can in fact defined an invariant for the MASA in that direction. Consider a separable, abelian C^*-subalgebra $D \subset A$, which is weakly dense. One can associate an equivalence relation in Y that we denote by $\mathcal{WR}_D$. Consider the equivalence class for $\equiv$ that is $\widehat{\mathcal{WR}_D}$. If we follow the argument of [60] we have that $\widehat{\mathcal{WR}_D}$ is an invariant for the MASA, hence it does not depend of the choice of the algebra D. In particular, $\mathcal{WR}_D \equiv \mathcal{WR}_{\mathcal{C}(Y)}$, hence by the main result of this paper, we have proved that this invariant coincides with the Takesaki invariant in the type II_1 case. It would be interesting to see if it is still the case in the type III case.

Note that the information encoded by the set of atoms is contained in the bimodule structure of $L^2(Y^2,\mu)$. Hence, the Takesaki invariant is a bimodule information.

3.4.1 The equivalence relation of Feldman and Moore

Let $A \subset M$ be a Cartan subalgebra. In [15, 16], the authors characterize a Cartan subalgebra by an equivalence relation $\mathcal{R}_{FM}$ and a 2-cocycle σ. Let us show, in the case of a type II_1 factor, that this equivalence relation is equivalent for "$\equiv$" to the Takesaki equivalence relation.

Let us recall the construction of Feldman and Moore, see also Connes [6][Section 5.4] or Renault [56]. Consider a standard probability space $(Y,\mathcal{D},\nu)$ and an equivalence relation $\mathcal{R}_{FM} \subset Y \times Y$. We suppose that $\mathcal{R}_{FM}$ is a Borel subset, the orbits of $\mathcal{R}_{FM}$ are countable and the measure ν is quasi-invariant. It means that the saturation of a null set by $\mathcal{R}_{FM}$ is still a null set.

Let

$$\mathcal{R}^{(2)}_{FM} = \{(s,t,u) \in Y^2, (s,t),(t,u) \in \mathcal{R}_{FM}\}$$

and $\sigma : \mathcal{R}^{(2)}_{FM} \longrightarrow \mathbb{T}$ a 2-cocycle in value in the circle group $\mathbb{T}$. It means that σ is a Borel function and

$$\sigma(s,t,u)\sigma(s,u,v) = \sigma(s,t,v)\sigma(t,u,v).$$

Consider the space $\mathcal{C}$ of bounded, Borel functions $f : \mathcal{R}_{FM} \longrightarrow \mathbb{C}$ such that for some integer $n \geqslant 0$, the cardinal of the set $\{t, f(s,t) \neq 0\}$ is smaller than n for any t. We define a multiplication on $\mathcal{C}$ that is:

$$(f * h)(s,t) = \sum_u f(s,u)h(u,t)\sigma(s,u,t).$$

Let $\tilde{\nu}$ be the measure on $\mathcal{R}_{FM}$ defined as follows:

$$\tilde{\nu}(f) = \int_Y \sum_t f(s,t)d\nu(s),$$

for any positive Borel function f on $\mathcal{R}_{FM}$.

Let $L^2(\mathcal{R}_{FM},\tilde{\nu})$ be the Hilbert space of measurable, square integrable functions from $\mathcal{R}_{FM}$ to $\mathbb{C}$. The algebra $\mathcal{C}$ acts by left multiplication on this Hilbert space. We denote the von Neumann algebra equal to the bicommutant of this action by $M(\mathcal{R}_{FM},\sigma)$. Let $A = \{f \in M(\mathcal{R}_{FM},\sigma), f(s,t) = 0 \text{ if } s \neq t\}$ be the subalgebra of diagonal function. We have that A is isomorphic to $L^\infty(Y,\nu)$ and $A \subset M(\mathcal{R}_{FM})$ is a Cartan subalgebra. This is the construction of Feldman and Moore, by [16][Theorem 1], every Cartan subalgebra is of this form.

Let us denote this MASA by $A \subset M$ and suppose that M is a II_1 factor in order to fit with the setting of this article. This is equivalent to assuming that $\mathcal{R}_{FM}$ has finite type (definition 3.3 of [15]).

Corollary 3.4.8. *Let $\mathcal{R}$ be the Takesaki equivalence relation of $A \subset M$, then the equivalence relation $\mathcal{R}_{FM}$ is equivalent to $\mathcal{R}$,i.e.*

$$\mathcal{R} \equiv \mathcal{R}_{FM}.$$

Proof. By [15][Theorem 1], there exists a countable group K and a group homomorphism

$$\begin{aligned} \Xi : K &\longrightarrow \mathfrak{I}(Y,\nu) \\ k &\longmapsto \Xi_k \end{aligned}$$

such that $\mathcal{R}_{FM}$ is the equivalence relation induced by this action

$$\mathcal{R}_{FM} = \{(\Xi_k(t),t), t \in Y, k \in K\}.$$

Let $k \in K$, we can define the function $u_k(s,t) = \delta_{t,\Xi_k(s)}$, where δ is the Kronecker symbol. Here, we identify element in M and the corresponding function from $\mathcal{R}_{FM}$ to $\mathbb{C}$. The element u_k is a unitary of M that satisfies that for any $f \in M$,

$$(u_k f u_k^*)(s,t) = f(\Xi_k(s),\Xi_k(t)).$$

In particular, $u_k \in N_M(A)$. Consider $G < N_M(A)$ the countable subgroup generated by the u_k. We have that $\{G \cup A\}'' = M$, hence $\{G \cup A\}'' = N_M(A)''$. Therefore, by remark 3.4.4, we have that the equivalence relation induced by the action of G, $\mathcal{N}_G$, is equivalent to the Takesaki equivalence relation $\mathcal{R}$. By construction $\mathcal{N}_G = \mathcal{R}_{FM}$, hence $\mathcal{R} \equiv \mathcal{R}_{FM}$. $\square$

3.5 Tensor product of MASAs

The objective of this section is to compute the Takesaki equivalence relation for MASAs constructed from tensor product of MASAs. After characterizing it with the set of atoms, we will deduce a corollary on the tensor product of von Neumann algebras generated by the normalizers.

Let Λ be a countable set and $\{A_l \subset M_l, l \in \Lambda\}$ a family of MASAs in some II_1 factors. Consider the infinite tensor products of von Neumann algebras

$$\bigotimes_{l\in\Lambda} A_l \subset \bigotimes_{l\in\Lambda} M_l.$$

We denote those von Neumann algebras by $A = \bigotimes_l A_l$ and by $M = \bigotimes_l M_l$. By the Tomita commutant theorem we have that:

$$A' \cap M = \bigotimes_l (A_l' \cap M_l) = \bigotimes_l A_l = A,$$

and

$$M' \cap M = \bigotimes_l (M_l' \cap M) = \bigotimes_l (\mathbb{C}.1) = \mathbb{C}.1.$$

Hence $A \subset M$ is still a MASA in a II_1 factor.

Consider some standard probability spaces $(Y_l,\mathcal{D}_l,\nu_l)$ such that A_l is isomorphic to $L^\infty(Y_l,\nu_l)$. We denote by $L^2(M_l)$ the GNS Hilbert space and by π_l,ρ_l the left and right action of the von Neumann algebra M_l on this Hilbert space. Consider the abelian von Neumann algebra generated by $\pi_l(A_l)$ and $\rho_l(A_l)$ that is

$$\mathcal{A}_l = \{\pi_l(A_l),\rho_l(A_l)\}'' \subset \mathcal{B}(L^2(M_l)).$$

We get an isomorphism of von Neumann algebras

$$\mathcal{A}_l \simeq L^\infty(Y_l^2,\mu_l)$$

where μ_l is a symmetric Borel measure on Y_l^2. The pushforward measure $p_{l,2_*}\mu_l$ is equivalent to the measure ν_l, where $p_{l,2}$ is the second coordinate projection. Hence, we get a decomposition of the measure as follows:

$$\mu_l = \int_{Y_l} \mu_{l,t} d\nu_l(t),$$

where $\mu_{l,t}$ is a Borel probability measure on the space Y_l, see section 3.1. We consider the set of atoms

$$\mathcal{Y}_l = \{(s,t) \in Y_l^2, \mu_{l,s}(\{t\}) > 0, \mu_{l,t}(\{s\}) > 0\}.$$

On the other hand, consider the infinite product of spaces $Y = \prod_l Y_l$ and the associate product measure $\nu = \bigotimes_l \nu_l$. We denote an element of Y by $\underline{t}$ and its l-component by t_l. Let $L^2(M)$ be the GNS Hilbert space and π,ρ the left and right action of the von Neumann algebra M on it. We denote the von Neumann subalgebra of $\mathcal{B}(L^2(M))$ generated by $\pi(A)$ and $\rho(A)$ by $\mathcal{A}$. Consider the subset $\mathcal{Y} \subset Y \times Y$ which is the set of couples $(\underline{s},\underline{t})$ such that for any $l \in \Lambda$, we have that $(s_l,t_l) \in \mathcal{Y}_l$.

Theorem 3.5.1. *We have that*

$$\mathcal{Y} \equiv \mathcal{R},$$

where $\mathcal{R}$ is the Takesaki equivalence relation of the MASA $A \subset M$.

Proof. We have the equalities $A = \bigotimes_l A_l$ and $M = \bigotimes_l M_l$, this induce a unitary transformation between the Hilbert spaces $L^2(M)$ with the tensor product of Hilbert spaces $\bigotimes_l L^2(M_l)$. This unitary transformation conjugates the actions π and ρ with the tensor product of actions $\bigotimes_l \pi_l$ and $\bigotimes_l \rho_l$. This implies that we can identify the von Neumann algebra $\mathcal{A}$ with the tensor product of von Neumann algebras $\bigotimes_l \mathcal{A}_l$. Furthermore, we get that the inclusion $\pi(A) \subset \mathcal{A}$ is isomorphic to the inclusion

$$(\bigotimes \pi_l(A_l)) \subset (\bigotimes_l \mathcal{A}_l). \tag{3.2}$$

Let μ be the product measure $\bigotimes_l \mu_l$ on Y^2. The inclusion 3.2 induces the coordinate projection $p(\underline{s},\underline{t}) = \underline{t}$ from (Y^2,μ) into (Y,ν). We get that $p_*\mu$ is equivalent to the measure ν. Consider an element $\underline{t} \in Y$ and the product measure $\mu_{\underline{t}} = \bigotimes_l \mu_{l,t_l}$ on Y. It is easy to check that

$$\mu = \int_Y \mu_{\underline{t}} d\nu(\underline{t}),$$

and this decomposition of the measure is a (p,Y) decomposition, see definition 3.0.18. Let $\underline{s} \in Y$, we have that $\mu_{\underline{t}}(\{\underline{s}\}) = \prod_l \mu_{l,t_l}(\{s_l\})$. Therefore, the set of atoms of $A \subset M$ is equal to $\mathcal{Y}$. By the theorem 3.4.1, we get that the Takesaki equivalence relation $\mathcal{R}$ is equivalent to $\mathcal{Y}$ for the relation "$\equiv$". $\square$

Corollary 3.5.2. *Let $N_{M_l}(A_l)$ be the group normalizer of the MASA $A_l \subset M_l$, and consider the von Neumann algebra generated by those normalizers*

$$\bigotimes_{l \in \Lambda} N_{M_l}(A_l)''.$$

Then,

$$N_M(A)'' = \bigotimes_{l \in \Lambda} N_{M_l}(A_l)''.$$

In particular, a tensor product of singular MASAs is a singular MASA.

Proof. Consider $l \in \Lambda$ and the map from the normalizer to the group of measure-preserving Borel isomorphisms,

$$\begin{aligned} \Theta^l : N_{M_l}(A_l) &\longrightarrow \mathfrak{I}(Y_l,\nu_l) \\ u_l &\longmapsto \Theta^l_{u_l}, \end{aligned}$$

such that

$$(u_l^* f u_l)(t) = f(\Theta^l_{u_l}(t))$$

for any $f \in A_l$, $u_l \in N_{M_l}(A_l)$ and $t \in Y_l$. Let us choose some countable subgroups $G_l < N_{M_l}(A_l)$ such that

$$\{G_l \cup A\}'' = N_{M_l}(A_l)''.$$

Such groups exist by proposition 3.3.3, and satisfy that the equivalence relation

$$\mathcal{N}_{G_l} = \{(\Theta^l_{u_l}(t),t), u_l \in G_l, t \in Y_l\} \equiv \mathcal{Y}_l,$$

for any $l \in \Lambda$. Let $G < N_M(A)$ be the countable subgroup generated by the $u = \bigotimes_l u_l$, where $u_l \in G_l$ and $u_l \neq 1$ only for finitely many $l \in \Lambda$. Consider the group homomorphism

$$\Theta : N_M(A) \longrightarrow \mathfrak{I}(Y,\nu),$$

it satisfies that

$$f(\Theta_u(t)) = (u^* f u)(\underline{t}) = \bigotimes_l (u_l^* f_l u_l)(t_l) = \bigotimes_l f_l(\Theta^l_{u_l}(t_l)),$$

for any $f = \bigotimes_l f_l \in A$, any unitary $u = \bigotimes_l u_l \in G$ and for almost everywhere $\underline{t} = (t_l)_l \in Y$. We have that the orbital equivalence relation associated is equal to

$$\mathcal{N}_G = \{(\Theta_u(\underline{t}),\underline{t}), \underline{t} \in Y, u \in G\}.$$

But, the l-component of $(\Theta_u(\underline{t}),\underline{t})$ is equal to $(\Theta^l_{u_l}(t_l),t_l)$ if $u = \bigotimes_l u_l$. Hence,

$$\mathcal{N}_G = \{(\underline{s},\underline{t}) \in Y^2, \forall l \in \Lambda, (s_l,t_l) \in \mathcal{N}_{G_l}\}.$$

By theorem 3.5.1 and theorem 3.4.1, we get that

$$\mathcal{N}_G \equiv \mathcal{R}.$$

Therefore, by remark 3.4.4, we have that $\{G \cup A\}'' = N_M(A)''$. On the other hand, G is generated by the G_l. Hence,

$$\{G \cup A\}'' = \bigotimes_l \{G_l \cup A_l\}'' = \bigotimes_l N_{M_l}(A_l)''.$$

Therefore,

$$N_M(A)'' = \bigotimes_l N_{M_l}(A_l)''.$$

$\square$

Remark 3.5.3. Corollary 3.5.2 has been already proved by Chifan in [4], but using different technics.

3.6 The group case

Let $H < G$ be an inclusion of countable discrete groups such that H is abelian and the inclusion of group von Neumann algebras $L(H) \subset L(G)$ is a MASA. We don't require that G has infinite conjugacy classes (ICC), which is equivalent to ask that $L(G)$ is a II_1 factor. Hence, $L(G)$ is just a finite von Neumann algebra with the trace that sends the component on the identity of the group. The generalization of the different invariants and previous results for finite von Neumann algebras with a fixed trace are obvious.

We review a characterization of the fact that $H < G$ gives a MASA:

Proposition 3.6.1. *The inclusion $L(H) \subset L(G)$ is maximal abelian if and only if for any $g \in G \backslash H$, the set*

$$\{hgh^{-1}, h \in H\}.$$

is infinite

Proof. See Godement [17] $\square$

Let $S \subset G$ be a system of left cosets representatives of H. We denote the unity of G by 1 and assume that $1 \in S$. Consider the two functions $\sigma : G \to S$ and $\eta : G \to H$ such that for any $g \in G$ we have $g = \sigma(g)\eta(g)$.

Notation 3.6.2. If I is a set we denote the Hilbert space of square summable complex valued function on I by $\ell^2(I)$. We denote the standard basis of $\ell^2(G)$ (resp. $\ell^2(H)$) by $\{e_g, g \in G\}$ (resp. $\{e_h, h \in H\}$). We denote the standard basis of $\ell^2(S)$ by $\{\varepsilon_s, s \in S\}$ to avoid any confusions.

Those two functions induce a unitary transformation:

$$\begin{aligned} \varphi_1 : \ell^2(G) &\longrightarrow \ell^2(S) \otimes \ell^2(H) \\ e_g &\longmapsto \varepsilon_{\sigma(g)} \otimes e_{\eta(g)}. \end{aligned}$$

Let $(\widehat{H},\mathcal{D},\nu)$ be the dual group of H with its Borel σ-algebra $\mathcal{D}$ and its left invariant Haar measure ν,
i.e. $\widehat{H} = \mathrm{Hom}(H,\mathbb{T})$ is the compact group of group homomorphisms from H with values in the circle group $\mathbb{T}$. Consider the Fourier transform:

$$\mathcal{F} : \ell^2(H) \longrightarrow L^2(\widehat{H},\nu),$$

such that $\mathcal{F}(e_h)(t) = t(h)$ for any $h \in H$ and any character $t \in \widehat{H}$. The Fourier transform is a unitary transformation. Let $L^2(\widehat{H},\ell^2(S),\nu)$ be the Hilbert space of measurable functions $f : \widehat{H} \longrightarrow \ell^2(S)$ such that

$$\int_{\widehat{H}} \|f(t)\|^2_{\ell^2(S)} d\nu(t) < \infty.$$

It is isomorphic to the tensor product of Hilbert spaces $\ell^2(S) \otimes L^2(\widehat{H},\nu)$ via the unitary transformation

$$\varphi_2 : \ell^2(S) \otimes L^2(\widehat{H},\nu) \longrightarrow L^2(\widehat{H},\ell^2(S),\nu)$$

defined as follows: $\varphi_2(x \otimes f)(t) = f(t)x$, where $x \in \ell^2(S)$, $f \in L^2(\widehat{H},\nu)$ and $t \in \widehat{H}$.

Consider

$$\phi := \varphi_2 \circ (1_{\ell^2(S)} \otimes \mathcal{F}) \circ \varphi_1,$$

where the identity function on $\ell^2(S)$ is denoted by $1_{\ell^2(S)}$. It is a unitary transformation from $\ell^2(G)$ into $L^2(\widehat{H},\ell^2(S),\nu)$ that satisfies $\phi(e_g)(t) = t(\eta(g))\varepsilon_{\sigma(g)}$, for any $g \in G$ and any $t \in \widehat{H}$. This unitary ϕ, defined a decomposition of the Hilbert space $\ell^2(G)$ in the constant field of Hilbert spaces $\ell^2(S)$ on the standard probability space $(\widehat{H},\mathcal{D},\nu)$. Consider the right action of G on the Hilbert space $\ell^2(G)$ that is $\rho(g)(e_{\tilde{g}}) = e_{\tilde{g}g^{-1}}$. The right action of H, $\rho(H)$, generates the diagonal algebra in this decomposition. Not that the von Neumann algebra generated by H is isomorphic to the von Neumann algebra of complex-valued, Borel, bounded functions $L^\infty(\widehat{H},\nu)$.

Let $C_r^*(G) \subset L(G)$ be the reduced C^*-algebra of G, it is dense and separable for the ultraweak topology in $L(G)$. Let π be the restriction of the left regular representation of $C_r^*(G)$ in $l^2(G)$.

Proposition 3.6.3. *The decomposition of $\ell^2(G)$ in $L^2(\widehat{H},\ell^2(S),\nu)$ induces a disintegration of the representation π of the C^*-algebra $C_r^*(G)$ that is:*

$$\pi = \int_{\widehat{H}}^{\oplus} \pi_t d\nu(t),$$

where the representation $\pi_t : C_r^*(G) \to \mathcal{B}(l^2(S))$ is defined as follow:

$$\pi_t(g)(\varepsilon_s) = t(\eta(gs))\varepsilon_{\sigma(gs)},$$

where $t \in \widehat{H}$ is a character, $s \in S$, $g \in G$.

Proof. Let $g \in G$, $s \in S$, we identify $\ell^2(G)$ and $L^2(\widehat{H},\ell^2(S),\nu)$ via ϕ. We denote, as in the previous section, the elements of $L^2(\widehat{H},\ell^2(S),\nu)$ as direct integral of vectors in $\ell^2(S)$. We have that

$$e_s = \int_{\widehat{H}}^{\oplus} \varepsilon_s d\nu(t)$$

is the constant vector, and

$$\pi(g)(e_s) = e_{gs} = \int_{\widehat{H}}^{\oplus} t(\eta(gs))\varepsilon_{\sigma(gs)} d\nu(t).$$

By uniqueness of the disintegration, we get that π_t, as defined in the proposition, is a representation almost everywhere of C^*-algebra and that

$$\pi = \int_{\widehat{H}}^{\oplus} \pi_t d\nu(t).$$

Consider K the set of t such that π_t is a representation, it is measurable and we have that $\nu(K) = \nu(\widehat{H})$, which is finite because $\widehat{H}$ is compact. Suppose there exists $\gamma \in \widehat{H}$ such that π_γ is not a representation. It is easy to see that $\gamma.K \cap K = \emptyset$ where $\gamma.K = \{\gamma t, t \in K\}$. The Haar measure is left invariant and so satisfies that $\nu(K) = \nu(\gamma.K)$, then $\nu(\widehat{H}) \geqslant \nu(K \cup \gamma.K) = 2\nu(K)$, a contradiction. □

Remark 3.6.4. This proposition could be proved algebraically, we choose to use measure theory to do it.

Let us begin by a lemma on the normalizer

$$N_G(H) = \{g \in G, gHg^{-1} = H\} :$$

Lemma 3.6.5. *Let $H < G$ be an inclusion of countable discrete groups such that $L(H) \subset L(G)$ is a MASA. Let $p : G \twoheadrightarrow G/H$ be the canonical projection from G to the set of right cosets. Then $p(HgH)$ is finite if and only if $p(HgH)$ is a point, i.e. g is in the normalizer $N_G(H)$. In other terms, the quasi-normalizer is equal to the normalizer.*

Proof. Let $g \in G$ such that $p(HgH)$ is finite. Thus, there exists a finite subset F of H such that

$$FgH = HgH.$$

Consider $h_0 \in H$ and $g_0 = gh_0g^{-1}$. We need to show that g_0 belongs to H. We have supposed that $L(H) \subset L(G)$ is a MASA, hence by proposition 3.6.1, for any $\widehat{g} \in G \backslash H$, the set $\{h\widehat{g}h^{-1}, h \in H\}$ is infinite. Let us show that $\{hg_0h^{-1}, h \in H\}$ is finite. Let $h \in H$, consider the product hg. It is in HgH, but $HgH = FgH$, hence there exists $f \in F$ and $k \in H$ such that $hg = fgk$. Therefore,

$$hg_0h^{-1} = fgkh_0k^{-1}g^{-1}f^{-1} = fgh_0g^{-1}f^{-1} = fg_0f^{-1} \in \{lg_0l^{-1}, l \in F\}.$$

Hence,

$$\{hg_0h^{-1}, h \in H\}$$

is finite, that implies that $g_0 \in H$. Thus, g is in the normalizer $N_G(H)$. □

Let $\mathcal{R}$ be the equivalence relation of $\widehat{H}$ defined such that $(t,\gamma) \in \mathcal{R}$ if and only if the representations of the C^*-reduced algebra of G, π_t and π_γ, are unitarily equivalent. It is the Takesaki equivalence relation.

Let $\mathcal{WR}$ be the equivalence relation of $\widehat{H}$ defined such that $(t,\gamma) \in \mathcal{WR}$ if and only if the representations restricted to the group H, $\pi_t|_H$ and $\pi_\gamma|_H$, are unitarily equivalent. It is the weak Takesaki equivalence relation.

Consider the action of the group normalizer

$$\begin{aligned} ad : N_G(H) &\longrightarrow Aut(\widehat{H},\nu) \\ k &\longmapsto ad_k \end{aligned}$$

defined as follows:

$$ad_k(t)(h) = t(khk^{-1}),$$

where $t \in \widehat{H}$, $k \in N_G(H)$, and $h \in H$. Let $\mathcal{N}$ be the equivalence relation induced by this action, i.e.

$$\mathcal{N} = \{(ad_k(t),t), t \in \widehat{H}, k \in N_G(H)\}.$$

Theorem 3.6.6. *The three equivalence relations $\mathcal{R}$, $\mathcal{WR}$ and $\mathcal{N}$ are equivalent for the relation "$\equiv$", i.e.*

$$\mathcal{R} \equiv \mathcal{WR} \equiv \mathcal{N}.$$

Proof. By definition, $\mathcal{R} \subset \mathcal{WR}$.

Let us show that $\mathcal{WR} \subset \mathcal{N}$. Let $(t,\gamma) \in \mathcal{WR}$, and v a unitary of $\ell^2(S)$ such that $v^*\pi_\gamma(h)v = \pi_t(h)$ for any $h \in H$. Consider $h \in H$,

$$\pi_\gamma(h)v(\varepsilon_1) = v\pi_t(h)(\varepsilon_1) = v(t(h)\varepsilon_1) = t(h)v(\varepsilon_1).$$

We denote by $y = \sum_{s \in S} y_s\varepsilon_s$ the vector $v(\varepsilon_1) \in \ell^2(S)$, where $y_s \in \mathbb{C}$. Thus, for any h in H we have

$$\pi_\gamma(h)(y) = t(h)y.$$

If we decompose this identity in the orthogonal basis $\{\varepsilon_s\}_{s \in S}$ we obtain for any $h \in H$ and any $s \in S$,

$$y_s\gamma(\eta(hs)) = y_{\sigma(hs)}t(h). \tag{3.3}$$

Let $s \in S$ such that $y_s \neq 0$. The last identity tell us that for any h in H, $|y_{\sigma(hs)}| = |y_s|$. Therefore $p(HsH)$ is finite, where $p : G \twoheadrightarrow G/H$ is the canonical projection on the set of the right cosets. By lemma 3.6.5, s is in the normalizer $N_G(H)$ thus $\sigma(hs) = s$ and $\eta(hs) = s^{-1}hs$. In particular, equation 3.3 implies that $\gamma = ad_s(t)$. Hence $\mathcal{WR} \subset \mathcal{N}$.

Let us show that $\mathcal{N} \prec \mathcal{R}$. The equivalence relation $\mathcal{N}$ is induced by the action of a countable subgroup of the normalizer $N_M(A)$ where the MASA $A \subset M$ is equal to the inclusion of group von Neumann algebras $L(H) \subset L(G)$. Hence, by theorem 3.4.1, we have that $\mathcal{N} \prec \mathcal{R}$. Therefore, we have proved the theorem. $\square$

Remark 3.6.7. We show that the three sets are equivalent under "$\equiv$", we can in fact show that they are equal. To do this, we proceed in a similar way that the end of the proof of proposition 3.6.3 where we used the left invariance of the Haar measure ν. Hence, an irreducible representation of the group G that appears in the decomposition given in proposition 3.6.3 is determined by its restriction to the abelian subgroup H.

Corollary 3.6.8. *Let $H < G$ be an inclusion of countable discrete groups such that $L(H) \subset L(G)$ is a MASA, then the von Neumann algebras generated by $N_G(H)$ and $N_{L(G)}(L(H))$ are equals, i.e.*

$$N_G(H)'' = N_{L(G)}(L(H))''.$$

Proof. Consider the countable subgroup of the normalizer of the algebras $N_G(H) < N_M(A)$, where $A \subset M = L(H) \subset L(G)$. We have shown, in theorem 3.6.6, that the equivalence relation induced by $N_G(H)$ is equivalent to the Takesaki equivalence relation $\mathcal{R}$. Hence, by remark 3.4.4, we have that

$$\{N_G(H) \cup L(H)\}'' = N_M(A)''.$$

But $N_G(H)'' = \{N_G(H) \cup L(H)\}''$. Therefore,

$$N_G(H)'' = N_M(A)''.$$

$\square$

Remark 3.6.9. If we drop the condition that $L(H) \subset L(G)$ is a MASA, the last corollary is false.

Remark 3.6.10. Suppose that $H < G$ is an inclusion of countable discrete groups such that $L(H) \subset L(G)$ is a MASA. It means that for any $g \in G \backslash H$, the set $\{hgh^{-1}, h \in H\}$ is infinite. One can define a continuous family of representations of G indexed by the dual group $\widehat{H}$. Suppose that the group normalizer $N_G(H)$ is equal to H. This implies, by theorem 3.6.6 and remark 3.6.7 that all of those representations are pairwise non unitarily equivalent. They are irreducible almost everywhere, it is not hard to show that they are in fact all irreducible. This provides us a way to produce explicitly many irreducible representations of a group.

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